LIMIT THEOREMS FOR CONVOLUTION POWERS
OF RANDOM FUNCTIONS

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FSU Statistics Report No. M117

August, 1966

This work was prepared while a Visiting Professor in the Department of Statistics, Florida State University, Tallahassee, Florida.
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I. Introduction

In a forthcoming paper Bergstrom (2) I have introduced a generalized
L^p-convolution \( \xi \ast \eta \) of two random functions \( \xi \) and \( \eta \). (I shall here use the
notation \( \ast \) for the generalized convolution.)

Let \( \overline{V}^p \) be the class of \( L^p \)-mean-continuous random functions of bounded
variation in respect to the \( L^p \)-norm. I have proved in that paper that
\( \xi \ast \eta \) exists and belongs to \( \overline{V}^p \) if \( \xi \epsilon \overline{V}^{q_1} \) and \( \eta \epsilon \overline{V}^{q_2} \) where \( l/q_1 \div 1/q_2 = 1/p \)
\( (q_1 \geq 1, q_2 \geq 1, p \geq 1) \). The convolution \( \ast \) is then a commutative operation.

Further it is associative in the sense that

\[ (\xi \ast \eta) \ast \zeta = \xi \ast (\eta \ast \zeta) \]  \( (L^p) \)

if \( \xi \epsilon \overline{V}^{q_1}, \eta \epsilon \overline{V}^{q_2}, \zeta \epsilon \overline{V}^{q_3} \) where \( l/q_1 + l/q_2 \div 1/q_3 = 1/p \leq 1, \ q_1 \geq 1, \ q_2 \geq 1, \ q_3 \geq 1 \)  \( (L^p \) after a relation means that it holds in respect to the \( L^p \)-norm,

cf. Bergstrom (2).)

We may now consider convolutions powers \( \xi_n \ast^k \eta_n \) of random functions \( \xi_n \)
of bounded variation, where \( \xi_n \) depends on the integer \( n \) and \( \{ k_n \} \) is a
sequence of positive integers tending to \( +\infty \) as \( n \rightarrow +\infty \). Then we are con-
fronted with limit theorems of the same typus as for functions (cf. Bergstrom
(1)), i.e. we may ask if \( \xi_n \ast^k \eta_n \) tends completely to a random function \( \eta \) as \( n \)
tends to \( +\infty \). Guided by the classical limit problems where we consider

conversions \( \overline{F}_n \) of distribution functions \( \overline{F}_n \) we may also ask if the limit

\( \eta \), if it exists, is infinitely divisible in the sense that to any positive
integer \( n \) belongs a random function \( \eta_n \) such that
\[ \eta_n^* = \eta \quad (L^p) \]

Mainly we shall deal with a.s. nondecreasing random functions. A random function is said to be a.s. nondecreasing on an interval if \( \xi(t_1) \leq \xi(t_2) \) a.s. for any pair \( t_1 < t_2 \). The class of all random functions which are a.s. nondecreasing on \([\omega, +\infty)\) and belong to \( \overline{V}_p \) is denoted by \( \overline{M}_p \). A special element in \( \overline{M}_p \) is the unit distribution function \( e \) which is 0 on \((-\infty, 0)\), 1 on \((0, +\infty)\) and \(1/2\) at 0.

A main tool in our investigation is the Gaussian \( L^p \)-norm defined by

\[
\left\| \xi \right\|_p = \sup_t \left\| \xi * \phi(\cdot/t) \right\|_p
\]

for any \( \xi \in \overline{V}_p \), where \( \phi \) denotes the normalized Gaussian distribution function and \( \sigma \) is any positive number. We shall show in Section 4 that there is a close connection between the \( L^p \)-complete convergence of a sequence \( \{\xi_n^*k_n\} \) of convolutions of random functions \( \xi_n^*k_n \in \overline{M}_p \) and the mutual convergence of \( \{k_n[\xi_n - e]\} \) in the Gaussian \( L^q \)-norm for some \( q \geq 1 \). Such a relation for functions was given in Bergström (1) p. 101.

The random functions \( k_n[\xi_n - e] \) belong to a class \( \overline{Q}^q \) consisting of all random functions \( \xi \), which belong to \( \overline{Q}^q \) and are a.s. nondecreasing on \([\omega, 0)\) and \((0, +\infty)\). Clearly \( \overline{M}_p \subset \overline{Q}^p \). \( \xi \in \overline{Q}^p \) is a.s. nonnegative on \([\omega, 0)\] since by definition \( \xi(\omega) = 0 \). In Section 3 we give necessary and sufficient conditions for the mutual convergence in the \( L^p \)-norm of a sequence \( \{\xi_n\} \) of random functions \( \xi_n \) in \( \overline{Q}^p \). In Section 2 we give some properties of the Gaussian \( L^p \)-norm, which correspond to properties of the usual Gaussian norm (cf. Bergström (1) p. 59).

In the same way as we generalize limit theorems for convolution powers we can generalize limit theorems for convolution products so that
they hold for convolution products of a.s. nondecreasing random functions.

2. The Gaussian $l^p$-norm.

The following concepts and theorems are generalizations of the corresponding concepts and theorems in Bergström (1) and the generalizations are so immediate that we content ourselves with some complementary remarks on the proofs. Thus we define the Gaussian $l^p$-norm of $\xi \in l^p$ by

$$\| \xi \|^p = \sup_t \| \xi * \phi(t) \|^p$$

for $\sigma > 0$, where $\phi$ denotes the normalized Gaussian d.f. (cf. Bergström (1), p. 59). It is easily verified that it satisfies the following norm relations

$$\| \xi + \eta \|^p \leq \| \xi \|^p + \| \eta \|^p \quad \text{for } \xi, \eta \in l^p \quad (1)$$

$$\| a \xi \|^p = |a| \| \xi \|^p \quad (2)$$

for any complex number $a$. Further

$$\| \xi \|^p = 0 \iff \| \xi \|^p = 0 \text{ for all } \sigma > 0. \quad (3)$$

The implication $\Rightarrow$ is immediate. The implication $\Leftarrow$ follows as $\sigma \to 0^+$ according to Bergström (2), Theorem 7.2. Thus $\| \xi \|^p$ is a norm if we say that a relation holds in this norm when the relation is true for any $\sigma > 0$.

