CONVOLUTIONS OF RANDOM FUNCTIONS

by

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1. Introduction

Let \([X, \Omega, P]\) be a probability space, \(X\) being the sample-space, \(\Omega\) a \(\sigma\)-field of subsets of \(X\) and \(P\) a probability measure on \(\Omega\). All complex random variables \(\xi\) on \(\Omega\) which have finite \(p^{th}\) moment, \(p \geq 1\), form a Banach space \(L^p\) with the norm

\[ ||\xi||_p = \sqrt[p]{E[|\xi|^p]} \]


\[ ||\xi||_{+\infty} = \lim_{p \to +\infty} ||\xi||_p \]

is also a norm and all \(\xi\) such that \(||\xi||_{+\infty} < +\infty\) form a Banach space \(L^{+\infty}\) with this norm [Loève, loc. cit]. It may be observed that \(||\xi||_{+\infty}\) is the a.s. supremum of \(\xi\) [Loève [2] 9.4].

In the following we consider Banach spaces \(L^p\), \(p \geq 1\) and also \(p = +\infty\), and we shall study families of random variables called random functions. Let \(R\) be the real line. A family \(\xi = \{\xi(t), t \in R\}\) of complex valued random variables \(\xi(t)\) is called a random function on \(R\). If \(\xi(t)\) is a number for any \(t\) it is a function (not random) on \(R\). If \(\xi(t)\) belongs to \(L^p\) for every \(t \in R\) we call \(\xi\) a \(L^p\)-random function. We are going to study the random variables \(\xi(t)\) in respect to the dependence of \(t\). The probability space \([X, \Omega, P]\) is of interest only in that it determines the \(L^p\)-space and the \(L^p\)-norm. When this norm has been introduced we can study the random functions on \(R\) very much in the same way as functions on \(R\), (which of course are special cases of random functions). Thus we introduce concepts such as limit, continuity, derivative, Riemann-integral, Riemann-Stieltjes integral
and other concepts for random functions in the same way as for functions
but these concepts have then to be defined in respect to the $L^p$-norm instead
of the absolute norm (absolute value). Accordingly we shall use the notations
$L^p$-limit, $L^p$-continuity and so on in order to point out which norm we use.

In this paper I am mainly going to study $L^p$-convolutions of random
functions and I shall show that these convolutions can be defined directly
by the help of $L^p$-RS-integrals (Read $L^p$-Riemann-Stieltjes integrals.).
In a forthcoming paper I shall also show how they are given indirectly by
the help of $L^p$-Fourier-Stieltjes transforms. In order to be able to use
the latter definition I shall then give a generalization of Bochner's
well-known theorem, that only continuous positive-semidefinite functions
are Fourier-Stieltjes transforms of some non-decreasing bounded functions.
Having introduced convolutions of random functions I can study limit
theorems for $L^p$-convolutions of infinitely many random functions. In
another paper I shall give generalizations of limit theorems for convolu-
tions of functions of bounded variation to $L^p$-convolutions of random
functions of bounded variation in respect to the $L^p$-norm and then I shall
follow the same pattern as in my book, Limit Theorems for Convolutions (1).
I shall also here frequently refer to this book on the following pages and
therefore I use the abbreviation LTC for it. In fact, the theory given
there for functions, can easily be transformed to a theory for random functions
by transforming relations in the absolute value into corresponding relations
in the $L^p$-norm. Therefore, in many cases, I shall refer to LTC and only
give some complementary remarks.

In order to emphasize the $L^p$-norm we usually add $(L^p)$ after a
relation. Thus for instance $\xi - \eta = 0 (L^p)$ means that $||\xi - \eta||_p = 0$. 
We say that $\xi$ and $\eta$ are $L^p$-equivalent or equal ($L^p$) if this relation is satisfied.

A random function $\xi$ is said to be of $L^p$-bounded variation on an interval $[a, b]$, if there exists a number $\alpha$ such that

$$\sum_{i=1}^{n} \frac{\left| \xi(t_i) - \xi(t_{i-1}) \right|}{\left| t_i - t_{i-1} \right|} \leq \alpha$$

(1)

for any net

$$N: a = t_0 < t_1 < t_2 < \ldots < t_n = b$$

fitted on $[a, b]$. The smallest value $\alpha$ having this property is called the $L^p$-variation of $\xi$ on $[a, b]$ and is denoted by

$$\frac{b}{a} \int \left| \frac{d}{dt} \xi(t) \right|_p$$

If (1) holds for any interval $[c, b]$ for any net on $[a, b]$, we say that $\xi$ is of $L^p$-bounded variation on $\mathbb{R}$ and the smallest value $\alpha$ satisfying (1) for any interval $[a, b]$ and any net on $[a, b]$ is called the $L^p$-variation of $\xi$ on $\mathbb{R}$ and is denoted by

$$\frac{1}{\mathbb{R}} \int \left| \frac{d}{dt} \xi(t) \right|_p$$

The class of all random functions of $L^p$-bounded variation is denoted by $W^p$.

