Supplemental material for analysis of the proportional hazards model with sparse longitudinal covariates

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Abstract

This supplementary material provides details on the proofs of Theorem 1, Theorem 2 and Corollary 1, 2 and 3.

1 Proofs of Lemmas

1.1 Proof of Lemma 1

Lemma 1 *Under conditions of Theorem 1, we have* $B_n(\beta, \tau) = o_p(1)$, *where*

$$\begin{aligned} B_n(\beta, t) &= n^{-2} \sum_{i=1}^{n} \int_0^t \Big(\sum_{k=1}^{M_i} K_{h_n}(u - R_{ik}) I(R_{ik} \le u) \Big[(\beta - \beta_0)^T Z_i(R_{ik}) \\ &- log \Big\{ \frac{S_n^{(0)}(\beta, u)}{S_n^{(0)}(\beta_0, u)} \Big\} \Big] \Big)^2 Y_i(u) e^{\beta_0^T Z_i(u)} \lambda_0(u) du. \end{aligned}$$

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Proof. We can write $B_n(\beta, t)$ as

$$
\begin{aligned}
B_n(\beta,t) &= n^{-1}\int_0^t n^{-1}\sum_{i=1}^n \int_0^\infty \int_0^\infty \{K_{h_n}(u-r_1)I(r_1\le u)dN^*(r_1)K_{h_n}(u-r_2)I(r_2\le u)dN^*(r_2)\} \\
& \quad \Big((\beta-\beta_0)^T Z_i(r_1)Z_i(r_2)^T(\beta-\beta_0) - (\beta-\beta_0)^T Z_i(r_1)\log\Big\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\Big\} \\
& - (\beta-\beta_0)^T Z_i(r_2)\log\Big\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\Big\} + \Big[\log\Big\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\Big\}\Big]^{\otimes 2}\Big)Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du.
\end{aligned}
$$

This can be expanded further:

$$
\begin{aligned}
B_n(\beta,t) &= n^{-1}\int_0^t n^{-1}\sum_{i=1}^n \int_0^\infty \int_0^\infty K_{h_n}(u-r_1)K_{h_n}(u-r_2)I(r_1\le u)I(r_2\le u)dN^*(r_1)dN^*(r_2) \\
& \quad (\beta-\beta_0)^T Z_i(r_1)Z_i(r_2)^T(\beta-\beta_0)Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du \\
& - n^{-1}\int_0^t n^{-1}\sum_{i=1}^n \int_0^\infty \int_0^\infty K_{h_n}(u-r_1)K_{h_n}(u-r_2)I(r_1\le u)I(r_2\le u)dN^*(r_1)dN^*(r_2) \\
& \quad (\beta-\beta_0)^T Z_i(r_1)\log\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\}Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du \\
& - n^{-1}\int_0^t n^{-1}\sum_{i=1}^n \int_0^\infty \int_0^\infty K_{h_n}(u-r_1)K_{h_n}(u-r_2)I(r_1\le u)I(r_2\le u)dN^*(r_1)dN^*(r_2) \\
& \quad (\beta-\beta_0)^T Z_i(r_2)\log\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\}Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du \\
& + n^{-1}\int_0^t n^{-1}\sum_{i=1}^n \int_0^\infty \int_0^\infty K_{h_n}(u-r_1)K_{h_n}(u-r_2)I(r_1\le u)I(r_2\le u)dN^*(r_1)dN^*(r_2) \\
& \quad [\log\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\}]^{\otimes 2}Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du \\
&= I + II + III + IV.
\end{aligned}
$$

Assume that for $t \neq r$, $pr(dN^*(t) = 1 \mid N^*(r) - N^*(r-) = 1) = g(t,r)dt$, where $g(t,r)$ is continuous for $t \neq r$ and $g(t\pm, t\pm)$ exists. After taking expectation, we obtain

$$
\begin{aligned}
E[I] &= n^{-1}\int_0^t n^{-1}\sum_{i=1}^n \Big[\int_{r\neq r'} K_{h_n}(u-r)K_{h_n}(u-r')I(r\le u)I(r'\le u)E\{dN^*(r)dN^*(r')\} \\
&\quad E\Big\{(\beta-\beta_0)^T Z_i(r)Z_i(r')^T(\beta-\beta_0)Y_i(u)e^{\beta_0^T Z_i(u)}\Big\} + \int_r K_{h_n}(u-r)^2 I(r\le u)E\{dN^*(r)\} \\
&\quad E\Big\{(\beta-\beta_0)^T Z_i(r)Z_i(r)^T(\beta-\beta_0)Y_i(u)e^{\beta_0^T Z_i(u)}\Big\}\Big]\lambda_0(u)du \\
&= n^{-1}\int_0^t n^{-1}\sum_{i=1}^n \Big[\int_{r\neq r'} K_{h_n}(u-r)K_{h_n}(u-r')I(r\le u)I(r'\le u)g(r,r')E\{dN^*(r')\}dr \\
&\quad E\Big\{(\beta-\beta_0)^T Z_i(r)Z_i(r')^T(\beta-\beta_0)Y_i(u)e^{\beta_0^T Z_i(u)}\Big\} + \int_r K_{h_n}(u-r)^2 I(r\le u)\lambda^*(r)dr \\
&\quad E\Big\{(\beta-\beta_0)^T Z_i(r)Z_i(r)^T(\beta-\beta_0)Y_i(u)e^{\beta_0^T Z_i(u)}\Big\}\Big]\lambda_0(u)du \\
&= n^{-1}\int_0^t \Big(\int_r K_{h_n}(u-r)I(r\le u)\lambda^*(u)E\Big\{(\beta-\beta_0)^T Z(r)Z(u)^T(\beta-\beta_0)Y(u)e^{\beta_0^T Z(u)}\Big\} \\
&\quad \{g(r,u+)+g(r,u-)\}/4dr + o(1) + h_n{}^{-1}\Big\{\int_0^\infty K(z)^2\lambda^*(u)dz + o(1)\Big\} \\
&\quad [(\beta-\beta_0)^T n^{-1}\sum_{i=1}^n E\Big\{Z_i(u-h_nz)Z_i(u-h_nz)^T Y_i(u)e^{\beta_0^T Z_i(u)}\Big\}(\beta-\beta_0)]\Big)\lambda_0(u)du.
\end{aligned}
$$

Using change of variables, we have

$$
\begin{aligned}
E\{I\} &= n^{-1}\int_0^t \Big[\{(\beta-\beta_0)^T s^{(2)}(\beta_0,u)(\beta-\beta_0)\}\{g(u+,u+)+g(u+,u-)+g(u-,u+) \\
&+ g(u-,u-)\}/8 + o(1) + 2h_n{}^{-1}\{\int_0^\infty K(z)^2dz + o(1)\}\{(\beta-\beta_0)^T s^{(2)}(\beta_0,u)(\beta-\beta_0) \\
&+ O(h_n)\}\Big]\lambda_0(u)du \\
&= O\{(nh_n)^{-1}\}.
\end{aligned}
$$

Similar order can be obtained for II, III and IV and we omit the details here. Therefore, $B_n(\beta,\tau)$ converges in probability to 0.