We also state the following relations

$$\| \xi \|^p_{\sigma_1} \leq \| \xi \|^p_{\sigma_2} \quad (4)$$

for $\sigma_2 \leq \sigma_1$,

$$\| \xi * \eta \|^p \leq \| \xi \|^q_1 \int_{\infty}^{+\infty} |d\eta(t)| \| \xi \|^q_2 \quad (5)$$

for $1/q_1 + 1/q_2 = 1/p \leq 1$, $q_1 \geq 1$, $q_2 \geq 1$, where $\int_{\infty}^{+\infty} d(t)$ denotes the total
variation of $\eta$ in respect to the $L^p$-norm (For the definition, see Bergström (2), Section 4.). These inequalities may be proved as in Bergström 1, p. 60.

For the proof of (5) we observe that

$$(\xi * \eta) * \phi(\sigma) = [\xi * \phi(\sigma')] * \eta$$

where the convolutions on the right-hand side can be given directly as $L^p$-RS-integrals (cf. Bergström (2), Lemma 3).

We need a special norm-inequality for random functions in the class $\mathcal{Q}^p$.

**Theorem 1.** There exists a positive number $\gamma$ such that

$$\| \xi * \eta \|_p^\sigma \leq \gamma \| \xi \|_q^\sigma \| \eta \|_p^q$$

if $1/p = 1/q_1 + 1/q_2$, $q_1 \geq 1$, $q_2 \geq 1$, $p \geq 1$, $\xi \in \mathcal{Q}^{1}$, $\eta \in \mathcal{Q}^{2}$ ($\gamma$ does not depend on $\xi$, $\eta$, $q_1$, $q_2$ or $\sigma$).

For the proof of this inequality we proceed as in Bergström (1) p. 61, but we have to apply Hölder's inequality when we make estimations in the norm. Thus we define the major norm $|\xi|_p^\sigma$ of the Gaussian norm as the largest number among

$$\| \int_\sigma^{+\infty} d\xi(t) \|_p^\sigma, \| \int_{-\infty}^\sigma d\xi(t) \|_p^\sigma, \| \xi(t) \|_p, \frac{1}{\sigma} \int_{|t| \leq 0} t^2 d\xi(t) \|_p.$$  

Then generalizing the lemma in Bergström (1) p. 62 we get

**Lemma 1.** Let $\mathcal{V}^{q_1}$ be strongly uniformly $L^{q_1}$-continuous 1) on the interval $(a,b)$ and let it have continuous $L^{q_1}$-derivatives $\xi'$ and $\xi''$ of the first and second orders on the finite interval $(-\sigma, \sigma)$. Then the inequality

$$\frac{1}{\sigma^2} \| \int \tau^2 d\xi(t) \|_p$$

is called strongly uniformly $L^{q_1}$-continuous when the interval $(-\infty, +\infty)$ can be divided into finitely many subintervals such that the $L^{q_1}$ oscillation of $\xi$ is smaller than any given number $c > 0$ on any such closed subinterval.
\[ \| \int_a^b \xi(t) \, d\eta(t) \|_p \leq \left[ 3 \sup_{t \in (a, b)} \| \xi(t) \|_{q_1} + \sigma \| \xi'(0) \|_{q_1} + \frac{1}{2} \sigma^2 \sup_{t \in (-\sigma, \sigma)} \| \xi''(t) \|_{q_1} \right] \eta_{q_2} \]

holds for \( \eta_{q_2} \), \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p} \leq 1 \), \( q_1 \geq 1 \), \( q_2 \geq 1 \), provided that the whole interval \([-\sigma, \sigma]\) belongs to \([a, b]\) or \((a, b)\) has no point in common with \((\sigma, \sigma)\).

Remark: The interval \((a, b)\) may here be \((\infty, +\infty)\), \((\infty, a)\) or \((a, +\infty)\), with finite \(a\) (since we permit \(a\) and \(b\) to be the formal point \(\infty\)).

As a corollary of this lemma we get

**Lemma 2.** There exists a positive number \( \sigma \) such that

\[ \| \eta \|_p^\sigma \leq \alpha \| \eta \|_{p \sigma} \]

for \( \eta \in V_p^p \) and any \( \sigma > 0 \).

Indeed, we get the lemma if we choose \( a = \infty \), \( b = +\infty \), \( \xi = \phi \left( \frac{t - e}{\sigma} \right) \) in the inequality of Lemma 1 and take supremum in respect to \( t \).

Proceeding as in Bergström (1) p. 64 we then prove

**Lemma 3.** There exists a positive number \( \gamma \) such that

\[ \| \eta \|_p^\sigma \leq \gamma \| \xi \|_{p \sigma} \]

for \( \xi \in \Omega^p \), \( \sigma > 0 \), \( p \geq 1 \).

By the help of these lemmas we get Theorem 1 in the same way as in Bergström (1) p. 64.
3. **Mutual convergence in the Gaussian norm.**

A sequence of random functions is said to converge \( L^P \)-weakly to a random function \( \xi \) on an interval \( I \) and we write

\[
\xi_n \rightharpoonup \xi \quad (L^P) \text{ on } I
\]  

if

\[
\| \xi_n(t) - \xi(t) \|_p \to 0
\]

at all points of \( I \) except at most a denumerable set. We say that \( \xi_n \) tends \( L^P \)-completely to \( \xi \) on \( I \) and we write

\[
\xi_n \rightharpoonup \xi \quad (L^P) \text{ on } I
\]

if \( \xi_n \rightharpoonup \xi \) on \( I \) and furthermore \( \xi_n \) tends to \( \xi \) in the \( L^P \)-norm at the endpoint of the interval where it is closed and at the limiting endpoint where it is open. Hence for instance, we require that

\[
\xi_n^{(+\infty)} \to \xi^{(+\infty)}, \quad \xi_n^{(-\infty)} \to \xi^{(-\infty)} \quad L^P \text{ when we write } \xi_n \rightharpoonup \xi \quad L^P \text{ on } (-\infty, +\infty).
\]

When we use the formal point \( \infty \) as a lower endpoint we also require that \( \xi_n^{(-\infty)} \to \xi^{(-\infty)} \quad L^P \).