A random function $\xi$ is said to be of bounded variation on $[a, b]$ in respect to the $L^p$-norm, if there exists a number $\alpha$ such that

$$\left\| \sum_{i=1}^{n} \frac{\left| \xi(t_i) - \xi(t_{i-1}) \right|}{\left| t_i - t_{i-1} \right|} \right\|_p \leq \alpha$$

(2)

for any net $N$ and $\xi$ is said to be of bounded variation on $\mathbb{R}$ in respect to $L^p$ if (2) holds with the same $\alpha$ for all intervals $[a, b]$ and for any net fitted on $[a, b]$. The class of all random functions of bounded variation in respect to the $L^p$-norm is denoted by $\mathcal{W}^p$. 
As in LTC [p. 19] we formally use the infinite number \( \infty \) and by definition the formal relations \( a + (\infty) = \infty + a = \infty, \quad a - (\infty) = \infty, \quad \infty - (\infty) = \infty, \quad a \cdot \infty = \infty \cdot a = \infty, \quad \infty \cdot \infty = \infty \). Further we define \( \xi(\infty) = 0 \) for any random function. Then we also permit an interval of the form \([\infty, b]\) which consists of all real numbers smaller than \( b \) the number \( b \) and the formal point \( \infty \). [Cf. LTC p. 19.]

The theory given in the following for random functions on \( \mathbb{R} \) can be generalized to random functions on a \( \kappa \)-dimensional vector-space \( \mathbb{R}_\kappa \) and that can essentially be done in the same way as the corresponding generalization for functions in LTC, part II.

2. Some essential properties of the \( L^p \)-norm.

We shall here collect some properties of the \( L^p \)-norm.

**Theorem 1.** The \( L^p \)-norm satisfies Hölder's inequality.

\[
\left| \xi_1 \cdot \xi_2 \cdots \xi_n \right|_r \leq \left| \xi_1 \right|_{r/\alpha_1} \cdot \left| \xi_2 \right|_{r/\alpha_2} \cdots \left| \xi_n \right|_{r/\alpha_n}
\]

for \( \xi_\nu \in L^r, \quad \alpha_\nu \geq 0, \quad \nu = 1, 2, \ldots, n, \quad \sum_{\nu=1}^n \alpha_\nu \leq 1, \quad r \geq 1.

(For \( \alpha_\nu = 0 \) we put \( L^r = L^{+\infty} \).)

**Proof:** See Loeve [2], p. 156.

A consequence of this inequality is

**Theorem 2.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and \( r \) be non-negative numbers and

\[
\sum_{\nu=1}^n \alpha_\nu \leq 1, \quad r \geq 1,
\]

and suppose that \( \xi_\nu \) belongs to \( L^{r/\alpha_\nu} \) for \( \nu = 1, 2, \ldots, n \). Then

\[
\xi_1 \cdot \xi_2 \cdot \cdots \cdot \xi_n \text{ belongs to } L^r.
\]
Corollary: If $1 \leq r \leq s$, then $L^r \supset L^s$.

Theorem 3. The $L^p$-norm satisfies Minkowski's inequality

$$||\xi - \eta||_p \leq ||\xi||_p + ||\eta||_p$$

Corollary: $||\xi - \eta||_p \geq ||\xi||_p - ||\eta||_p$

Proof: See Loeve [2], p. 150.

We shall have use for the following special inequality which goes in the opposite direction to Minkowski's inequality.

Theorem 4. If $\xi$ and $\eta$ are real and belong to $L^p$ and if $\xi \geq \eta \geq 0$ a.s., then

$$||\xi - \eta||_p \leq \left( ||\xi||_p^p - ||\eta||_p^p \right)^{1/p}$$

Proof: Since $p \geq 1$ we get

$$\xi^p = (\eta + \xi - \eta)^p \geq \eta^p + (\xi - \eta)^p \ a.s.$$ and thus

$$||\xi||_p^p \geq ||\eta||_p^p + ||\xi - \eta||_p^p$$

We also quote the well-known inequality of Markov

Theorem 5. $P[|\xi| \geq \epsilon] \leq \frac{c}{\epsilon^p} ||\xi||_p^p$

for any positive number $\epsilon$ and $\xi \in L^p$.

Using this inequality we get

Theorem 6. If $\left\{\xi_n\right\}$ is a sequence of real random variables in $L^p$ such that $\xi_n \geq 0$ a.s. for every $n$ and if $\xi = L^p$-lim $\xi_n$, then $\xi \geq 0$ a.s.

Proof: By Markov's inequality we obtain

$$P[\xi \leq - \epsilon] \leq P[\xi - \xi_n < - \epsilon] \leq \frac{1}{c^p} ||\xi - \xi_n||_p^p \rightarrow 0 \ (n \rightarrow +\infty).$$

A set $T$ of elements is called a direction if a) it is partially ordered, b) every pair $t_1$ and $t_2$ of elements of $T$ is followed by an element $t_3$ in $T$ (cf. Loeve (1), p. 67). Denote the partial ordering by $\preceq$ (we write $t_1 \prec t_2$ if $t_1$ precedes $t_2$). Now let $\{\xi(t) : t \in T\}$ be a
family of random variables where T is a direction. (We also say that $\xi$ is a random function defined on the direction T). Then we call a random variable $\xi_0$ the $L^p$-limit of this family in the direction T, if $\xi_0$ has the following properties: To any $\varepsilon > 0$ there belongs at least one element $t_\epsilon \in T$ such that

$$||\xi_0 - \xi(t)||_p < \varepsilon \quad \text{for} \quad t > t_\epsilon$$

Further this family of random variables is called $L^p$-convergent in the direction T when such a limit exists. Clearly the $L^p$-limit is unique (in respect to $L^p$-equivalence).

The family $\{\xi(t): t \in T\}$ is called mutually $L^p$-convergent in the direction T if there to any $\varepsilon > 0$ belongs at least one element $t_\epsilon \in T$ such that

$$||\xi(t) - \xi(\tau)||_p < \varepsilon$$

for $t_\epsilon < t$, $t_\epsilon < \tau$.