1.2 Proof of Lemma 2

Lemma 2 *Under assumptions of Theorem 1,*

$$(nh_n)^{1/2}U_n(\beta_0) \to N\{0, \Sigma(\beta_0)\},$$

where $\Sigma(\beta_0)) = \int_0^\infty K(z)^2 dz \int_0^\tau \Big\{ s^{(2)}(\beta_0, u) - \frac{s^{(1)}(\beta_0,u)^{\otimes 2}}{s^{(0)}(\beta_0,u)} \Big\} \lambda_0(u)du$

Proof. We can decompose $(nh_n)^{1/2}U_n(\beta_0)$ into two parts:

$$\begin{aligned} (nh_n)^{1/2}U_n(\beta_0) &= h_n^{1/2} n^{-1/2} \sum_{i=1}^n \int_0^\tau \int_0^\infty K_{h_n}(u-r)I(r \le u)dN_i^*(r)[Z_i(r) - \bar{Z}(\beta_0, u)]dM_i(u) \\ &+ h_n^{1/2} n^{-1/2} \sum_{i=1}^n \int_0^\tau \int_0^\infty K_{h_n}(u-r)I(r \le u)dN_i^*(r)\{Z_i(r) - \bar{Z}(\beta_0, u)\}Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du \\ &= I(\beta_0, \tau) + II(\beta_0, \tau), \end{aligned}$$

where

$$dM_i(u) = dN_i(u) - Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du.$$

Note that

$$\begin{aligned} II(\beta_0, \tau) &= h_n^{1/2} n^{-1/2} \int_0^\tau \sum_{i=1}^n \sum_{j=1}^{M_i} K_{h_n}(u - R_{ij})I(R_{ij} \le u)\{Z_i(R_{ij}) - \bar{Z}(\beta_0, u)\} \\ &\quad Y_i(u)\{e^{\beta_0^T Z_i(u)} - e^{\beta_0^T Z_i(R_{ij})}\}\lambda_0(u)du. \end{aligned}$$

Define $F(u, s) = E\Big[\{Z(u-s) - \bar{Z}(\beta_0, u)\}Y(u)\{e^{\beta_0^T Z(u)} - e^{\beta_0^T Z(u-s)}\}\Big]$. After taking expectation together with Taylor expansion we have

$$\begin{aligned}E\{II(\beta_0,\tau)\} &= n^{1/2}h_n^{1/2}\int_0^\tau\int_0^\infty K(z)E\Big[\{Z(u-h_nz)-\bar Z(\beta_0,u)\}Y(u)\\ &\quad \{e^{\beta_0^TZ(u)}-e^{\beta_0^TZ(u-h_nz)}\}dz\lambda_0(u)du\\ &= n^{1/2}h_n^{1/2}\{F(u,0)+\frac{\partial F(u,s)}{\partial s}|_{s=0}h_nz+o(h_n)\}\\ &= O(n^{1/2}h_n^{3/2})=o(1)\qquad \text{by (A5).}\end{aligned}$$

Therefore $II(\beta_0,\tau)$ converges to 0 in probability.

We now derive the asymptotic normality of the term $I(\beta_0,\tau)$. By the martingale property, we have

$$\begin{aligned}< I(\beta_0), I(\beta_0) > (\tau) &= h_nn^{-1}\sum_{i=1}^n\int_0^\tau\Big[\int_0^\infty K_{h_n}(u-r)I(r\le u)dN_i^*(r)\{Z_i(r)-\bar Z(\beta_0,u)\}\Big]^2\\ &\quad Y_i(u)e^{\beta_0^TZ_i(u)}\lambda_0(u)du.\end{aligned}$$

Similar as in the derivation of $E\{B_n(\beta,t)\}$, after taking expectation and change of variables, we have

$$\begin{aligned}E\{I(\beta_0,\tau)^2\} &= E< I(\beta_0), I(\beta_0) > (\tau)\\ &= n^{-1}\sum_{i=1}^n\int_0^\tau\int_0^\infty K(z)^2\lambda^*(u)dz\\ &\quad E\Big[Z_i(u)Z_i(u)^T-2Z_i(u)\frac{S_n^{(1)}(\beta_0,u)}{S_n^{(0)}(\beta_0,u)}+\frac{S_n^{(1)}(\beta_0,u)^{\otimes 2}}{S_n^{(0)}(\beta_0,u)^2}\Big]Y_i(u)e^{\beta_0^TZ_i(u)}\lambda_0(u)du+o(1)\\ &= 2\int_0^\infty K(z)^2dz\int_0^\tau\Big[s^{(2)}(\beta_0,u)-\frac{s^{(1)}(\beta_0,u)^{\otimes 2}}{s^{(0)}(\beta_0,u)}\Big]\lambda_0(u)du+o(1).\end{aligned}$$

Next, we verify that Lindeberg condition holds for $I(\beta_0, u)$. For $\forall\epsilon > 0$, consider

$$\int_0^\tau \sum_{l=1}^n \Big[h_n^{1/2} n^{-1/2} \sum_{k=1}^{M_l} K_{h_n}(u - R_{lk}) I(R_{lk} \le u)\{Z_l(R_{lk}) - \bar{Z}(\beta_0, u)\}\Big]^2$$
$$Y_l(u) e^{\beta_0^T Z_l(u)} \lambda_0(u) I\Big\{|h_n^{1/2} n^{-1/2} \sum_{k=1}^{M_l} K_{h_n}(u - R_{lk}) I(R_{lk} \le u)\{Z_l(R_{lk}) - \bar{Z}(\beta_0, u)\}| > \epsilon\Big\} du.$$

This can be decomposed into two parts.