We shall here only consider sequences \( \{\xi_n\} \) of random functions in \( \overline{V^P} \) for which \( \xi_n^{(+)} \) and \( \xi_n^{(-)} \) exist and we are mainly interested in sequences in \( \overline{\Gamma^P} \) (which contains \( \overline{M^P} \)). Generalizing theorems in Bergström (1) (p. 38 and p. 74-75) we state

**Theorem 1.** A sequence \( \{\xi_n\} \) of a.s. nondecreasing random functions on an interval \((a, b)\) tends \( L^P \)-weakly to a \( L^P \)-bounded random function \( \xi \) (which is then a.s. nondecreasing) on \((a, b)\) if and only if it tends \((L^P)\) to this random function \( \xi \) at all \( L^P \)-continuity points of \( \xi \).

**Theorem 2.** A sequence \( \{\xi_n\}, \xi_n \in \overline{\Gamma^P} \), is mutually convergent in the Gaussian \( L^P \)-norm if and only if the following conditions hold
(i) There exists a random function \( \xi \) which is a.s. nondecreasing and bounded in the \( L^p \)-norm on \( |t| > \delta \) for any \( \delta > 0 \) and such that \( \{ \xi_n \} \) tends \( L^p \)-completely to \( \xi \) on \( (-\infty, -\delta], [\delta, +\infty) \) if \( \pm \delta \) are \( L^p \)-continuity points of \( \xi \).

(ii) \( \left\{ \int \frac{t^k d\xi_n(t)}{|t| < \delta} \right\} \) converges in the \( L^p \)-norm for \( k = 1, 2 \) provided that \( \pm \delta \) are \( L^p \)-continuity points of \( \xi \).

Remark 1. The convergence of \( \{ \xi_n \} \) in the Gaussian \( L^p \)-norm obviously means uniform convergence of \( \{ \xi_n \ast \phi(\cdot/\delta)(t) \} \) in the \( L^p \)-norm and in respect to \( t e^{-|t|^2/\delta} \) for every \( \delta > 0 \).

Remark 2. The "if" assertion even follows when (ii) is only required for some \( \delta \) such that \( \pm \delta \) are \( L^p \)-continuity points of \( \xi \).

Proof: The proof of Theorem 1 follows as in Bergström 1, p. 38. The proof of Theorem 2 is not such a direct generalization. We first prove the necessity of the conditions (i) and (ii). The sufficiency will follow more easily. For the proof we need some lemmas.

Lemma 1. If \( \{ \xi_n \} \), \( \xi \in Q^p \) is mutually convergent in the Gaussian \( L^p \)-norm, then for \( \delta > 0 \)

\[ 1^0 \limsup_{n \to \infty} \left\| t^v d\xi_n(t) \right\|_p < +\infty \text{ for } v = 0, 1, 2, \]

\[ 2^0 \limsup_{n \to \infty} \sup_{|t| > \delta} \left\| \xi_n(t) \right\|_p < +\infty. \]

Proof: If \( \{ \xi_n \} \) is mutually convergent in the Gaussian \( L^p \)-norm then it is bounded in this norm. The inequalities \( 1^0 \) and \( 2^0 \) then follow from Lemma 2.4.
Lemma 2. If \( \{ \xi_n \} \) is a sequence in \( \overline{Q} \), and \( \{ \| \xi_n(t) \|_p \} \) is uniformly bounded in respect to all \( n \) and \( t \) on \( |t| \geq \delta \) for any \( \delta > 0 \), then there exists a sequence \( N \) of positive integers such that
\[
\lim_{n \to +\infty} \sup_{n \in N} \| \xi_n(t) - \xi_n(t) \|_p = g(t)
\]
exists on \((-\infty, +\infty)\) and \( g \) is nondecreasing on \((-\infty, 0)\) and nonincreasing on \((0, +\infty)\) and it is uniformly bounded on \( |t| \geq \delta \) for any \( \delta > 0 \). To any continuity point \( t \) of \( g \) and any \( c > 0 \), there exists a positive number \( h \) such that
\[
\lim_{n \to +\infty} \sup_{n \in N} \| \xi_n(t+h) - \xi_n(t-h) \|_p \leq c.
\]

Proof: Clearly the random function \( \xi_n - \xi_n(t) \) is a.s. nonnegative and a.s. nondecreasing on \((-\infty, 0)\) and a.s. nonincreasing on \((0, +\infty)\).

Hence
\[
g_n = \| \xi_n - \xi_n(t) \|_p
\]
is a sequence of functions which are nondecreasing on \((0, -\infty)\) and nonincreasing on \((0, +\infty)\). Further the sequence of these functions is uniformly bounded on \( |t| \geq \delta \) since \( \| \xi_n \| \) has this property. The existence of \( N \) and \( g \) with the required properties then follows from Helly's first theorem (cf. Bergström (1), p. 71). According to an inequality in Bergström (2) [Theorem 2.4] we have
\[
\| \xi - \xi' \|_p \leq \left( \| \xi \|_p^p - \| \xi' \|_p^p \right)^{1/p}
\]
if \( \xi \) and \( \xi' \) are random variables in \( L^p \) and \( \xi > \xi' \geq 0 \) a.s. Hence we get

1) \( w-lim \) means weak limit, hence here convergence to \( g \) at all continuity points \( t \) of \( g \).
\[ \left\| \xi_n(t+h) - \xi_n(t-h) \right\|_p \leq \left( \left\{ [g_n(t+h)]^P - [g_n(t-h)]^P \right\} \right)^{1/p} \]

on \( t + h < 0 \) and \( t - h < 0, \ h > 0 \).