**Theorem 7.** If T is a direction then the family $\{\xi(t): t \in T\}$ of random variables is $L^p$-convergent in the direction T if and only if it is mutually $L^p$-convergent in this direction.

**Proof:** If (1) holds we can choose a sequence $\{t_n\}$ in T such that

$t_n < t_{n+1}$ and

$$||\xi(t_n) - \xi(t_{n-1})||_p < 1/2^n$$

for $n, k = 1, 2, \ldots$ and $n > k$ and hence the sequence $\{\xi(t_n)\}$ is mutually convergent in respect to the $L^p$-norm and thus convergent in this norm, since $L^p$ is a Banach space. It is easily seen from (1) that $L^p$-lim $\xi(t_n)$ is the limit of the family in the direction T. Clearly the family is mutually $L^p$-convergent in the direction T if it is $L^p$-convergent in this direction.
We call the real family \( \{ \xi(t) : t \in T \} \) a.s. non-decreasing (non-increasing) in the direction \( T \) if \( \xi(t) \leq \xi(\tau) \) (\( \xi(t) \geq \xi(\tau) \)) a.s. for \( t \neq \tau \). If the family is either non-decreasing or nonincreasing in the direction \( T \) it is called monotone in that direction.

Theorem 3. If \( T \) is a direction and the family \( \{ \xi(t) : t \in T \} \) is real and a.s. monotone in the direction \( T \) and
\[
\sup_{t \in T} ||\xi(t)||_p < +\infty
\]
then the family is \( L^p \)-convergent in the direction \( T \).

Proof: It is sufficient to consider that case when the family is a.s. nondecreasing in the direction \( T \). Further we may assume that \( \xi(t) \geq 0 \) a.s., since we may as well consider the family \( \{ \xi(t) - \xi(t_0) : t \in T, t \geq t_0 \} \)
where \( t_0 \) is some element in \( T \). Clearly we then have \( ||\xi(t)||_p \leq ||\xi(\tau)||_p \)
for \( t \geq \tau \). Put
\[
\sup_{t \in T} ||\xi(t)||_p = \alpha
\]
To any given positive number \( \varepsilon \) we may choose \( t_\varepsilon \) such that
\[
||\xi(t_\varepsilon)||_p > \alpha^p - \varepsilon^p
\]
Applying Theorem 4, we get for \( t \geq t_\varepsilon \)
\[
||\xi(t) - \xi(t_\varepsilon)|| \leq \varepsilon
\]
Hence \( \{ \xi(t) : t \in T \} \) is \( L^p \)-mutually convergent in the direction \( T \) and thus \( L^p \)-convergent in that direction.

3. Almost strongly uniformly \( L^p \)-continuous \( L^p \)-processes.

A random function \( \xi \) is said to be almost strongly uniformly \( L^p \)-continuous
on an interval \([a, b]\) \((a = \infty \text{ permitted, cf. Section } 1)\), if for any positive number \(c\) there exists at least one net

\[
N: \quad a = t_0 < t_1 < t_2 < \ldots < t_n = b
\]

fitted on \([a, b]\) such that the \(L^p\)-oscillation of \(\xi\) on the interior of every mesh, \((t_{i-1}, t_i)\), is smaller than \(c\). (The \(L^p\)-oscillation of \(\xi\) on a set \(T\) is equal to \(\sup_{t, \tau \in T} \|\xi(t) - \xi(\tau)\|_p\).)

**Theorem 1.** A \(L^p\)-random function \(\xi\) is almost strongly uniformly \(L^p\)-continuous on an interval \([a, b]\) if and only if the \(L^p\)-limits \(\xi(t^+)\) and \(\xi(t^-)\), \(t \in (a, b)\), and \(\xi(a^+)\) and \(\xi(b^-)\) exist.\(^1\)

**Remark:** M. Loève has proved that under the assumption of separability of \(\xi\), the \(L^p\)-limits \(\xi(t^+)\) and \(\xi(t^-)\) are equal at all points except at most an enumerable subset when they exist [Loève I, p. 511]. However, we do not assume separability here and therefore the set of points where \(\xi(t^+) \neq \xi(t^-)\) may not be enumerable.

**Proof:** Clearly it is sufficient to consider a real random function \(\xi\). The set \(T = \{t : t \leq \tau, \tau \in (a, b), \tau \in (a, b)\}\) is a direction. If \(\xi\) is almost strongly uniformly \(L^p\)-continuous, given \(c > 0\), we can find \(t_c \in T\) such that

\[
\|\xi(t) - \xi(c)\|_p \leq c
\]

for \(t \in T, s \in T\) \(t \supset t_c, s \supset t_c\). Thus \(\{\xi(t) : t \in T\}\) is mutually \(L^p\)-convergent in the direction \(T\) and then \(L^p\)-convergent in that direction [cf. Theorem 2.7], i.e. the \(L^p\)-limit \(\xi(\tau^-)\) exist. In the same way we find that \(\xi(\tau^+)\) exist for \(\tau \in (c, b)\). We prove the inverse statement indirectly and make the following assumption.