$$\begin{aligned}
& \int_0^\tau \sum_{l=1}^n \Big[h_n^{1/2} n^{-1/2} \sum_{k=1}^{M_l} K_{h_n}(u - R_{lk}) I(R_{lk} \le u)\{Z_l(R_{lk}) - \bar{Z}(\beta_0, u)\}\Big]^2 I(M_l \le a) \\
& Y_l(u) e^{\beta_0^T Z_l(u)^T} \lambda_0(u) I\Big\{|h_n^{1/2} n^{-1/2} \sum_{k=1}^{M_l} K_{h_n}(u - R_{lk}) I(R_{lk} \le u)\{Z_l(R_{lk}) - \bar{Z}(\beta_0, u)\}| > \epsilon\Big\} du \\
+ & \int_0^\tau \sum_{l=1}^n \Big[h_n^{1/2} n^{-1/2} \sum_{k=1}^{M_l} K_{h_n}(u - R_{lk}) I(R_{lk} \le u)\{Z_l(R_{lk}) - \bar{Z}(\beta_0, u)\}\Big]^2 I(M_l > a) \\
& Y_l(u) e^{\beta_0^T Z_l(u)^T} \lambda_0(u) I\Big\{|h_n^{1/2} n^{-1/2} \sum_{k=1}^{M_l} K_{h_n}(u - R_{lk}) I(R_{lk} \le u)\{Z_l(R_{lk}) - \bar{Z}(\beta_0, u)\}| > \epsilon\Big\} du \\
= & \; I + II.
\end{aligned}$$

By (A1),

$$I = O_p(1) \int_0^\tau I(O_p(h_n^{1/2} n^{-1/2}) > \epsilon) \lambda_0(u) du \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad h_n \to 0.$$

Since $\mathrm{pr}(M_l > a) \to 0$ as $a \to \infty$ by (A2), we have

$$II \to 0.$$

This shows that $(nh_n)^{1/2} U_n(\beta_0, \cdot)$ converges weakly to a certain continuous Gaussian process.

Since this process evaluated at time $t = \tau$ has covariance matrix $\Sigma(\beta_0)$, therefore, we have

$$(nh_n)^{1/2} U_n(\beta_0, u) \to N\{0, \Sigma(\beta_0)\}, \tag{1.1}$$

where $\Sigma(\beta_0) = 2\int_0^\infty K(z)^2 dz \int_0^\tau \left\{ s^{(2)}(\beta_0, u) - \frac{s^{(1)}(\beta_0,u)^{\otimes 2}}{s^{(0)}(\beta_0,u)} \right\} \lambda_0(u) du$.

1.3 Proof of Lemma 3

Lemma 3 *Under conditions of Theorem 2, we have*

$$\begin{aligned} & (nh_n)^{1/2} E\left[\int_0^\tau \int_0^\infty K_{h_n}(t-r) \left\{ Z(r) - \frac{\tilde{S}_n^{(1)}(\beta,t)}{\tilde{S}_n^{(0)}(\beta,t)} \right\} dN^*(r) dN(t) \right] \\ = & \ (nh_n)^{1/2} A(\beta_0)(\beta - \beta_0) + D n^{1/2} h_n^{5/2} + o\{(nh_n)^{1/2} |\beta - \beta_0|\}. \end{aligned}$$

Proof. After change of variables, we have

$$\begin{aligned} K \ \equiv & \ (nh_n)^{1/2} E\left[\int_0^\tau \int K_{h_n}(t-s) \left\{ Z(s) - \frac{\tilde{S}_n^{(1)}(\beta,t)}{\tilde{S}_n^{(0)}(\beta,t)} \right\} dN^*(s) dN(t) \right] \\ = & \ (nh_n)^{1/2} \int_0^\tau \Big[\int_z K(z) E\{Z(t - h_n z) Y(t) e^{\beta_0^T Z(t)}\} \lambda^*(t - h_n z) dz \\ - & \ \int_z K(z) \{ \frac{\tilde{s}^{(1)}(\beta,t)}{\tilde{s}^{(0)}(\beta,t)} + o_p(1) \} E\{Y(t) e^{\beta_0^T Z(t)}\} \lambda^*(t - h_n z) dz \Big] \mu_0(t) dt \end{aligned}$$

Denote $G(t,s) = E\{Z(t-s)Y(t)e^{\beta_0^T Z(t)}\}$. We then do Taylor expansions

$$\begin{aligned} K \ = & \ (nh_n)^{1/2} \int_0^\tau \Big[E\{Z(t) Y(t) e^{\beta_0^T Z(t)} \lambda^*(t)\} \Big] \mu_0(t) dt \\ - & \ (nh_n)^{1/2} \int_0^\tau \frac{\tilde{s}^{(1)}(\beta,t)}{\tilde{s}^{(0)}(\beta,t)} E\{Y(t) e^{\beta_0^T Z(t)} \lambda^*(t)\} \mu_0(t) dt + (nh_n^5)^{1/2} D_1 + o(1), \end{aligned}$$

where we used the fact that $\int_{-\infty}^{\infty} K(z)dz = 1$, $\int_{-\infty}^{\infty} zK(z)dz = 0$ and

$$D_1 = \int_z K(z)z^2dz \int_0^\tau \Big[\frac{1}{2}\frac{\partial G^2(t,s)}{\partial s^2}|_{s=0}\lambda^*(t) - \frac{\partial G(t,s)}{\partial s}|_{s=0}\frac{\partial \lambda^*(x)}{\partial x}|_{x=t}\Big]\mu_0(t)dt + o(|\hat{\beta} - \beta_0|).$$

We do a further Taylor expansion of K around β_0 and obtain

$$\begin{aligned} K &= -(nh_n)^{1/2}\int_0^\tau \{\tilde{s}^{(2)}(\beta_0,t) - \frac{\tilde{s}^{(1)}(\beta_0,t)^{\otimes 2}}{\tilde{s}^{(0)}(\beta_0,t)}\}\mu_0(t)dt(\beta - \beta_0) + Dn^{1/2}h_n^{5/2} + o\{1 + (nh_n)^{1/2}|\beta - \beta_0|\} \\ &= (nh_n)^{1/2}A(\beta_0)(\beta - \beta_0) + Dn^{1/2}h_n^{5/2} + o\{1 + (nh_n)^{1/2}|\beta - \beta_0|\}, \end{aligned}$$