To any continuity point \( t \) of \( g \) we may choose \( h > 0 \) such that \( t - h \) and \( t + h \) are continuity points of \( g \) and

\[ \left\{ [g(t+h)]^P - [g(t-h)]^P \right\}^{1/p} \leq c. \]

Letting \( n \) pass to \( +\infty \) through \( n \) we then get

\[ \lim_{n \to +\infty} \sup_{n \in N} \left\| \xi_n(t+h) - \xi_n(t-h) \right\|_p \leq c. \]

We are now going to prove the necessity of the conditions (i) and (ii) in Theorem 2. Consider then the following inequality which we simply get by Taylor's formula. (\( \delta \) is any given positive number.)

\[ \left\| \xi_n - \xi_m \right\|_p = \sup_t \left\| \int_{-\infty}^{+\infty} \phi(t/\sigma) d[H_n(t) - \xi_n(t)] \right\|_p \]

\[ = \left\| \int_{|\tau| \leq \sigma} \phi(t/\sigma) \left[ \frac{t}{\sigma} \phi(t/\sigma) + \frac{1}{\sigma} \phi(t/\sigma)^2 \frac{t}{\sigma} \frac{2}{\sigma} \phi(t/\sigma)^2 - \frac{1}{\sigma} \frac{3}{\sigma} \phi(t/\sigma)^3 \right] d[H_n(t) - \xi_m(t)] \right\|_p \]

\[ + \int_{|\tau| > \delta} \phi(t/\sigma) d[H_n(t) - \xi_m(t)] \right\|_p \]

(6)

where \( 0 < \theta < 1 \). At first let \( |t| \geq \delta \). Then as \( \sigma \to 0^+ \) we get

\[ \frac{1}{\sigma} \phi\left( \frac{t}{\sigma} \right) \leq \frac{1}{\sigma} \phi\left( \frac{2\delta}{\sigma} \right) \to 0 \quad (\sigma \to 0^+) \]

\[ \frac{1}{\sigma^2} \phi''\left( \frac{t}{\sigma} \right) \leq \frac{1}{\sigma^2} \phi''\left( \frac{2\delta}{\sigma} \right) \to 0 \quad (\sigma \to 0) \]

\[ \frac{1}{\sigma^3} \phi''\left( \frac{t}{\sigma} \right) \leq \frac{1}{\sigma^3} \phi''\left( \frac{2\delta}{\sigma} \right) \to 0 \quad (\sigma \to 0^+) \]

and all these convergences are uniform in respect to \( t \) and \( \tau \) for
\[ |t| \leq 2\delta, \ |\tau| \leq \delta. \] Hence regarding Lemma 1 we get from (6)

\[
\lim_{n \to +\infty} \| \int_{|\tau| \leq \delta} e(t-\tau) d[\xi_n(\tau) - \xi_m(\tau)] + \int_{|\tau| \leq \delta} \phi(\frac{t-\tau}{\sigma}) d[\xi_n(\tau) - \xi_m(\tau)] \|_p \\
\to 0 \ (\sigma \to 0+) \quad (7)
\]

and this convergence is uniform in respect to \( t \) and \( \tau \) for \(|t| \geq 2\delta\). Now consider the second integral in (7) and choose \( h \) such that \( 0 < h < \frac{\delta}{2} \).

Since

\[ |\phi(\frac{t-\tau}{\sigma}) - e(t-\tau)| \to 0 \ (\sigma \to 0+) \]

uniformly in respect to \( t \) and \( \tau \) for \(|t-\tau| \geq h\) we get, also regarding Lemma 1

\[
\limsup_{n \to +\infty} \left( \int_{|\tau| \geq \delta, \ |t-\tau| \geq h} [\phi(\frac{t-\tau}{\sigma}) - e(t-\tau)] d[\xi_n(\tau) - \xi_m(\tau)] \right) \|_p \\
\to 0 \ (\sigma \to 0+) \quad (8)
\]

uniformly in respect to \( t \) for \(|t| \geq 2\delta\). Further we have (since \(|t| \geq 2\delta\), \( 0 < n < \frac{\delta}{2} \))

\[
\| \int_{|\tau| \geq \delta, \ |t-\tau| \leq h} [\phi(\frac{t-\tau}{\sigma}) - e(t-\tau)] d[\xi_n(\tau) - \xi_m(\tau)] \|_p \\
\leq \| \xi_n(t+h) - \xi_n(t-h) \|_p \\
+ \| \xi_m(t+h) - \xi_m(t-h) \|_p \quad (9)
\]

Now choose \( N, \ g \) and \( h \) to given \( \varepsilon > 0 \) and \(|\delta| \geq 2\delta\) according to Lemma 2 such that

\[
\limsup_{n \to +\infty} \| \xi_n(t+h) - \xi_n(t-h) \|_p \leq \varepsilon. \quad (10)
\]

Combining the relations (7), (8) and (9) we then get

\[
\lim_{n \to +\infty} \| \int_{n \to +\infty} e(t-\tau) d[\xi_n(\tau) - \xi_m(\tau)] \|_p \\
\to 0 \ (\sigma \to 0+) \quad (11)
\]
Here $c$ is arbitrary. Hence observing that the integral is equal to $\xi_n(t) - \xi_n(t)$ we get

$$\| \xi_n(t) - \xi_n(t) \|_p \to 0 \quad \text{as} \quad n \to \infty, \quad m \to \infty, \quad n, m \in \mathbb{N}$$

(12)

and thus

$$L^p - \lim_{n \to \infty} \xi_n(t) = \xi(t)$$

(13)

exists. Obviously (12) holds for all $t$ which are continuity points of the function $g$, constructed in Lemma 2. Hence $\xi_n$ tends $L^p$-weakly to $\xi$ on $(-\infty, 0)$, $(0, +\infty)$ as $n \to +\infty$, $n \in \mathbb{N}$. We show that the convergence is complete on $(-\infty, -\delta]$, $[\delta, +\infty)$ if $\pm \delta$ are $L^p$-continuity points of $\xi$. In fact, all we have to prove is that

$$\| \xi_n(\infty) - \xi(\infty) \|_p \to 0, \quad \| \xi_n(-\infty) - \xi(-\infty) \|_p \to 0 \quad (n \to \infty).$$

