Lucas Tree Models

Financial Economics II

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Consider an economy with a representative consumer with preferences described by $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ where $u(c_t) = \ln(c_t + \gamma)$ where $\gamma \geq 0$ and $c_t$ denotes consumption of the fruit in period $t$. The sole source of the single good is an everlasting tree that produces $d_t$ units of the consumption good in period $t$. The dividend process $d_t$ is Markov, with $\text{prob}\{d_{t+1} \leq d' \mid d_t = d\} = F(d', d)$. Assume the conditional density $f(d', d)$ of $F$ exists. There are competitive markets in the title of trees and in state-contingent claims. Let $p_t$ be the price at $t$ of a title to all future dividends from the tree.

(a) Prove that the equilibrium price $p_t$ satisfies

$$p_t = (d_t + \gamma) \sum_{j=1}^{\infty} \beta^j E_t \left( \frac{d_{t+j}}{d_{t+j} + \gamma} \right)$$

Consumer optimizes the following household problem.

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Budget constraint:

$$A_{t+1} = R_t (A_t + y_t - c_t)$$

where $c_t, y_t, A_t, R_t$ indicate the consumption of an agent at time $t$, the agent’s labor income, the amount of a single asset valued in units of consumption good, and the real gross rate of return on the asset between time $t$ and $t+1$. The Euler equation gives the following condition.

$$u'(c_t) = E_t \beta R_t u'(c_{t+1})$$

The above equation does not spell out complete general equilibrium setups. Lucas’s asset pricing model does use general equilibrium reasoning.

Lucas model assumptions:
The labor income is zero.
The durable good in the economy is only a set of trees.
Representative agent assumption.
The fruit is nonstorable.

Recall \( c_t = d_t \) in a general equilibrium.
Letting \( R_t = \frac{p_{t+1} + d_{t+1}}{p_t} \), the Euler equation will be:

\[
E_t \beta \frac{u'(c_{t+1})}{u'(c_t)} \left( \frac{p_{t+1} + d_{t+1}}{p_t} \right) = 1
\]

\[
p_t = E_t \beta \frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1})
\]

Using the equilibrium condition \( c_t = d_t \).

\[
p_t = E_t \beta \frac{u'(d_{t+1})}{u'(d_t)} (p_{t+1} + d_{t+1})
\]

Since \( u(c_t) = \ln(c_t + \gamma) \),

\[
p_t = E_t \beta \frac{d_t + \gamma}{(d_{t+1} + \gamma)} (p_{t+1} + d_{t+1})
\]

The price at time \( t+1 \) is as follows:

\[
p_{t+1} = E_t \beta \frac{d_{t+1} + \gamma}{(d_{t+2} + \gamma)} (p_{t+2} + d_{t+2})
\]

By plugging \( p_{t+1} \) back into \( p_t \),

\[
p_t = E_t \beta \frac{d_t + \gamma}{(d_{t+1} + \gamma)} \left( \beta \frac{d_{t+1} + \gamma}{(d_{t+2} + \gamma)} (p_{t+2} + d_{t+2}) + d_{t+1} \right)
\]

\[
= E_t \left( \beta \frac{d_t + \gamma}{(d_{t+1} + \gamma)} d_{t+1} + \beta^2 \frac{d_t + \gamma}{(d_{t+2} + \gamma)} d_{t+2} + \beta^2 \frac{d_t + \gamma}{(d_{t+2} + \gamma)} p_{t+2} \right)
\]

Recursively,

\[
p_{t+1} = E_t \sum_{j=1}^{\infty} \beta^j \frac{d_t + \gamma}{(d_{t+j} + \gamma)} d_{t+j} + \lim_{j \to \infty} E_t \beta^j \frac{d_t + \gamma}{(d_{t+j} + \gamma)} p_{t+j}
\]
Since \( \lim_{j \to \infty} E_t \beta^j \frac{(d_t + \gamma)}{(d_{t+j} + \gamma)} p_{t+j} = 0 \), we obtain the final pricing formula.