where

$$D = \int_z K(z)z^2dz \int_0^\tau \Big[\frac{1}{2}\frac{\partial G^2(t,s)}{\partial s^2}|_{s=0}\lambda^*(t) - \frac{\partial G(t,s)}{\partial s}|_{s=0}\frac{\partial \lambda^*(x)}{\partial x}|_{x=t}\Big]\mu_0(t)dt$$

and

$$\begin{aligned} A(\beta_0) &= -\int_0^\tau \Big\{\tilde{s}^{(2)}(\beta_0,t) - \frac{\tilde{s}^{(1)}(\beta_0,t)^{\otimes 2}}{\tilde{s}^{(0)}(\beta_0,t)}\Big\}\mu_0(t)dt \\ &= -\int_0^\tau E\Big[\{Z(t) - \frac{\tilde{s}^{(1)}(\beta_0,t)}{\tilde{s}^{(0)}(\beta_0,t)}\}\{Z(t) - \frac{\tilde{s}^{(1)}(\beta_0,t)}{\tilde{s}^{(0)}(\beta_0,t)}\}^T Y(t)e^{\beta_0^T Z(t)}\Big]\mu_0(t)dt. \end{aligned}$$

It is a non-negative definite matrix. From (C5), $A(\beta_0)$ is non-singular.

1.4 Proof of Lemma 4

Lemma 4 *Under conditions of Theorem 2,*

$$\begin{aligned} & var\Big[\int_0^\tau \int h_n^{1/2}K_{h_n}(t-r)\{Z(r) - \frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\}dN^*(r)dN(t)\Big] \\ = & \int K(z)^2dz \int_0^\tau \Big\{\tilde{s}^{(2)}(\beta_0,t) - \frac{\tilde{s}^{(1)}(\beta_0,t)^{\otimes 2}}{\tilde{s}^{(0)}(\beta_0,t)}\Big\}\mu_0(t)dt. \end{aligned}$$

Proof. This can be calculated as follows:

$$\begin{aligned}
\Sigma &= \text{var}\Big[\int_0^\tau \int h_n^{1/2} K_{h_n}(t-r)\{Z(r) - \frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\} dN^*(r) dN(t)\Big] \\
&= E h_n \iint \iint K_{h_n}(t_1-r_1) K_{h_n}(t_2-r_2)\{Z(r_1) - \frac{\tilde{S}_n^{(1)}(\beta_0,t_1)}{\tilde{S}_n^{(0)}(\beta_0,t_1)}\}\{Z(r_2) - \frac{\tilde{S}_n^{(1)}(\beta_0,t_2)}{\tilde{S}_n^{(0)}(\beta_0,t_2)}\} \\
& \quad dN^*(r_1) dN^*(r_2) dN(t_1) dN(t_2) - \Big[E\int_0^\tau \int h_n^{1/2} K_{h_n}(t-r)\{Z(r) - \frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\} dN^*(r) dN(t)\Big]^2 \\
&= I - II.
\end{aligned}$$

For II, we get

$$\begin{aligned}
II &= h_n\Big[E\int_0^\tau \int K_{h_n}(t-r)\{Z(r) - \frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\}\lambda^*(r) dr e^{\beta_0^T Z(t)} Y(t)\mu_0(t) dt\Big]^2 \\
&= h_n\Big(\int_0^\tau \int K(z)\Big[E\{Z(t-hz)Y(t)e^{\beta_0^T Z(t)}\}\lambda^*(t-hz)\mu_0(t) dt \\
&- E\{\frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)} Y(t) e^{\beta_0^T Z(t)}\}\lambda^*(t-h_n z)\mu_0(t) dt\Big] dz\Big)^2 \\
&= h_n\Big(\int_0^\tau [\tilde{s}^{(1)}(\beta_0,t) - \tilde{s}^{(1)}(\beta_0,t) + O_p\{(nh_n)^{-1}\}]\mu_0(t) dt + O(h_n^2)\Big)^2 \\
&= o(h_n).
\end{aligned}$$

Next we decompose I into four parts.

$$\begin{aligned}
I &= h_n E\int_{t_1\neq t_2}\int_{r_1\neq r_2} K_h(t_1-r_1)K_h(t_2-r_2)\{Z(r_1)-\frac{\tilde{S}_n^{(1)}(\beta_0,t_1)}{\tilde{S}_n^{(0)}(\beta_0,t_1)}\}\{Z(r_2)-\frac{\tilde{S}_n^{(1)}(\beta_0,t_2)}{\tilde{S}_n^{(0)}(\beta_0,t_2)}\}\\
&\quad E\{dN^*(r_1)dN^*(r_2)dN(t_1)dN(t_2)\}\\
&+ h_n E\int_{t_1\neq t_2}\int K_h(t_1-r)K_h(t_2-r)\{Z(r)-\frac{\tilde{S}_n^{(1)}(\beta_0,t_1)}{\tilde{S}_n^{(0)}(\beta_0,t_1)}\}\{Z(r)-\frac{\tilde{S}_n^{(1)}(\beta_0,t_2)}{\tilde{S}_n^{(0)}(\beta_0,t_2)}\}\\
&\quad E\{dN^*(r)dN(t_1)dN(t_2)\}\\
&+ h_n E\int\int_{r_1\neq r_2} K_h(t-r_1)K_h(t-r_2)\{Z(r_1)-\frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\}\{Z(r_2)-\frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\}\\
&\quad E\{dN^*(r_1)dN^*(r_2)dN(t)\}\\
&+ h_n E\int\int K_h(t-r)^2\{Z(r)-\frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\}^{\otimes 2}E\{dN^*(r)dN(t)\}\\
&= I_1+I_2+I_3+I_4.
\end{aligned}$$

It is easy to see that $I_1=O(h_n), I_2=O(h_n)$ and $I_3=O(h_n)$. Now we look at I_4 :

$$\begin{aligned}
I_4 &= h_n E\int_0^\tau\int h_n^{-2}K(z)^2\Big\{Z(t-h_nz)-\frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\Big\}^{\otimes 2}\lambda^*(t-h_nz)h_n dz Y(t)e^{\beta_0^T Z(t)}\mu_0(t)dt\\
&= \int_0^\tau\int K(z)^2 E\Big[Z(t-h_nz)Z(t-h_nz)^T Y(t)e^{\beta_0^T Z(t)}\lambda^*(t-h_nz)-2\frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\\
&\quad Z(t-h_nz)Y(t)e^{\beta_0^T Z(t)}\lambda^*(t-h_nz)+\{\frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\}^{\otimes 2}Y(t)e^{\beta_0^T Z(t)}\lambda^*(t-h_nz)\Big]dz\mu_0(t)dt\\
&= \int K(z)^2dz\int\Big\{\tilde{s}^{(2)}(\beta_0,t)-\frac{\tilde{s}^{(1)}(\beta_0,t)^{\otimes 2}}{\tilde{s}^{(0)}(\beta_0,t)}\Big\}\mu_0(t)dt+O(h_n)+O\{(nh_n)^{-1}\}.
\end{aligned}$$