Now let first $n$ tend to $+\infty$ through $N$, then $t$ tend to $+\infty$ and at last $n$ tend to $+\infty$ through $N$. Then obviously the right-hand side of (9) tends to 0. Combining (7), (8) and (9) we thus find that

$$\lim_{m \to \infty} \sup_{t \to +\infty} \lim_{n \to +\infty} \sup_{m \in \mathbb{N}} \left\| \xi_n(t) - \xi_m(t) \right\|_p = 0.$$

Hence

$$\| \xi_n(+\infty) - \xi(+\infty) \|_p \to 0 \quad (m \to +\infty).$$

In the same way we find that

$$\| \xi_m(-\infty) - \xi(-\infty) \|_p \to 0 \quad (m \to +\infty).$$

Having thus proved that $\xi_n$ tends $L^p$-completely to $\xi$ as $n \to +\infty$, $n \in \mathbb{N}$, we may consider (6) for arbitrary $t$ and fixed $\sigma$. Letting $n$ and $m$ tend to $+\infty$ through $N$ we find by Helly's generalized second theorem (cf. Bergström (2), Theorem 7.1) that the second integral in (6) tends to 0 in the $L^p$-norm, uniformly in
respect to $t$, $t \in (-\infty, +\infty)$. Further
\[
\lim_{n \to \infty} \sup_{m \to \infty} \left\| \int_{|\tau| \leq \delta} \phi''\left(\frac{\tau - \Theta_0}{o}\right) d[\xi_n(\tau) - \xi_m(\tau)] \right\|_p \\
\leq 2 \delta \sup |\phi''(t)| \cdot \frac{1}{o^3} \lim_{n \to \infty} \sup_{m \to \infty} \left\| \int_{|\tau| \leq \delta} \tau^2 d[\xi_n(\tau) - \xi_m(\tau)] \right\|_p 
\to 0 \quad \delta \to 0^+ 
\]
since the last integral remains bounded as $n \to \infty$ according to Lemma 1.

Since, by Helly's generalized second theorem,
\[
\left\| \int_{|\tau| \leq \delta_1} \tau^\nu d[\xi_n(\tau) - \xi_m(\tau)] \right\| \to 0 \quad n \to \infty \\
m \to \infty \\
n, m \in \mathbb{N}
\]
for $\nu = 0, 1, 2$, if $\pm \delta_0$ and $\pm \delta_1$ are $L^p$-continuity points of $\xi$, we thus obtain from (6) for any $\delta$ such that $\pm \delta$ are $L^p$-continuity points of $\xi$.
\[
\left\| -\frac{1}{o} \phi\left(\frac{\tau}{o}\right) \int_{|\tau| \leq \delta} d[\xi_n(\tau) - \xi_m(\tau)] + \frac{1}{2} \phi''\left(\frac{\tau}{o}\right) \int_{|\tau| \leq \delta} \tau^2 d[\xi_n(\tau) - \xi_m(\tau)] \right\|_p \\
\to 0 \quad n \to \infty \\
m \to \infty \\
n, m \in \mathbb{N}
\]
(14)

For $t = 0$ we have $\phi''\left(\frac{t}{o}\right) = 0$ and thus
\[
\left\| \int_{|\tau| \leq \delta} \tau d\xi_n(\tau) - \int_{|\tau| \leq \delta} \tau d\xi_m(\tau) \right\|_p \to 0 \quad n \to \infty \\
m \to \infty \\
n, m \in \mathbb{N}
\]
Hence
\[
L^p-\lim_{n \to \infty} \int_{|\tau| \leq \delta} \tau d\xi_n(\tau) = \alpha_1(\delta) \quad n \in \mathbb{N}
\]
(15)
exists. Regarding this convergence we then find by (14) that
\[
\left\{ \int_{|\tau| \leq \delta} \tau^2 d\xi_n(\tau) \right\} 
\]
is mutually convergent in the $L^p$-norm and thus that
\[
L^p-\lim_{n \to \infty} \int_{|\tau| \leq \delta} \tau^2 d\xi_n(\tau) = \alpha_2(\eta) \quad n \in \mathbb{N}
\]
(16)
exists.

Thus we have proved that the limits (i) and (ii) exist as \( n \) tends to \( +\infty \) through a certain sequence of positive integers and from the proof it follows that such a sequence may be selected as a subsequence of any infinite sequence of positive integers. Hence we have only to show that the limits in (i) and (ii) are the same for any such sequence. Now consider two such sequences \( N \) and \( N' \) selected according to Lemma 2 with corresponding functions \( g \) and \( g' \). Clearly we may then choose \( h \) to given \( \varepsilon > 0 \) such that (10) holds for \( N \) and also remains true if \( N \) is changed into \( N' \). Then the right side of (9) tends to \( 2\varepsilon \) as \( n \to +\infty \) through \( N \) and \( m \) tends to \( +\infty \) through \( N' \).

Since (7) and (8) hold (independently of \( N \) and \( N' \)) we then conclude that (11) and (12) remain true if \( n \to +\infty \) through \( N \) and \( m \to +\infty \) through \( N' \). Hence

\[
\operatorname{L}^{P}\lim_{n \to +\infty} \xi_n(t) = \operatorname{L}^{P}\lim_{m \to +\infty} \xi_m(t). \quad n \in N \quad m \in N',
\]

In the same way we find by (14) that the limits (ii) are independent of the sequence \( N \). Thus we have proved the necessity of the conditions (i) and (ii). Their sufficiency easily follows from (6). (The left-hand side of this inequality is equal to the supremum in respect to \( t \) of the right-hand side.)