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\(^1\) The corresponding theorem for functions on \(R\) was not given in LTC.
\( \Lambda_c \): The \( L^p \)-limits \( \xi(t^+), \xi(t^-), t \in (a, b) \) and \( \xi(a^+), \xi(b^-) \) exist but \( \xi \) is not almost uniformly \( L^p \)-continuous on \([a, b]\). If we divide \([a, b]\) into two subintervals by its midpoint this assumption must hold for at least one of these subintervals, say for \([a^{(2)}, b^{(2)}]\). Proceeding in a classical way we may find a sequence \([a^{(n)}, b^{(n)}]\) of subintervals for which the assumption \( \Lambda_c \) holds. This sequence of subintervals has a limiting point \( \tau \) on \([a, b]\). Clearly the \( L^p \)-limits \( \xi(\tau^+) \) and \( \xi(\tau^-) \) cannot exist if \( \tau \in (a, b) \), \( \xi(\tau^+) \) cannot exist if \( \tau = a \), and \( \xi(\tau^-) \) cannot exist if \( \tau = b \).

4. The Class \( V^p \).

In section 1 we have defined the class \( V^p \) of random functions of bounded variation in respect to the \( L^p \)-norm. We are now going to study this class more closely. To begin with we show that \( \xi \in V^p \) determines another random function of this class which corresponds to the total variation of a function and which we therefore call the total variation of \( \xi \) in respect to the \( L^p \)-norm.

Consider nets

\[ N : a = t_0 < t_1 < t_2 < \ldots < t_n = b \quad (1) \]

fitted on an interval \([a, b]\). We say that the net \( N' \) is a refinement of \( N \) and we put \( N' \succ N \quad (N \prec N') \) if the set of points in \( N' \) belong to the set of points in \( N \). To \( \xi \in V^p \) and such a net \( N \) there corresponds the sum

\[ \sigma(N)(\xi) = \sum_{v=1}^{n} |\xi(t_i) - \xi(t_{i-1})| \quad (2) \]
If $\mathcal{N}' \triangleright \mathcal{N}$, then clearly
\[ \sigma^{(N')}(\xi) \geq \sigma^{(N)}(\xi) \geq 0 \]
Further
\[ \sup_N \| \sigma^{(N)}(\xi) \|_P \]
is finite, if $\xi$ is of bounded variation on $[a, b]$ in respect to the
$L^P$-norm. Thus the set $\{\mathcal{N}\}$ of all nets is directed by $\triangleright$ and the family
$\{\sigma^{(N)}(\xi)\}$ is nondecreasing in that direction. Applying Theorem 2.8 we
conclude that this family has a $L^P$-limit, which we denote by $\int_a^b |d\xi|$ and
call the total variation of $\xi$ on $[a, b]$ in respect to the $L^P$-norm. According
to the definition of the $L^P$-limit and Theorem 2.8 this total variation has
the following properties.

**Theorem 1.** If $\xi$ is of bounded variation on $[a, b]$ in respect to the $L^P$-norm,
then
\[ \left\| \int_a^b |d\xi| \right\|_P = \sup_N \| \sigma^{(N)}(\xi) \|_P \]
Further we have

**Theorem 2.** Let $\xi$ be of bounded variation on $[a, b]$ in respect to the
$L^P$-norm and let $\mathcal{N}$ be the net (I). Then
\[ \int_a^b |d\xi| = \sum_{v=1}^n \int_{t_{v-1}}^{t_v} |d\xi| \quad (L^P) \]

**Proof:** We may fit a net $\mathcal{N}'$ on $[a, b]$ such that $\mathcal{N}' \triangleright \mathcal{N}$. Then $\mathcal{N}'$
determines
nets $\mathcal{N}'_v$ on the subintervals $[t_{v-1}, t_v]$. Clearly we may choose $\mathcal{N}'$ such that
$\sigma^{(N')}(\xi)$ and $\sigma^{(N'_v)}(\xi)$ approximates $\int_a^b |d\xi|$ and $\int_a^{t_v} |d\xi|$ for $v = 1, 2, \ldots, n$,
respectively arbitrarily closely in respect to the $L^P$-norm. Since we have
\[ \sigma^{(N)}(\xi) = \sum_{v=1}^n \sigma^{(N'_v)}(\xi) \]
the theorem follows.
For $a = \infty$, $b = t$ we put
\[ \eta(t) = \int_t^\infty |d\xi| \quad (L^p) \]
and call $\eta$ the total variation of $\xi$ in respect to the $L^p$-norm. Clearly $\eta \in V^p$.

A random function $\xi$ is called a.s. nonincreasing if
\[ \xi(t) - \xi(\tau) \geq 0 \text{ a.s. for any pair } (t, \tau) \text{ with } t \geq \tau. \]
(It may be observed that the sample functions need not be nondecreasing when the random function is nondecreasing, not even almost all sample functions need be nondecreasing.) It follows from Theorem 2 that the total variation of $\xi \in V^p$ in respect to the $L^p$-norm is a.s. nondecreasing (cf. also Theorem 2.6). Using this fact we can show

Theorem 3. A real random function $\xi$ belonging to $V^p$ is the difference between two a.s. nondecreasing random functions $\eta$ and $\zeta$ belonging to $V^p$.

Proof: Let $\eta$ be the total variation of $\xi$ in respect to the $L^p$-norm.

Putting $\zeta = \eta - \xi$ we get $\xi(t) - \xi(\tau) = \eta(t) - \eta(\tau) - [\xi(t) - \xi(\tau)]$
and the right hand side is a.s. non-negative for $t > \tau$ according to the definition of $\eta$.

By the help of this theorem we show

Theorem 4. $\xi \in V^p$ is almost strongly uniformly $L^p$-continuous.