Therefore, we have

$$\tilde{\Sigma}(\beta_0)=\int K(z)^2dz\int_0^\tau\Big\{\tilde{s}^{(2)}(\beta_0,t)-\frac{\tilde{s}^{(1)}(\beta_0,t)^{\otimes 2}}{\tilde{s}^{(0)}(\beta_0,t)}\Big\}\mu_0(t)dt. \tag{1.2}$$

2 Proofs of Main Results

2.1 Proof of Theorem 1

Our main tool is the martingale central limit theorem (Theorem 5.3.5 in Fleming and Harrington (2005)). First we need the following proposition:

Proposition 1 *Under (A1), (A2) and (A5), for any compact neighbourhood $\mathcal{B}$ of β_0, we have*

$$lim_{n\to\infty} sup_{0\le t\le\tau,\beta\in\mathcal{B}}||S_n^{(k)}(\beta,t)-s^{(k)}(\beta,t)||=0 \quad a.s. \quad for \quad k=0,1,2. \tag{2.3}$$

Proof. This follows from Theorem 37 of Polland (1984) and the observation that $S_n^{(k)}(\beta,t)$ is Lipschitz continuous in $\beta\in\mathcal{B}$. $\square$

To show the consistency of $\hat{\beta}_n$, first it follows from the definition of $u_1(\beta)$ that $u_1(\beta_0)=0$. Second, it follows from condition (A4) and the fact that $v_1(\beta)$ is semi-positive definite for any β that β_0 is the unique root to the equation $u_1(\beta)=0$. Finally we need to show that $U_n(\beta)$ converges in probability to $u_1(\beta)$ uniformly in $\mathcal{B}$. Consider the process

$$\begin{aligned}
F_n(\beta,t) &= l_n^*(\beta,t)-l_n^*(\beta_0,t)\\
&= n^{-1}\sum_{i=1}^n\int_0^t\sum_{k=1}^{M_i}K_{h_n}(u-R_{ik})I(R_{ik}\le u)(\beta-\beta_0)^TZ_i(R_{ik})dN_i(u)\\
&- n^{-1}\sum_{i=1}^n\int_0^t\sum_{k=1}^{M_i}K_{h_n}(u-R_{ik})I(R_{ik}\le u)\log\Big\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\Big\}dN_i(u),
\end{aligned}$$

and the process

$$\begin{aligned}
G_n(\beta,t) &= n^{-1}\sum_{i=1}^n\int_0^t\sum_{k=1}^{M_i}K_{h_n}(u-R_{ik})I(R_{ik}\le u)(\beta-\beta_0)^TZ_i(R_{ik})Y_i(u)e^{\beta_0^TZ_i(u)}\lambda_0(u)du\\
&- n^{-1}\sum_{i=1}^n\int_0^t\sum_{k=1}^{M_i}K_{h_n}(u-R_{ik})I(R_{ik}\le u)\log\Big\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\Big\}Y_i(u)e^{\beta_0^TZ_i(u)}\lambda_0(u)du.
\end{aligned}$$

Then for each β, $F_n(\beta,\cdot) - G_n(\beta,\cdot)$ is a local square integrable martingale with

$$< F_n(\beta,\cdot) - G_n(\beta,\cdot), F_n(\beta,\cdot) - G_n(\beta,\cdot) >= B_n(\beta,\cdot),$$

where

$$\begin{aligned} B_n(\beta,t) &= n^{-2}\sum_{i=1}^{n}\int_0^t \Big(\sum_{k=1}^{M_i} K_{h_n}(u-R_{ik})I(R_{ik}\le u)\Big[(\beta-\beta_0)^T Z_i(R_{ik}) \\ &- \log\Big\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\Big\}\Big]\Big)^2 Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du. \end{aligned}$$

From Lemma 1, we have $B_n(\beta,\tau)$ converges in probability to 0.

Now we look at $G_n(\beta,\tau)$. After taking expectation and change of variables, we have

$$E\{G_n(\beta,\tau)\} = n^{-1}\sum_{i=1}^{n}\int_0^\tau \int_0^{u/h_n} K(z)I(z\ge 0)\lambda^*(u-h_n z)dz$$
$$\Big((\beta-\beta_0)^T E\{Z_i(u-h_n z)Y_i(u)e^{\beta_0^T Z_i(u)}\} - E\Big[\log\Big\{\frac{S_n^{(0)}(\beta,u)}{S_n^{(0)}(\beta_0,u)}\Big\}\Big]Y_i(u)e^{\beta_0^T Z_i(u)}\Big)\lambda_0(u)du$$

It follows that for each $\beta \in \mathcal{B}$,

$$G_n(\beta,\tau) \to \int_0^\tau \Big[(\beta-\beta_0)^T s^{(1)}(\beta_0,u) - \log\Big\{\frac{s^{(0)}(\beta,u)}{s^{(0)}(\beta_0,u)}\Big\}s^{(0)}(\beta_0,u)\Big]\lambda_0(u)du \quad \text{in probability.} \tag{2.4}$$

Thus by the inequality of Lenglart (Corollary 3.4.1 in Fleming and Harrington (2005)), $F_n(\beta,\tau)$ converges in probability to the same limit as $G_n(\beta,\tau)$ for each $\beta \in \mathcal{B}$.