\[
p_{t+1} = E_t \sum_{j=1}^{\infty} \beta^j \frac{(d_t + \gamma)}{(d_{t+j} + \gamma)} d_{t+j}
\]

(b) Find a formula for the risk-free one-period interest rate \( R_{1t} \). Prove that in the special case in which \( \{d_t\} \) is independently and identically distributed, \( R_{1t} \) is given by \( R_{1t}^{-1} = \beta_k (d_t + \gamma) \), where \( k \) is a constant. Give a formula for \( k \).

We now suppose that there are markets in one- and two-period perfectly safe loans, which bear gross rates of return \( R_{1t} \) and \( R_{2t} \). At the beginning of time \( t \), the returns \( R_{1t} \) and \( R_{2t} \) are known with certainty and are risk free from the viewpoint of the agents. That is, at time \( t \), \( R_{1t}^{-1} \) is the price of a perfectly sure claim to one unit of consumption at time \( (t+1) \), and \( R_{2t}^{-1} \) is the price of a perfectly sure claim to one unit of consumption at time \( (t+2) \). The representative agent solves the following optimization problem:

\[
\max_{c_t, L_{1t+1}, L_{2t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)
\]

subject to the budget constraint:

\[
c_t + L_{1t} + L_{2t} \leq d_t + L_{1t-1}R_{1t-1} + L_{2t-2}R_{2t-2}
\]

where \( L_{jt} \) is the amount lent for \( j \) periods at time \( t \).

Using the Lagrange Multiplier method,

\[
L = E_0 \sum_{t=0}^{\infty} \beta^t (u(c_t) + \lambda_t (d_t + L_{1t-1}R_{1t-1} + L_{2t-2}R_{2t-2} - c_t - L_{1t} - L_{2t}))
\]

By taking differentiations with respect to \( \{c_t, L_{1t}, L_{2t}\} \),

\[
\begin{align*}
    c_t & : \quad E_0 \beta^t (u'(c_t) - \lambda_t) = 0 \\
    L_{1t} & : \quad E_0 \beta^t (\beta \lambda_{t+1} - \lambda_t) = 0 \\
    L_{2t} & : \quad E_0 \beta^t (\beta^2 \lambda_{t+2} - \lambda_t) = 0
\end{align*}
\]
Using the Markov property $E_0 = E_t$.

\begin{align*}
  c_t & : \lambda_t = u'(c_t) \\
  L_{1t} & : \lambda_t = E_t(\beta \lambda_{t+1} R_{1t}) \\
  L_{2t} & : \lambda_t = E_t(\beta^2 \lambda_{t+2} R_{2t})
\end{align*}

Combining the first-order conditions gives:

\begin{align*}
  E_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} R_{1t} \right) &= 1 \\
  E_t \left( \beta^2 \frac{u'(c_{t+2})}{u'(c_t)} R_{2t} \right) &= 1
\end{align*}

Assuming the risk-free interest rates,

\begin{align*}
  R_{1t}^{-1} &= E_t \left( \beta \frac{c_t + \gamma}{c_{t+1} + \gamma} \right) \\
  R_{2t}^{-1} &= E_t \left( \beta^2 \frac{c_t + \gamma}{c_{t+2} + \gamma} \right)
\end{align*}

Since $u(c_t) = \ln(c_t + \gamma)$,

\begin{align*}
  R_{1t}^{-1} &= E_t \left( \beta \frac{d_t + \gamma}{d_{t+1} + \gamma} \right) \\
  R_{2t}^{-1} &= E_t \left( \beta^2 \frac{d_t + \gamma}{d_{t+2} + \gamma} \right)
\end{align*}

Recall $c_t = d_t$ in a general equilibrium.