Proof: We may consider real random functions according to Theorem 3.1. It is sufficient to prove that the $L^p$-limits $\xi(t^-)$ and $\xi(t^+)$ exist for every number $t$ and for $t = \infty$ and according to Theorem 3 above it is sufficient to consider an a.s. nondecreasing random function. Then the family $\{\xi(t) : t < \tau\}$ is nondecreasing in the direction of increasing $t$ and clearly $\xi$ is bounded in respect to the $L^p$-norm. Applying Theorem 2.8 we conclude that $\xi(t^-)$ exists. Here we may permit $\tau = \infty$ (then $\tau^- = +\infty$).
In the same way we find that \( \xi(t^{+}) \) exists.

If \( \xi(t^{+}) = \xi(t^{-}) \) (L^p)

then \( t \) is called a L^p-continuity point otherwise it is called a L^p-discontinuity point. We have

**Theorem 5.** \( \xi \in V^p \) has at most an enumerable set of L^p-discontinuity points.

**Proof:** We may consider real random functions according to Theorem 3. It is sufficient to consider a real a.s. nondecreasing random function \( \xi \in V^p \).

If then \( t_v, \nu = 1, 2, \ldots, n \) are different points on the real line, we certainly have (cf. Theorem 2.4)

\[
||\xi(t^{+})||_p^p \geq \left| \sum_{\nu=1}^{n} [\xi(t_v^{+}) - \xi(t_v^{-})] \right|_p^p
\]

\[
\geq \sum_{\nu=1}^{n} ||\xi(t_v^{+}) - \xi(t_v^{-})||_p^p
\]

Hence there are at most finitely many L^p-discontinuity points such that

\[
2^{r-1} \leq ||\xi(t^{+}) - \xi(t^{-})||_p < 2^r
\]

for any integer \( r \) and therefore the theorem holds.

The enumerable set of all L^p-discontinuity points of \( \xi \in V^p \) is called the L^p-discontinuity lattice of \( \xi \) (cf. LTC, p. 19-20).

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**5. A representation formula**

We shall now consider the subclass \( \overline{V}^p \) of \( V^p \) consisting of all \( \xi \in V^p \) such that \( \xi(t) = \frac{1}{2}[\xi(t^{+}) + \xi(t^{-})] \) (L^p)

A random function satisfying this relation is called L^p-mean continuous.

**Theorem 1.** The total variation \( \eta \) of \( \xi \in V^p \) is L^p-continuous at \( t_0 \) if \( \xi \) is L^p-continuous at that point. If \( \xi \in \overline{V}^p \) then \( \eta \in \overline{V}^p \).
Corollary: A real random function belonging to $\mathcal{V}^p$ has a representation

$$\xi = \eta - \zeta$$

where $\eta$ and $\zeta$ belong to $\mathcal{V}^p$ and are a.s. nondecreasing and $\eta$ and $\zeta$ are $L^p$-continuous at every point where $\xi$ is $L^p$-continuous.

Proof: We observe that

$$\eta(t^+) = \eta(t) + |\xi(t^+) - \xi(t)| \quad (L^p)$$
$$\eta(t^-) = \eta(t) - |\xi(t) - \xi(t^-)| \quad (L^p)$$

In fact, if $a$ and $b$ are any finite points on the real line such that $a < b$

$$\eta(b) = \eta(a) + \eta(b) - \eta(a) \geq \eta(a) + |\xi(b) - \xi(a)| \quad \text{a.s.}$$

and thus, letting $a \to b$ we find by Theorem 2.6 and Theorem 2.6

$$\eta(b) \geq \eta(b^-) + |\xi(b) - \xi(b^-)| \quad \text{a.s.} \quad (1)$$

On the other hand $\eta(b) - \eta(a)$ is the $L^p$-limit of sums

$$\sum_{v=1}^{n} |\xi(t_v) - \xi(t_{v-1})|$$

for a directed set of nets

$$N : a = t_0 < t_1 < \ldots < t_n = b$$

and hence

$$|\xi(b) - \xi(a)| = L^p\lim_{n \to \infty} \sum_{v=1}^{n} |\xi(t_v) - \xi(t_{v-1})|$$

$$\leq L^p\lim_{n \to \infty} \int_{a}^{t_n} |d\xi(t)| + |\xi(b) - \xi(t_{n-1})| \quad \text{a.s.}$$

As $N$ passes through a refinement of nets $T_{n-1} \to b$ and thus we obtain, applying again the theorems 2.6 and 2.8

$$\eta(b) - \eta(a) \leq \int_{c}^{b} |d\xi(t)| + |\xi(b) - \xi(b^-)| \quad \text{a.s.}$$

If we here let a tend to $b$ and observe the theorems just mentioned we find that

$$\eta(b) - \eta(b^-) \leq |\xi(b) - \xi(b^-)| \quad \text{a.s.} \quad (2)$$
Hence, combining (1) and (2) we get \( \eta(b) - \eta(b-) = |\xi(b) - \xi(b-)| \text{ a.s.} \)

In the same we find that \( \eta(b+) - \eta(b) = |\xi(b+) - \xi(b)| \text{ a.s.} \). Using these relations we get the statements. The corollary follows from this theorem and Theorem 4.3.

We shall now give a representation of \( \xi \in \overline{V}^p \) which corresponds to the representation of a function of bounded variation (cf. LTC, Theorem 1.5). For that reason we introduce the unit distribution function \( c \),

\[
c(t) = \begin{cases} 
0 & \text{for } t < 0 \\
1/2 & \text{for } t = 0 \\
1 & \text{for } t > 0 
\end{cases}
\]

and put \( c^{(c)}(t) = c(t-c) \). Then we can state

**Theorem 2.** A real random function belonging to \( \overline{V}^p \) has a representation

\[
\xi = \xi_\infty + \sum_{c \in \Lambda(\xi)} c^{(c)}
\]

where \( \xi_\infty \in \overline{V}^p \) and \( \xi_\infty \) is \( L^p \)-discontinuity lattice of \( \xi \), and

\[
\alpha_c = \xi(c+) - \xi(c-).
\]

The series is uniformly convergent in respect to the \( L^p \)-norm. If furthermore \( \xi \) is a.s. nondecreasing, then \( \xi_\infty \) is a.s. nondecreasing and \( \alpha_c \geq 0 \) a.s.