Now by the boundedness condition we may evaluate the first and second derivatives of this limiting function of β by taking partial derivatives inside the integral. These derivatives equal to

$$\int_0^\tau \Big\{s^{(1)}(\beta_0,u) - s^{(0)}(\beta_0,u)\bar{z}(\beta,u)\Big\}\lambda_0(u)du = u_1(\beta)$$

and

$$\begin{aligned} & -\int_0^\tau \Big\{s^{(2)}(\beta,u)\frac{s^{(0)}(\beta_0,u)}{s^{(0)}(\beta,u)} - s^{(1)}(\beta,u)^{\otimes 2}\frac{s^{(0)}(\beta_0,u)}{s^{(0)}(\beta,u)^2}\Big\}\lambda_0(u)du \\ = & -\int_0^\tau \Big[\frac{s^{(2)}(\beta,u)}{s^{(0)}(\beta,u)} - \bar{z}(\beta,u)^{\otimes 2}\Big]s^{(0)}(\beta_0,u)\lambda_0(u)du = -v_1(\beta). \end{aligned}$$

The first derivative is zero at $\beta = \beta_0$; the second is minus a positive semi-definite matrix; and at $\beta = \beta_0$ is a minus positive definite matrix. Thus for each $\beta \in \mathcal{B}$, $F_n(\beta,\tau)$ converges in probability to a concave function of β with a unique maximum at $\beta = \beta_0$. Since $\hat{\beta}_n$ maximizes the random concave function $F_n(\beta,\tau)$, by the fact that pointwise convergence in probability of random concave functions implies uniform convergence on compact subspaces (Andersen and Gill (1982)), it follows that $\hat{\beta}_n \to \beta_0$ in probability.

Next we show the asymptotic normality of $\hat{\beta}_n$. By Taylor expansion of $U_n(\hat{\beta}_n,\tau)$, we have

$$0 = U_n(\hat{\beta}_n,\tau) = U_n(\beta_0,\tau) + \frac{\partial U_n(\beta,\tau)}{\partial \beta}\mid_{\beta=\beta^*}(\hat{\beta}_n - \beta_0), \tag{2.5}$$

where β^* lies in the segment between $\hat{\beta}_n$ and β_0. We have

$$(nh_n)^{1/2}(\hat{\beta}_n - \beta_0) = -\Big\{\frac{\partial U_n(\beta,\tau)}{\partial \beta}\mid_{\beta=\beta^*}\Big\}^{-1}(nh_n)^{1/2}U_n(\beta_0,\tau). \tag{2.6}$$

Thus, we have two tasks here: first to establish the asymptotic normality of $(nh_n)^{1/2}U_n(\beta_0,\tau)$ and second to find the limiting distribution of $\frac{\partial U_n(\beta,\tau)}{\partial \beta}\mid_{\beta=\beta^*}$ for any β^* between $\hat{\beta}$ and β_0. The first part follows from Lemma 2. For the second part of the proof, note that

$$
\begin{aligned}
& -\frac{\partial U_n(\beta,u)}{\partial \beta}\ |_{\beta=\beta^*} \\
= \ & n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\int_0^{u/h_n} K_{h_n}(u-r)I(r\le u)dN_i^*(r)\Big\{\frac{S_n^{(2)}(\beta^*,u)}{S_n^{(0)}(\beta^*,u)} - \frac{S_n^{(1)}(\beta^*,u)^{\otimes 2}}{S_n^{(0)}(\beta^*,u)^2}\Big\}dN_i(u)
\end{aligned}
$$

and that

$$
W(\beta_0) = \int_0^{\tau}\Big\{s^{(2)}(\beta_0,u) - \frac{s^{(1)}(\beta_0,u)^{\otimes 2}}{s^{(0)}(\beta_0,u)}\Big\}\lambda_0(u)du.
$$

Define $V_n(\beta,t) = \frac{S_n^{(2)}(\beta,t)}{S_n^{(0)}(\beta,t)} - \bar{Z}(\beta,t)^{\otimes 2}$ and $v(\beta,t) = \frac{s^{(2)}(\beta,t)}{s^{(0)}(\beta,t)} - \bar{z}(\beta,t)^{\otimes 2}$. Hence

$$
\begin{aligned}
& ||-\frac{\partial U_n(\beta,\tau)}{\partial \beta}|_{\beta=\beta^*} - W(\beta_0)|| \\
\le \ & ||n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\int_0^{\infty} K_{h_n}(u-r)I(r\le u)dN_i^*(r)\{V_n(\beta^*,u)-v(\beta^*,u)\}dN_i(u)|| \\
+ \ & ||n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\int_0^{\infty} K_{h_n}(u-r)I(r\le u)dN_i^*(r)\{v(\beta^*,u)-v(\beta_0,u)\}dN_i(u)|| \\
+ \ & ||n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\int_0^{\infty} K_{h_n}(u-r)I(r\le u)dN_i^*(r)v(\beta_0,u)\{dN_i(u)-Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du\}|| \\
+ \ & ||\int_0^{\tau}\{n^{-1}\sum_{i=1}^{n}\int_0^{\infty} K_{h_n}(u-r)I(r\le u)dN_i^*(r)Y_i(u)e^{\beta_0^T Z_i(u)} - s^{(0)}(\beta_0,u)\}v(\beta_0,u)\lambda_0(u)du|| \\
= \ & I + II + III + IV \qquad (2.7)
\end{aligned}
$$

By (A1) and Theorem III.1 in Andersen and Gill (1982), it follows that

$$
\sup_{t\in[0,\tau],\beta\in\mathcal{B}}||V_n(\beta,t)-v(\beta,t)|| \to 0 \quad \text{in probability.} \qquad (2.8)
$$

Hence $\beta^* \to \beta_0$ in probability. By Chebyshev's inequality,

$$\begin{aligned} &\text{pr}\{\int_0^\tau n^{-1}\sum_{i=1}^n \int_0^\infty K_{h_n}(u-r)I(r \le u)dN_i^*(r)dN_i(u) > c\} \\ \le\ & \frac{\int_0^\tau s^{(0)}(\beta_0,u)\lambda_0(u)du}{c} \to 0 \end{aligned} \tag{2.9}$$

as $c \to \infty$ by (A2) and (A3). Therefore, $I = o_p(1)$.

Again, (2.8), (2.9) together with the continuity of $v(\beta, t)$ in β, uniformly for t implies that II is also asymptotically negligible.

For III, using Lenglart's inequality as in Theorem I.1 in Andersen and Gill (1982) and Chebyshev's inequality. We have

$$\begin{aligned} &\text{pr}\Big\{\int_0^\tau n^{-1}\sum_{i=1}^n \int_0^\infty K_{h_n}(u-r)I(r \le u)dN_i^*(r)v(\beta_0,u)dM_i(u) > \delta\Big\} \\ \le\ & \eta\delta^{-2} + \text{pr}\Big[n^{-1}\int_0^\tau n^{-1}\sum_{i=1}^n\{\int_0^\infty K_{h_n}(u-r)I(r \le u)dN_i^*(r)\}^2 v(\beta_0,u)^2 Y_i(u)e^{\beta_0^T Z_i(u)}\lambda_0(u)du > \eta\Big] \\ \le\ & \eta\delta^{-2} + O\{(nh_n\eta)^{-1}\}. \end{aligned}$$

Thus, III disappears as $n \to \infty$.