\begin{align*}
  R_{1t}^{-1} &= E_t \left( \beta \frac{d_t + \gamma}{d_{t+1} + \gamma} \right) \\
  R_{2t}^{-1} &= E_t \left( \beta^2 \frac{d_t + \gamma}{d_{t+2} + \gamma} \right)
\end{align*}

By letting $k_{1t} = E_t \left( \frac{1}{d_{t+1} + \gamma} \right)$, the pricing formula can be expressed as:

\begin{equation*}
  R_{1t}^{-1} = \beta k_{1t}(d_t + \gamma)
\end{equation*}
(c) Find a formula for the risk-free two-period interest rate \( R_{2t} \). Prove that in the special case in which \( \{d_t\} \) is independently and identically distributed, \( R_{2t} \) is given by \( R_{2t}^{-1} = \beta^2 k (d_t + \gamma) \), where \( k \) is the same constant you found in part (b).

By letting \( k_{2t} = E_t \left( \frac{1}{d_{t+2} + \gamma} \right) \), the pricing formula can be expressed as:

\[ R_{2t}^{-1} = \beta^2 k_{2t} (d_t + \gamma) \]

Let me show that \( k_{1t} \) and \( k_{2t} \) are identical. Since \( d_t \) are identically distributed and follow the Markov chain,

\[ k_{2t} = E_t \left( \frac{1}{d_{t+2} + \gamma} \right) = E_{t+1} \left( \frac{1}{d_{t+2} + \gamma} \right) = k_{1t} \]

2. Consider the following version of the Lucas's tree economy. There are two kinds of trees. The first kind is ugly and gives no direct utility to consumers, but yields a stream of fruit \( \{d_{1t}\} \), where \( d_{1t} \) denotes a positive random process obeying a first-order Markov process. The second tree is beautiful and yields utility on itself. This tree also yields a stream of the same kind of fruit \( d_{2t} \), where it happens that \( d_{2t} = d_{1t} = \left( \frac{1}{2} \right) d_t \ \forall \ t \), so that the physical yields of the two kinds of trees are equal. There is one of each tree for each \( N \) individuals in the economy. Trees last forever, but the fruit is not storable. Trees are the only source of fruit.

Each of the \( N \) individuals in the economy has preferences described by

\[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, s_{2t}) \tag{1} \]

where \( u(c_t, s_{2t}) = \ln c_t + \gamma \ln(s_{2t}) \) where \( \gamma \geq 0 \), \( c_t \) denotes consumption of the fruit in period \( t \) and \( s_{2t} \) is the stock of beautiful trees owned at the beginning of the period \( t \). The owner of a tree of either kind \( i \) at the start of the period receives the fruit \( d_{it} \) produced by the tree during
that period.

Let \( p_t \) be the price of a tree of type \( i \) (where \( i = 1, 2 \)) during period \( t \). Let \( R_t \) be the gross rate of returns of tree \( i \) during that period held from period \( t \) to \( t + 1 \).

(a) Write down the consumer optimization problem in sequential and recursive form.

**Consumer optimization in a recursive form**

The Bellman’s equation is given by

\[
v(d_t, s_{1t}, s_{2t}) = \max_{\{c_t, s_{1t+1}, s_{2t+1}\}} \left( \ln(c_t) + \gamma \ln(s_{2t}) + E_t \beta v(d_t, s_{1t+1}, s_{2t+1}) \right)
\]

where \( c_t + p_{1t} s_{1t+1} + p_{2t} s_{2t+1} \leq (d_{1t} + p_{1t}) s_{1t} + (d_{2t} + p_{2t}) s_{2t} \).

**Consumer optimization in a sequential form**

The sequential form is given by

\[
\max_{\{c_t, s_{1t+1}, s_{2t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \left( \ln(c_t) + \gamma \ln(s_{2t}) \right)
\]

where \( c_t + p_{1t} s_{1t+1} + p_{2t} s_{2t+1} \leq (d_{1t} + p_{1t}) s_{1t} + (d_{2t} + p_{2t}) s_{2t} \).