**Proof:** According to the corollary of Theorem 1 it is sufficient to consider the case when \( \xi \) is a.s. nondecreasing. Then by appropriate use of Theorem 2.8 and Theorem 2.6 we can proceed in the same way as in LTC, Section 1.5.
6. $L^p$-Riemann-Stieltjes Integrals

The $L^p$-Riemann-Stieltjes integral for random functions will be defined in the same way as the RS-integral for functions (cf. LTC p. 26). Consider two random functions $\xi$ and $\eta$ on an interval $[a, b]$. Here $a = -\infty$ is permitted. Let

$$
\Pi : \ c = \tau_0 < \tau_1 < \ldots < \tau_n = b
$$

be a net fitted on $[a, b]$ and $\tau_k$ any point on the open mesh $(\tau_{k-1}, \tau_k)$. Then to $\Pi$ there corresponds the RS-sum

$$
\sigma^{(\Pi)}(\xi, \eta) = \sum_{k=1}^{n} \xi(\tau_k) [\eta(\tau_k) - \eta(\tau_{k-1})]
$$

A random variable $\sigma$ is called the $L^p$-RS-integral of $\xi$ in respect to $\eta$ on $[a, b]$ and is denoted by

$$
\sigma = \int_a^b \xi(t) \, d\eta(t)
$$

if there to any positive number $\varepsilon$ belongs at least one net $\Pi_{\varepsilon}$ such that

$$
||\sigma - \sigma^{(\Pi)}(\xi, \eta)||_p < \varepsilon
$$

for $\Pi \succ \Pi_{\varepsilon}$. It may be observed that this inequality should be satisfied independently of the assigned points $\tau_k$ on the open meshes. However, besides this $L^p$-RS-integral we shall consider two special $L^p$-RS-integrals called left- and right-$L^p$-RS-integrals respectively. The former one is defined in the same way as the $L^p$-RS-integral above but in the RS-sums we change $\xi(\tau_k)$ into the $L^p$-limit $\xi(\tau_{k-1}^+)$, which is assumed to exist. In the definition of the right-$L^p$-RS-integral we change $\xi(\tau_k)$ into $\xi(\tau_k^+)$. Clearly the left- and right-$L^p$-RS-integrals exist if the general $L^p$-RS-integral exists, and then are equal to the latter one.

We shall use the left- and right-$L^p$-RS-integrals in order to define generalized convolutions. Now we shall give some existence theorems.

---

1. Stochastic integrals as limits in probability of sums of the form (2) have been studied by K. Yto (3).
Theorem 1. If \( \xi \) is almost strongly \( L^q \)-continuous on \((a,b)\) and \( \eta \) is of \( L^q \)-bounded variation on \([a,b]\), where \( 1/q_1 + 1/q_2 = 1/p \leq 1 \), \( q_1 \geq 1 \), \( q_2 \geq 1 \), then \( \xi \) has a \( L^p \)-RS-integral on \([a,b]\) in respect to \( \eta \).

Proof: To a given positive number \( \varepsilon \) we choose a net \( \mathcal{N}_c \) such that the \( L^q \)-oscillation of \( \xi \) is smaller than \( \varepsilon / \alpha \) on every mesh of \( \mathcal{N}_c \), where

\[
\alpha = \int_a^b \left| \frac{d\eta(t)}{dt} \right|^p dt
\]

(We may assume that \( \alpha \neq 0 \).) Then for a refinement \( \mathcal{N} \) of \( \mathcal{N}_c \) let \( \sigma^{(N)}(\xi, \eta) \) and \( \sigma^{(E)}(\xi, \eta) \) be the corresponding RS-sums. Then it is easily seen that we can write

\[
\sigma^{(N)}_{(c)}(\xi, \eta) - \sigma^{(E)}(\xi, \eta) = \sum_{k=1}^{n} \left| \xi(u_k) - \xi(t_k) \right| \left| \eta(t_k) - \eta(t_{k-1}) \right| \quad (4)
\]

where \([t_{k-1}, t_k] \), \( k = 1, 2, \ldots, n \) are the meshes of \( \mathcal{N} \), \( t_k \in (t_{k-1}, t_k) \) and \( u_k \) is a point on the interior of that mesh of \( \mathcal{N} \) to which \([t_{k-1}, t_k] \) belongs. Hence applying Hölder's inequality (Theorem 2.1) we get

\[
\left| \left| \sigma^{(N)}_{(c)}(\xi, \eta) - \sigma^{(N)}(\xi, \eta) \right| \right|_p \leq \sum_{k=1}^{n} \left| \xi(u_k) - \xi(t_k) \right|_q \left| \eta(t_k) - \eta(t_{k-1}) \right|_q \quad (5)
\]

and

\[
\left| \left| \sigma^{(N)}_{(c)}(\xi, \eta) - \sigma^{(E)}(\xi, \eta) \right| \right|_p \leq \varepsilon
\]

for \( \mathcal{N} \succ \mathcal{N}_c \). Since then the family \( \{ \sigma^{(N)}(\xi, \eta) \} \) is mutually convergent and the family is directed in respect to refinements of nets. The existence of the integral follows from Theorem 2.7.