Finally, $IV = o_p(1)$ by (A2) and the uniform convergence of $S_n^{(0)}(\beta_0, u)$ to $s^{(0)}(\beta_0, u)$. □

2.2 Proof of Corollary 1

We next show the consistency of the variance estimate. It follows from the proof of Theorem 1 that

$$-\frac{\partial U_n(\beta,\tau)}{\partial\beta}|_{\beta=\hat\beta_n} \to W(\beta_0) \quad \text{in probability.}$$

On the other hand, by law of large numbers, consistency of $\hat{\beta}_n$ for β_0 and the continuous mapping theorem

$$\hat{\Sigma} = n^{-2} \sum_{i=1}^{n} \Big[\int_0^{\tau} \int_0^{\infty} K_{h_n}(u-r) I(r \leq u) \{Z_i(r) - \bar{Z}(\hat{\beta}_n, u)\} dN_i^*(r) dN_i(u) \Big]^{\otimes 2} \xrightarrow{p} E\{\hat{\Sigma}(\beta_0)\}. \tag{2.10}$$

Note that

$$E\{\hat{\Sigma}(\beta_0)\} = n^{-1} E\Big[\int_0^{\tau} \int_0^{\infty} K_{h_n}(u-r) I(r \leq u) \{Z(r) - \bar{Z}(\beta_0, u)\} dN^*(r) dN(u) \Big]^{\otimes 2}$$

After change of variables, and by (A1),

$$E\{\hat{\Sigma}(\beta_0)\} \;=\; \frac{1}{nh_n} \int_0^{\infty} K(z)^2 dz \int_0^{\tau} \{ s^{(2)}(\beta_0, u) - \frac{s^{(1)}(\beta_0, u)^{\otimes 2}}{s^{(0)}(\beta_0, u)} \} du.$$

Therefore,

$$(nh_n)\hat{\Sigma} \xrightarrow{p} \Sigma(\beta_0) \quad \text{as} \quad nh_n \to \infty.$$

The consistency of variance estimate follows.

2.3 Proof of Theorem 2

Our main tools are empirical processes (van der Vaart and Wellner (1996)).

The key idea is to establish the following relationship

$$\begin{aligned} & \sup_{|\beta-\beta_0|<M(nh_n)^{-1/2}} \Big| (nh_n)^{1/2} \tilde{U}_n(\beta) - (nh_n)^{1/2} [\tilde{U}_n(\beta_0) - E\{\tilde{U}_n(\beta_0)\}] - (nh_n)^{1/2} A(\beta_0)(\beta - \beta_0) \Big| \\ = \;& D n^{1/2} h_n^{5/2} + o_p\{1 + (nh_n)^{1/2} |\beta - \beta_0|\}, \end{aligned} \tag{2.11}$$

where $A(\beta_0)$ is given in Theorem 2.

To obtain (2.11), first, using $\mathcal{P}_n$ and $\mathcal{P}$ to denote the empirical measure and true probability measure respectively, we obtain

$$
\begin{aligned}
(nh_n)^{1/2}\tilde{U}_n(\beta) &= (nh_n)^{1/2}(\mathcal{P}_n - \mathcal{P})\left[\int_0^\tau \int_0^\infty K_{h_n}(t-r)\left\{Z(r) - \frac{\tilde{S}_n^{(1)}(\beta,t)}{\tilde{S}_n^{(0)}(\beta,t)}\right\} dN^*(r)dN(t)\right] \\
&+ (nh_n)^{1/2}E\left[\int_0^\tau \int_0^\infty K_{h_n}(t-r)\left\{Z(r) - \frac{\tilde{S}_n^{(1)}(\beta,t)}{\tilde{S}_n^{(0)}(\beta,t)}\right\} dN^*(r)dN(t)\right] \\
&= I + II. \qquad (2.12)
\end{aligned}
$$

For the second term on the right-hand side of (2.12), from Lemma 3, we have

$$
II = (nh_n)^{1/2}A(\beta_0)(\beta-\beta_0) + Dn^{1/2}h_n^{5/2} + o\{(nh_n)^{1/2}|\beta-\beta_0|\}, \qquad (2.13)
$$

where

$$
D = \int_{-\infty}^{\infty} z^2K(z)dz \int_0^\tau \left[E\{Z(t)'Y(t)e^{\beta_0^T Z(t)}\}\lambda^{*'}(t) + 2^{-1}E\{Z(t)''Y(t)e^{\beta_0^T Z(t)}\}\lambda^*(t)\right]\mu_0(t)dt
$$

and

$$
\begin{aligned}
A(\beta_0) &= -\int_0^\tau \left\{\tilde{s}^{(2)}(\beta_0,t) - \frac{\tilde{s}^{(1)}(\beta_0,t)^{\otimes 2}}{\tilde{s}^{(0)}(\beta_0,t)}\right\}\mu_0(t)dt \\
&= -\int_0^\tau E\left[\{Z(t) - \frac{\tilde{s}^{(1)}(\beta_0,t)}{\tilde{s}^{(0)}(\beta_0,t)}\}\{Z(t) - \frac{\tilde{s}^{(1)}(\beta_0,t)}{\tilde{s}^{(0)}(\beta_0,t)}\}^T Y(t)e^{\beta_0^T Z(t)}\right]\mu_0(t)dt.
\end{aligned}
$$

The matrix $A(\beta_0)$ is a non-negative definite and by assumption (C5) non-singular. For the first term on the right-hand side of (2.12), we consider the class of functions

$$
\left\{h_n^{1/2}\int_0^\tau \int K_{h_n}(t-r)\{Z(r) - \frac{\tilde{S}_n^{(1)}(\beta,t)}{\tilde{S}_n^{(0)}(\beta,t)}\}dN^*(r)dN(t) : |\beta-\beta_0| < \epsilon\right\}
$$

for a given constant ϵ. Note that the functions in this class are Lipschitz continuous in β and the Lipschitz constant is uniformly bounded by

$$M_1 \int_0^\tau \int h_n^{1/2} K_{h_n}(t-r) dN^*(r) dN(t),$$

which has finite second moment and M_1 is the upper bound of $\frac{\tilde{s}^{(2)}(\beta,t)}{\tilde{s}^{(0)}(\beta,t)} - \{\frac{\tilde{s}^{(1)}(\beta,t)}{\tilde{s}^{(0)}(\beta,t)}\}^{\otimes 2}$. Therefore, this class is P-Donsker class by the Jain-Marcus theorem (van der Vaart and Wellner (1996)). As the result, we obtain that the first term in the right-hand side of (2.12) for $|\beta - \beta_0| < M(nh_n)^{-1/2}$ is equal to

$$\begin{aligned} & (nh_n)^{1/2}(\mathcal{P}_n - \mathcal{P}) \left[\int_0^\tau \int K_h(t-r) \left\{ Z(r) - \frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)} \right\} dN^*(r) dN(t) \right] + o_p(1) \\ = \ & (nh_n)^{1/2} \Big[\tilde{U}_n(\beta_0) - E\{\tilde{U}_n(\beta_0)\} \Big] + o_p(1). \end{aligned} \quad (2.14)$$