4. Conditions for the \( \operatorname{L}^{P} \)-convergence of Convolutions of Random Functions.

Consider a sequence \( \{\xi_n^{*k_n}\} \) of convolutions, where \( \{\xi_n\} \) is a sequence of \( \operatorname{L}^{P} \)-mean-continuous a.s. nondecreasing random functions and \( \{k_n\} \) is a sequence of positive numbers tending to \( +\infty \) as \( n \to +\infty \). We ask for conditions for the \( \operatorname{L}^{P} \)-complete convergence of \( \{\xi_n^{*k_n}\} \) as \( n \to +\infty \). Of course,
we have then to require that \( \xi_n^{*k_n}(t) \) belongs to \( L^p \) and since \( \xi_n^{*k_n} \) is a.s. nondecreasing and \( L^p \)-mean-continuous it then belongs to \( M^p \). Further we have

\[
\xi_n^{*k_n}(t) \leq \xi_n^{(\infty)} \leq [\xi_n^{(\infty)}]^{k_n} \text{ a.s.}
\]

and thus it is sufficient to require that \([\xi_n^{(\infty)}]^{k_n}\) belongs to \( L^p \),

\[
\|\xi_n^{(\infty)}\|_{k_n}^k = \|\xi_n^{(\infty)}\|_{k_n}^k \leq +\infty.
\]

However if \( \xi_n^{*k_n} \) converges \( L^p \)-completely to a random function \( \eta \) then

\[
[\xi_n^{(\infty)}]^{k_n} = [\xi_n^{(\infty)}]^{k_n}
\]

must converge in the \( L^p \)-norm to \( \eta^{(\infty)} \) and then

\[
\|\xi_n^{(\infty)}\|_{k_n}^k = \|\xi_n^{(\infty)}\|_{k_n}^k
\]

converges. Applying the necessary and sufficient conditions for convergence of such powers of numbers we get

**Theorem 1.** Let \( \{k_n\} \) be a sequence of positive integers tending to \( +\infty \) as \( n \to +\infty \) and let \( \{\xi_n\} \) be a sequence of random functions belonging to \( M^p \) such that \( \xi_n^{*k_n} \) converges \( L^p \)-completely to a random function \( \xi \). Then in order that \( \xi \) is a.s. finite and a.s. \( \neq 0 \) it is necessary and sufficient that

\[
\lim_{n \to +\infty} k_n [\|\xi_n^{(\infty)}\|_{k_n}^p - 1]
\]

exists finite.

In order to get conditions for the \( L^p \)-complete convergence of \( \xi_n^{*k_n} \) we apply some Lemmas in Bergström (1) p. 84. We state them in a slightly different form, depending on the fact that we have to consider \( L^p \)-norms for different \( p \geq 1 \). Then we have to make estimations as follows.
Lemma 1. Let $\nu_1, \nu_2$ and $\nu = \nu_1 + \nu_2$ be nonnegative integers and $p$, $q_1$ and $q_2$ real numbers not smaller than 1 which satisfy the inequality

$$1/q_1 + 1/q_2 \leq 1/p.$$ 

Then, if $\xi$ is of bounded variation in respect to the $L^{1/q_1}$-norm and $\xi_i$ is of bounded variation in respect to the $L^{1/q_2}$-norm, we have

$$\| \xi * \xi_1^{\nu_1} * \xi_2^{\nu_2} \|_p \leq \| \xi \|_{q_1} \cdot K_1 K_2$$

with

$$K_i = \exp \nu \left| \int_{-\infty}^{+\infty} \| d\xi_i(t) \|_{q_2} - 1 \right|.$$ 

Proof: Let $\eta_i$ be the total variation of $\xi_i$ in respect to the $\nu q_2$ norm. Then for $\nu_2 > 1$

$$\| \xi * \xi_1^{\nu_1} * \xi_2^{\nu_2} \|_p \leq \| \xi * \xi_1^{\nu_1} * \xi_2^{\nu_2 - 1} \cdot \eta_2 \|_p$$

and we get by Hölder's inequality

$$\| \xi * \xi_1^{\nu_1} * \xi_2^{\nu_2} \|_p \leq \| \xi * \xi_1^{\nu_1} * \xi_2^{\nu_2 - 1} \|_{p_1} \| \eta_2 \|_{q_2}$$

with

$$1/p_1 \leq 1/p - 1/\nu q_2.$$ 

Hence by induction we obtain

$$\| \xi * \xi_1^{\nu_1} * \xi_2^{\nu_2} \|_p \leq \| \xi \|_{q_1} \cdot \| \eta_1 \|_{q_2} \| \eta_2 \|_{q_2}$$

with

$$1/q_1 \leq 1/p - \nu \cdot 1/\nu q_2 = 1/p - 1/q_2$$

and
\[ \| \eta_1^i \|_{vq_2} \leq \max \left[ 1, \| \eta_1^i \|_{vq_2} \right] \leq \exp \| \eta_1^i \|_{vq_2} - 1 \].

Using Lemma 1 we get the following lemmas corresponding to the lemmas in Bergström (1).

Lemma 2. If \( \xi_1 \in \bar{\mathbb{M}}_{1p}, \xi_2 \in \bar{\mathbb{M}}_{2p} \) and \( p, q_1 \) and \( q_2 \) are real numbers such that \( p \geq 1, q_1 \geq 1, q_2 \geq 1, 1/p \leq 1/q_1 + 1/q_2 \) then

\[ \| \xi_1 - \xi_2 \|_p \leq r \| \xi_1 - \xi_2 \|_{q_1, q_2} \]

with

\[ K_1 = \exp (r-1) \| \xi_i \|_{(r-1)q_2} - 1 \].

Proof: We get the inequality from the identity

\[ \xi_1 - \xi_2 = (\xi_1 - \xi_2) \sum \xi_1^v \xi_2^\nu = r \frac{1}{1+r} \]

if we estimate the terms on the right-hand side according to Lemma 1.