We observe that this theorem remains true if \( q_1 = +\infty \) or \( q_2 = +\infty \) and this particularly if \( \xi \) or \( \eta \) is a function (not random) on \((+\infty, +\infty)\) and if it satisfies the required conditions. However the conditions may be weakened in these cases as follows.
Theorem 2. If $\xi$ is an almost strongly $L^\infty$-continuous random function on $(a, b)$ and $\eta$ is a random function of bounded variation in respect to the $L^p$-norm then $\xi$ has a $L^p$-RS-integral in respect to $\eta$ on $[a, b]$.

Theorem 3. If $\xi$ is an almost strongly $L^p$-continuous random function on $(a, b)$ and $\eta$ is a function of bounded variation on $[a, b]$ then $\xi$ has a $L^p$-RS-integral in respect to $\eta$ on $[a, b]$.

Remark: If $\xi$ is a random function of bounded variation in respect to the $L^p$-norm and $\eta$ is of bounded variation and continuous on $[a, b]$, then in the RS-sums which define the $L^p$-RS-integral we may choose the assigned point on any mesh as any point on this closed mesh.

Proof of Theorem 2: Observing that the $L^\infty$-norm is the essential supremum we now get from (4)

$$||\xi^{(\mathcal{N}_c)}(\xi, \eta) - \sigma^{(\mathcal{N})}(\xi, \eta)||_p \leq ||\sum_{k=1}^{n} ||\xi(t_k) - \eta(t_{k-1})||_{L^\infty}||\eta(t_k) - \eta(t_{k-1})||_p$$

Hence, if we let $\alpha$ be the $L^p$-norm of the total variation of $\eta$ in respect to the $L^p$-norm on $[a, b]$ and choose $\mathcal{N}_c$ such that the $L^\infty$-oscillation of $\xi$ on every mesh of $\mathcal{N}_c$ is smaller than $\epsilon/\alpha$, we get the inequality (5) also in this case and can proceed as in the proof of Theorem 1.

Proof of Theorem 3: Choose $\mathcal{N}$ such that the $L^p$-oscillation of $\xi$ is smaller than $\epsilon/\alpha$ where $\alpha$ is the total variation of $\eta$ on $[a, b]$. Then again we get the inequality (5).

In order to show the statement of the remark we have only to prove that we may choose $t_{i-1}$ and $t_i$ as assigned points of the mesh $[t_{i-1}, t_i]$ for any $i$. According to the definition of the RS-sum we may choose a net $\mathcal{N}_c$ such that $\sigma^{(\mathcal{N}_c)}(\xi, \eta)$ approximates the $L^p$-RS-integral of $\xi$ in respect to $\eta$ on $[a, b]$ with an error in the $L^p$-norm which is smaller than $\epsilon$. Then let $\mathcal{N}$ be any refinement of $\mathcal{N}_c$ and form $\sigma^{(\mathcal{N})}(\xi, \eta)$ as before
but permit the assigned point of a mesh to be any point on this closed mesh. Since $\xi$ is of bounded variation in respect to the $L^p$-norm it is almost strongly $L^p$-continuous and hence there are at most finitely many points $t$ such that $||\xi(t+) - \xi(t-)||_p > \frac{\epsilon}{2}$.

Hence we can choose $N_\epsilon$ such that the $L^p$-oscillation of $\xi$ is smaller than $\epsilon$ on every closed mesh of $\Pi_\epsilon$ except at most finitely many. Further we may choose $N_\epsilon$ such that the total variation of $\eta$ is smaller than $\frac{1}{2}\epsilon$ on the set of all those closed meshes, such that the $L^p$-oscillation of $\xi$ on the mesh is larger than $\epsilon$. We also choose $N_\epsilon$ such that the oscillation of $\eta$ on every closed mesh is smaller than $\epsilon$. Then regarding (4) we get

$$||\sigma^H(\xi, \eta) - \sigma^H(\xi(t)\eta)||_p \leq c[\sup |\xi(t)||_p + \int_0^b |d\eta(t)|]$$

for $H > H_\epsilon$. Since $\epsilon$ is arbitrary the statement follows. We again get the inequality (5) and can then proceed as before.

The existence theorems stated above are not sufficient to guarantee the existence of the $L^p$-RS-integral of $\xi$ and $\eta$ if both are of bounded variation in respect to norms. We shall show that the right- and left-$L^p$-RS-integrals exist in this case according to

**Theorem 4.** If $\xi$ is of bounded variation in respect to the $L^{q_1}$-norm and $\eta$ is of bounded variation in respect to the $L^{q_2}$-norm on $[a,b]$, where $1/p = 1/q_1 + 1/q_2 \leq 1$, $q_1 \geq 1$, $q_2 \geq 1$, then $\xi$ has a left-$L^p$-RS-integral and a right-$L^p$-RS-integral in respect to $\eta$ on $[a,b]$.

**Proof:** We may assume that $\xi$ and $\eta$ are real according to Theorem 4.3.

It is sufficient to consider that case when $\xi$ and $\eta$ are a.s. nondecreasing.