(b) Define a rational expectations equilibrium.

**Definition** The following is called the *market clear condition*

\[
\sum_{i=1}^{I} c^i_t = \sum_{i=1}^{I} d^i_t \tag{2}
\]

\[
\sum_{i=1}^{I} s^i_{1t} = \sum_{i=1}^{I} s^i_{10} = I
\]

\[
\sum_{i=1}^{I} s^i_{2t} = \sum_{i=1}^{I} s^i_{20} = I
\]
where \( s_{1t}^i \) and \( s_{2t}^i \) are each agent’s number of trees at initial time.

**Definition** A *sequential household problem* is defined by each agent’s utility optimization problem:

\[
\max_{\{c_t, s_{1t+1}, s_{2t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t (\ln(c_t) + \gamma \ln(s_{2t}))
\]

where \( c_t + p_{1t} s_{1t+1} + p_{2t} s_{2t+1} \leq (d_{1t} + p_{1t}) s_{1t} + (d_{2t} + p_{2t}) s_{2t} \).

**Definition** A *rational competitive equilibrium* is an allocation, \( \{\{c_t^i\}_{t=0}^\infty\}_{i=1}^I, \{\{s_{1t}^i, s_{2t}^i\}_{t=0}^\infty\}_{i=1}^I \), and a price system, \( \{p_{1t}, p_{2t}\}_{t=0}^\infty \), such that the allocation solves each household problem and satisfies the market clear condition.

(c) Find the pricing functions mapping the state of the economy at \( t \) onto \( p_{1t} \) and \( p_{2t} \) (give precise formulas). [Hint: You should be able to directly derive \( p_{1t} \) from the example seen in class, then since pricing function have to be linear you can guess a pricing function \( p_{2t} = kd_t \) and solve for \( k \) parameter using Euler equation of the second stock.]

I am going to use the sequential form to find a solution. Using the Lagrange Multiplier,

\[
L = E_0 \sum_{t=0}^{\infty} \beta^t ((\ln(c_t) + \gamma \ln(s_{2t}))
+ \lambda_t ((d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t} - c_t - p_{1t}s_{1t+1} - p_{2t}s_{2t+1}))
\]

By taking differentiations with respect to \( \{c_t, s_{1t+1}, s_{2t+1}\} \).

\[
\begin{align*}
c_t & : E_0 \beta^t (\frac{1}{c_t} - \lambda_t) = 0 \\
s_{1t+1} & : E_0 \beta^t (\beta \lambda_{t+1} (d_{1t+1} + p_{1t+1}) - \lambda_t p_{1t}) = 0 \\
s_{2t+1} & : E_0 \beta^t \left( \beta \lambda_{t+1} (d_{2t+1} + p_{2t+1}) - \lambda_t p_{2t} + \beta \frac{\gamma}{s_{2t+1}} \right) = 0
\end{align*}
\]
Using the Markov property $E_0 = E_t$.

\[ c_t : \frac{1}{c_t} = \lambda_t \]

\[ s_{1t+1} : p_{1t} = E_t \beta \frac{\lambda_{t+1}}{\lambda_t} (d_{1t+1} + p_{1t+1}) \]

\[ s_{2t+1} : p_{2t} = E_t \beta \frac{\lambda_{t+1}}{\lambda_t} (d_{2t+1} + p_{2t+1}) + \beta \frac{\gamma}{s_{2t+1}} \]

Combining the first-order conditions, we can obtain the pricing formula.

\[ p_{1t} = E_t \beta \frac{c_t}{c_{t+1}} (d_{1t+1} + p_{1t+1}) \]

\[ p_{2t} = E_t \beta \frac{c_t}{c_{t+1}} (d_{2t+1} + p_{2t+1}) + \beta \frac{\gamma c_t}{s_{2t+1}} \]

Recall \( c_t = d_t \) in a general equilibrium.