Lemma 3. Let \( p, p_1, p_2, q_1 \) and \( q_2 \) be real numbers not smaller than 1 which satisfy the condition

\[ 1/p_1 + 1/q_1 \leq 1/p, 1/p_2 + 1/q_2 \leq 1/p \]

and let \( m_1, m_2 \) and \( r \) be positive integers, \( m_1 > r, m_2 > r \). Then if \( \xi \in \bar{\mathbb{M}}_{m,p} \)

and \( \eta \in \bar{\mathbb{M}}_{2p} \), we have

\[ \| \xi_1^{m_1} - \xi_2^{m_2} \|_p \leq \| \xi_1 (\xi_1 - e) - m_2 (\xi_2 - e) \|_{p_1 K_1 K_2} ^\sigma \]

\[ + r \| \xi_1 - e \|_{p_1 (K_1 K_2 + K_1)} + r \| \xi_2 - e \|_{q_1 (K_1 K_2 + K_2)} \]

\[ + \frac{1}{2} \frac{m_1}{r} \| (\xi_1 - e)^2 \|_{p_2 K_1 K_2} + \frac{1}{2} \frac{m_2}{r} \| (\xi_2 - e)^2 \|_{K_1 K_2} \]
with

\[ K_i = \exp{(r-1)} \frac{\| \xi_i^{(t+\infty)} \|_{(r-1)}}{q_1} - 1 \]

\[ K_i' = \exp{(r-1)} \frac{\| \xi_i^{(t+\infty)} \|_{(r-1)}}{q_2} - 1 \]

Proof: Let \( s_i = \frac{m_i}{r} \) = the largest integer \( \leq \frac{m_i}{r} \). Hence

\[ m_i = rs_i + t_i, \quad 0 \leq t_i \leq r-1 \]

and we have identically

\[ \xi_i^{*m_1} - \xi_i^{*m_2} = \xi_i^{*rs_1} - \xi_i^{*t_1-e} + \xi_i^{*rs_2} - \xi_i^{*t_2-e}. \]  \hspace{1cm} (3)

According to the identity (2) we have

\[ \xi_i^{*rs_1} - \xi_i^{*rs_2} = \xi_i^{*s_1} - \xi_i^{*s_2} * \sum_{v_1+v_2=r-1}^{1} \xi_i^{*v_1 s_1} * \xi_i^{*v_2 s_2}. \]  \hspace{1cm} (4)

Further by this identity for \( \xi_1 = \xi, \; \xi_2 = e \) we easily get (cf. Bergström (1), p. )

\[ \xi_i^{*s} - e = s(\xi-e) + (\xi-e) \sum_{\nu=0}^{s-2} (s-\nu-1) \xi_i^{*\nu}. \]  \hspace{1cm} (5)

Combining this identity (for \( \xi_1 = \xi_1 \) and \( \xi_2 = s_1 \) and \( s_2 \)) with (4) we get

\[ \xi_i^{*rs_1} - \xi_i^{*rs_2} = [s_1(\xi_1-e)-s_2(\xi_2-e)] * \sum_{v_1+v_2=r-1}^{1} \xi_i^{*v_1 s_1} * \xi_i^{*v_2 s_2} \]

\[ + (\xi_1-e) \sum_{\nu=0}^{s_1-2} (s_1-\nu-1) * \sum_{v_1+v_2=r-1}^{1} \xi_i^{*v_1 s_1} * \xi_i^{*v_2 s_2} \]

\[ + (\xi_2-e) \sum_{\nu=0}^{s_2-2} (s_2-\nu-1) * \sum_{v_1+v_2=r-1}^{1} \xi_i^{*v_1 s_1} * \xi_i^{*v_2 s_2}. \]
Estimating the terms on the right-hand side according to Lemma 1, we get
\[ \| \xi_1^{*rs_1} - \xi_2^{*rs_2} \|_p \leq r \| s_1 (\xi_1 - e) - s_2 (\xi_2 - e) \|_{p_1} K_1 K_2 \]
\[ + \frac{1}{2} rs_1 (s_1 - 1) \| (\xi_1 - e)^2 \|_{p_2} K_1' K_2 + \frac{1}{2} rs_2 (s_2 - 1) \| (\xi_2 - e)^2 \|_{p_2} K_1' K_2'. \]

In the same way we obtain
\[ \| \xi_1^{*t_i} (\xi_1 - e) \|_p \leq (r-1) \| \xi_1 - e \|_{p_1} K_i. \]

Regarding the identity (3) and the estimations (6) and (7) and also observing that
\[ 0 \leq m_1 - rs_1 = t_i - r-1, \quad \sum_{\nu=0}^{s-2} (s-\nu-1) = \frac{1}{2} s(s-1) \]
we get the inequality of the lemma.

We are now in the position to prove

**Theorem 2.** Let \( p, p_1 \) and \( q_1 \) be real numbers such that
\[ p \geq 1, \quad p_1 \geq 1, \quad q_1 \geq 1, \quad 1/p_1 + 1/q_1 > 1/p. \]

Further let \( k_n \) be a sequence of positive integers tending to \( +\infty \) as \( n \to +\infty \),
and let \( \{ \xi_n \} \) be a sequence of random functions satisfying the following conditions

1° \( \xi_n^{mek} q_1^{k_1} \)
2° \( \| k_n (\xi_n - e) - k_m (\xi_m - e) \|_{p_1} \to 0 \quad (n \to +\infty) \)
3° \( \limsup_{n \to +\infty} k_n^2 \| (\xi_n - e)^2 \|_{p_1} < +\infty \)
4° \( \limsup_{n \to +\infty} k_n \| \xi_n (\xi_n \to +\infty) \|_{k_n q_i} - 1 \| < +\infty, \quad i = 1, 2. \)

Then \( \{ \xi_n^{*k_n} \} \) converges \( L^p \)-completely to a random function \( \eta \) belonging to \( \mathcal{M}^p \)
and \( \neq 0 \) a.s. Further \( \eta \) is infinitely divisible in respect to the \( L^p \)-norm.
Remark 1. The condition 2° can be given in a more explicit form according to Theorem 3.2.