Combining (2.13) and (2.14), we obtain (2.11). Consequently,

$$\begin{aligned} & (nh_n)^{1/2} A(\beta_0)(\tilde{\beta}_n - \beta_0) + O_p(n^{1/2} h_n^{5/2}) + o_p\{1 + (nh_n)^{1/2} |\tilde{\beta}_n - \beta_0|\} \\ = \ & (nh_n)^{1/2} [\tilde{U}_n(\beta_0) - E\{\tilde{U}_n(\beta_0)\}]. \end{aligned} \quad (2.15)$$

On the other hand, from Lemma 4, we obtain

$$\begin{aligned} \tilde{\Sigma}(\beta_0) &= \mathrm{var}\Big[\int_0^\tau \int h_n^{1/2} K_h(t-r) \{Z(r) - \frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\} dN^*(r) dN(t) \Big] \\ &= \int K(z)^2 dz \int_0^\tau \Big\{ \tilde{s}^{(2)}(\beta_0,t) - \frac{\tilde{s}^{(1)}(\beta_0,t)^{\otimes 2}}{\tilde{s}^{(0)}(\beta_0,t)} \Big\} \mu_0(t) dt. \end{aligned}$$

To prove the asymptotic normality, we verify that Lyapunov condition holds. Define

$$\psi_i = (nh_n)^{1/2} n^{-1} \int \int K_{h_n}(t-r) \{Z(r) - \frac{\tilde{S}_n^{(1)}(\beta_0,t)}{\tilde{S}_n^{(0)}(\beta_0,t)}\} dN^*(r) dN(t).$$

Similar to the calculation of $\Sigma(\beta_0)$,

$$\sum_{i=1}^{n} E\Big(|\psi_i - E\psi_i|^3\Big) = nO\{(nh_n)^{3/2}n^{-3}h_n^{-2}\} = O\{(nh_n)^{-1/2}\}.$$

Thus,

$$(nh_n)^{1/2}\Big[\tilde{U}_n(\beta_0) - E\{\tilde{U}_n(\beta_0)\}\Big] \to N\{0, \tilde{\Sigma}(\beta_0)\}.$$

Combing with (2.15), we finish the proof of Theorem 2. $\square$

2.4 Proof of Corollary 3

To begin with, we have

$$-\frac{\partial \tilde{U}_n(\beta, u)}{\partial \beta} = n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\int_0^{u/h_n} K_{h_n}(u-r)dN_i^*(r)\Big\{\frac{\tilde{S}_n^{(2)}(\beta, u)}{\tilde{S}_n^{(0)}(\beta, u)} - \frac{\tilde{S}_n^{(1)}(\beta, u)^{\otimes 2}}{\tilde{S}_n^{(0)}(\beta, u)^2}\Big\}dN_i(u).$$

Using the similar argument to obtain equation (2.14), we show

$$\Big\{\int\int K_{h_n}(u-r)dN^*(r)\Big\{\frac{\tilde{S}_n^{(2)}(\beta, u)}{\tilde{S}_n^{(0)}(\beta, u)} - \frac{\tilde{S}_n^{(1)}(\beta, u)^{\otimes 2}}{\tilde{S}_n^{(0)}(\beta, u)^2}\Big\}dN(u) : |\beta - \beta_0| < \epsilon\Big\}$$

is a P-Glivenko-Cantelli class. Therefore, $\sup_{|\beta-\beta_0|<\epsilon}|\frac{\partial \tilde{U}_n(\beta)}{\partial \beta}|_{\beta=\tilde{\beta}_n} - E\{\frac{\partial \tilde{U}_n(\beta)}{\partial \beta}|_{\beta=\tilde{\beta}_n}\}| \to 0$ in probability. Since $\tilde{\beta}_n$ is consistent for β_0, by continuous mapping theorem, $\frac{\partial \tilde{U}_n(\beta)}{\partial \beta}|_{\beta=\tilde{\beta}_n}$ converges in probability to $A(\beta_0)$. Similarly, let $\hat{\tilde{\Sigma}}(\beta) = n^{-2}\sum_{i=1}^{n}\Big[\int_0^{\tau}\int_0^{\infty} K_{h_n}(u-r)\{Z_i(r) - \tilde{Z}(\beta, u)\}dN_i^*(r)dN_i(u)\Big]^{\otimes 2}$, then $\sup_{|\beta-\beta_0|<\epsilon}|\hat{\tilde{\Sigma}}(\beta) - E\{\hat{\tilde{\Sigma}}(\beta)\}| \to 0$ in probability. On the other hand,

$$E\{\hat{\tilde{\Sigma}}(\beta)\} = n^{-1}E\Big[\int_0^{\tau}\int_0^{\infty} K_{h_n}(u-r)\{Z_i(r) - \tilde{Z}(\beta, u)\}dN_i^*(r)dN_i(u)\Big]^{\otimes 2}.$$

After change of variables, and by (C4),

$$E\{\hat{\hat{\Sigma}}(\beta)\} \quad = \quad \frac{1}{nh_n}\int_0^{\infty} K(z)^2 dz \int_0^{\tau} \{\tilde{s}^{(2)}(\beta_0, u) - \frac{\tilde{s}^{(1)}(\beta_0, u)^{\otimes 2}}{\tilde{s}^{(0)}(\beta_0, u)}\} du.$$

Therefore,

$$(nh_n)\hat{\hat{\Sigma}} \xrightarrow{p} \tilde{\Sigma}(\beta_0) \quad \text{as} \quad nh_n \to \infty.$$

The consistency of variance estimate follows.

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