\[ p_{1t} = E_t \beta \frac{d_t}{d_{t+1}} (d_{1t+1} + p_{1t+1}) \]

\[ p_{2t} = E_t \beta \frac{d_t}{d_{t+1}} (d_{2t+1} + p_{2t+1}) + \beta \frac{\gamma d_t}{s_{2t+1}} \]

Let us assume the linear form of the first tree’s pricing function.

\[ p_{1t} = k_{1t} d_t \]

Next apply the above linear form to the Euler equation.

\[ k_{1t} d_t = E_t \beta \frac{d_t}{d_{t+1}} (d_{1t+1} + k_{1t+1} d_{t+1}) \]

\[ k_{1t} = E_t \beta k_{1t+1} + E_t \beta \frac{d_{1t+1}}{d_{t+1}} \]

Recursively we obtain \( k_{1t} \).

\[ k_{1t} = E_t \sum_{j=1}^{\infty} \beta^j \frac{d_{1t+j}}{d_{t+j}} \]
Since we are given \( d_{1t} = \frac{1}{2} d_t \).

\[
\begin{align*}
k_{1t} &= \frac{1}{2} E_t \sum_{j=1}^{\infty} \beta^j \\
&= \frac{\beta}{2(1 - \beta)}
\end{align*}
\]

Let us assume the linear form of the second tree’s pricing function.

\[
p_{2t} = k_{2t} d_t
\]

Next apply the above linear form to the Euler equation.

\[
k_{2t} d_t &= E_t \beta \frac{d_t}{d_{t+1}} (d_{2t+1} + k_{2t+1} d_{t+1}) + \beta \frac{\gamma d_t}{s_{2t+1}} \\
k_{2t} &= E_t \beta k_{2t+1} + E_t \beta \left( \frac{d_{2t+1}}{d_{t+1}} + \frac{\gamma}{s_{2t+1}} \right)
\]

Recursively we obtain \( k_{2t} \).

\[
k_{2t} = E_t \sum_{j=1}^{\infty} \beta^j \left( \frac{d_{2t+j}}{d_{t+j}} + \frac{\gamma}{s_{2t+1}} \right)
\]

Since we are given \( d_{2t} = \frac{1}{2} d_t \) and \( s_{2t+1} = 1 \) in equilibrium.

\[
k_{2t} = E_t \sum_{j=1}^{\infty} \beta^j \left( \frac{1}{2} + \gamma \right)
= \left( \frac{1}{2} + \gamma \right) \frac{\beta}{1 - \beta}
\]

Finally, we have got the pricing equations.

\[
p_{1t} = \left( \frac{\beta}{2(1 - \beta)} \right) d_t \\
p_{2t} = \left( \left( \frac{1}{2} + \gamma \right) \frac{\beta}{1 - \beta} \right) d_t
\]
(d) Prove that if $\gamma > 0$, then $R_{1t} > R_{2t} \forall t$

The returns $R_{1t}, R_{2t}$ are defined as:

$$R_{1t} = \frac{p_{1t+1} + d_{1t+1}}{p_{1t}}$$

$$R_{2t} = \frac{p_{2t+1} + d_{2t+1}}{p_{2t}}$$

From the derived pricing equations,

$$R_{1t} = \frac{1}{2} \left( \frac{\beta}{1 - \beta} \right) d_{t+1} + \frac{1}{2} d_{t+1}$$

$$R_{2t} = \frac{\left( \frac{1}{2} + \gamma \right) \left( \frac{\beta}{1 - \beta} \right) d_{t+1} + \frac{1}{2} d_{t+1}}{\left( \frac{1}{2} + \gamma \right) \left( \frac{\beta}{1 - \beta} \right) d_{t}}$$

By rearranging the equations,

$$R_{1t} - R_{2t} = \left( 1 - \frac{1}{1 + 2\gamma} \right) \frac{1 - \beta}{\beta} \frac{d_{t+1}}{d_{t}}$$

If $\gamma > 0$, then $R_{1t} > R_{2t}$