Remark 2. I $\limsup \frac{k}{n} \| \xi^{*}_{n} - e \|^{\sigma}_{2p_{1}} < +\infty$. Then we have

$\| (\xi - e)^{\ast 2} \|^{\sigma}_{q_{2}} \leq \gamma (\| \xi - e \|^{\sigma}_{p_{1}})^{2}$

with a constant $\gamma$ (according to Theorem 3.2). Hence if 2° holds for any $p_{1} \geq 1$ and 4° holds for any $q_{1} \geq 1$, then 3° follows from 2° and thus $\xi^{*}_{n}$ will tend $L^{p}$-completely to $\eta$ for any $p \geq 1$. It may be observed that 4° is a necessary condition for such a convergence.

Proof of the Theorem: Applying the Lemma with $m_{1} = k_{n}$, $m_{2} = k_{m}$ we get

$\| \xi^{*}_{n} - \xi^{*}_{m} \|^{\sigma}_{p_{n}} \leq \| k_{n} (\xi^{*}_{n} - e) - k_{m} (\xi^{*}_{m} - e) \|^{\sigma}_{p_{n}} + k_{n}^{2} \| K_{n} \|^{\sigma}_{p_{n}} + k_{m}^{2} \| K_{m} \|^{\sigma}_{p_{m}}$

$+ \frac{1}{2} \frac{k_{n}^{2}}{r} \| (\xi^{*}_{n} - e)^{\ast 2} \|^{\sigma}_{p_{n}} + \frac{1}{2} \frac{k_{m}^{2}}{r} \| (\xi^{*}_{m} - e)^{\ast 2} \|^{\sigma}_{p_{m}}$

with

$K_{n} = \exp (k_{n} - 1) \| \xi_{n}^{(+\infty)} \| (k_{n} - 1) q_{1} - 1 \|.$

According to 4°, $K_{n}$ remains bounded as $n \to +\infty$. Hence in the inequality above the first and the second terms tend to 0 as $n \to +\infty$, $m \to +\infty$, according to 2° and by 3° the third and the fourth terms then tend to quantities which are arbitrarily small for sufficiently large $r$. Thus we conclude that

$\| \xi^{*}_{n} - \xi^{*}_{m} \|_{p} \to 0 \quad n \to +\infty, \quad m \to +\infty.$
By Theorem 3.2 we then have

$$L^p\text{-}\lim \xi_n^{*k} = \eta$$

with $\eta \in \mathcal{M}^p$.

In order to prove that $\eta$ is infinitely divisible we choose any integer $r > 1$ and put $\left[\frac{r}{k} \right] = s_n$. It follows from that part of the theorem that has just been proved that

$$\xi_n^{*s} \rightarrow \eta, \text{ (}L^p\text{)}$$

Now define $p'$ by

$$\frac{1}{p'} = \frac{1}{p_1} + \frac{1}{r q_1}.$$ 

Then the conditions $l^o - 4^o$ are satisfied if we change $k_n$, $\sigma$ and $q_1$ into $s_n$, $\sigma'$ and $r q_1$ respectively and keep $p_1$. Hence $\xi_n^{*s}$ tends $L^{p'}$-completely to a random function $\eta_r$ in $\mathcal{M}^{p'}$.

Put $q' = \frac{rq_1}{r-1}$. Then

$$\frac{1}{p'} + \frac{1}{q'} = \frac{1}{p_1} + \frac{1}{r q_1} r + r-1/q_1 r = \frac{1}{p_1} + \frac{1}{q_1} \leq \frac{1}{p}.$$ 

Applying Lemma 2 we then get

$$\|\xi_n^{*s} \eta_s - \eta_r \|_p \leq \|\xi_n^{*s} \eta_s - \eta_r \|_p K_1 K_2$$

where

$$K_1(n) = \exp (r-1) \left| \|\xi_n^{*s} \eta_s^{(\infty)} \|_{(r-1)q} - 1 \right|$$

$$K_2 = \exp (r-1) \left| \|\eta_r^{(\infty)} \|_{(r-1)q'} - 1 \right|.$$ 

Since $\xi_n^{*s} \eta_s^{(\infty)} = [\xi_n^{(\infty)}]^{s_n}, (r-1)q' = rq_1$ we have
\[ K_1(n) = \exp(r-1) \left| n \left( \xi_n^{(+\infty)} \right)^{n} - 1 \right| \]

and \[ \left| n \xi_n^{(+\infty)} \right|^{n}_{rs q_1} \] remains finite as \( n \to +\infty \) according to \( 4^\circ \). Hence \( K_1(n) \) remains finite as \( n \to +\infty \). We also observe that \( K_2 \) is defined. In fact \( \left[ n \xi_n^{(+\infty)} \right]^{n}_{rs q_1} \) tends to \( \eta_r^{(+\infty)} \) in the \( L^p \)-norm since \( \xi_n^{*s} \) converges \( L^p \)-completely.

But then \( \left[ n \xi_n^{(+\infty)} \right]^{n}_{rs q_1} \) converges in probability to \( \eta_r^{(+\infty)} \). Further

\[ \left| \left[ n \xi_n^{(+\infty)} \right]^{n}_{rs q_1} \right| \text{ converges; since} \]

\[ \left| n \xi_n^{(+\infty)} \right|^{n}_{rs q_1} = \left| n \xi_n^{(+\infty)} \right|^{n}_{rs q_1} \]

converges according to \( 4^\circ \) (and to a number which is different from 0). Hence \( \left[ n \xi_n^{(+\infty)} \right]^{n}_{rs q_1} \) converges to \( \eta_r^{(+\infty)} \) in the \( L^r \)-norm and \( \eta_r^{(+\infty)} \) belongs to \( L^r \) (cf. Loève (1), \( L_r \)-convergence theorem p. 113). Since \( \xi_n^{*s} \) tends to \( \eta_r \) in the Gaussian \( L^p \)-norm we now find by (8) that

\[ \left| n \xi_n^{*s} - \eta_r \right|^{n}_{r} \to 0 \text{ (} n \to +\infty \text{)} \]

and thus

\[ \eta = \eta_r^{*r} \text{ (} L^p \) \]
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