Then the RS-sums
\[ \sigma^{(N)}(\xi, \eta) = \sum_{i=1}^{n} \xi(t_{i-1}^+) [\eta(t_i) - \eta(t_{i-1})] \]

obviously form a family which is a.s. nondecreasing in the direction \( \xi \) in respect to \( \eta \). Further
\[ \sigma^{(N)}(\xi, \eta) \leq \xi(b) \sum_{i=1}^{n} [\eta(t_i) - \eta(t_{i-1})] = \xi(b) [\eta(b) - \eta(a)] \text{ a.s.} \]

and thus
\[ ||\sigma^{(N)}(\xi, \eta)||_p \leq ||\xi(b)||_q \bigg| \eta(b) - \eta(a) \bigg|_q \]

According to Theorem 2.8, the family \( \{\sigma^{(N)}(\xi, \eta)\} \) is then \( L^P \)-convergent in the direction \( \xi \) in respect to \( \eta \) and the limit is obviously the left-\( L^P \)-RS-integral of \( \xi \) in respect to \( \eta \) on \( [a, b] \). In the same way we find that \( \xi \) has a right-\( L^P \)-RS-integral in respect to \( \eta \) on \( [a, b] \).

The \( L^P \)-RS-integrals (even such left- and right-integrals) have the following properties, which correspond to properties of usual RS-integrals.

1° \[ b \int_a^c \xi(t)d\eta(t) + \int_b^c \xi(t)d\eta(t) = \int_a^b \xi(t)d\eta(t) \quad (L^P) \]

for \( a < b < c \), provided that the integrals on the left side are defined.

2° It is a bilinear functional
\[ b \int_a^b (\xi_1(t) + \xi_2(t))d\eta(t) = \int_a^b \xi_1(t)d\eta(t) + \int_a^b \xi_2(t)d\eta(t), \]
\[ b \int_a^b \xi(t)d\eta_1(t) + \int_a^b \xi(t)d\eta_2(t) = \int_a^b \xi(t)d\eta_1(t) + \int_a^b \xi(t)d\eta_2(t) \quad (L^P) \]

provided the integrals in such a relation exist. (cf. LTC, I 2.1 and I 2.3.)

Further \( L^P \)-RS-integrals on closed intervals are extended to corresponding integrals on open and semi-open intervals in the same way as RS-integrals (cf. LTC I2.2).
The $L^p$-RS-integral can be given as a $L^p$-R-integral ($L^p$-Riemann-integral) in the following case.

**Theorem 5.** Let $\xi$ be almost strongly $L^{q_1}$-continuous and $\zeta$ almost strongly $L^{q_2}$-continuous on the finite interval $(a,b)$ where

\[ \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} \leq 1, \quad q_1 \geq 1, \quad q_2 \geq 1. \]

Then

\[ \eta(t) = \int_a^t \xi(\tau) d\tau \quad (L^p) \]

is defined for $a \leq t \leq b$ and

\[ \int_a^b \xi(t) d\eta(t) = \int_a^b \xi(t) \zeta(t) dt. \quad (6) \]

**Proof:** $\eta$ is defined on $[a, b]$ according to Theorem 3 and then it easily follows that $\eta$ is of $L^{q_2}$-bounded variation on $[a, b]$ since $\xi$ is $L^{q_2}$-bounded on $[a, b]$ when it is almost strongly uniformly $L^{q_2}$-continuous.

Then applying again Theorem 3 we conclude that the integral on the left side of (6) exists. In the same way we find that the integral on the right side exists, if we notice that $\xi\zeta$ is almost strongly uniformly $L^p$-continuous on $[a, b]$. Only using the definition of these integrals we now conclude that we can approximate the two integrals arbitrarily closely in respect to the $L^p$-norm by the same RS-sum

\[ \sum_{k=1}^{n} \xi(\tau_k) [\eta(t_k) - \eta(t_{k-1})] = \sum_{k=1}^{n} \xi(\tau_k) \zeta(\tau_k)(t_k - t_{k-1}) \]

where

\[ \tau_k \in (t_{k-1}, t_k). \]

So far we have only considered $L^p$-RS-integrals on closed intervals.
The $L^p$-RS-integral on an open or semi-open interval (for instance $(\infty, +\infty)$) is defined as the $L^p$-limit of corresponding $L^p$-RS-integrals on closed intervals which belong to the given interval and tend to it, hence in the same way as RS-integrals on open and semi-open intervals.

7. Limits of $L^P$-RS-integrals.

A sequence $(\eta_n)$ of random functions is said to converge $L^P$-weakly to a random function $\eta$ on an open, closed or semi-closed interval if

$||\eta_n - \eta||_p$ tends to 0 on the interval, i.e. if this norm tends weakly to 0 at all points with the exception at most an enumerable set of points (cf. LTC, Section 2.7). It is said to converge $L^P$-completely to $\eta$, if $||\eta_n - \eta||_p$ tends completely to 0, which means that this norm tends to 0 $L^P$-weakly and at an endpoint of the interval, where it is closed and at the limiting endpoint where it is open. Further it is required that the norm tends to 0 at $-\infty$ if the lower endpoint of the interval is $\infty$ (cf. LTC loc. cit.).

The following generalization of a Theorem of Felly follows exactly in the same way as in LTC, Section 2.7.

Theorem 1. Let $\xi$ be a bounded and uniformly continuous function on $(-\infty, +\infty)$ and $(\eta_n)$ a sequence of random functions of bounded variation in respect to the $L^p$-norm on the finite interval $[a, b]$ and tending $L^P$-completely to a random function $\eta$ of bounded variation in respect to the $L^p$-norm on $[a, b]$. Further suppose that

$$\limsup_{n \to +\infty} \frac{b}{a} \int_a^b |\eta_n|_p < +\infty.$$
Then
\[ \int_a^b \xi(t+\tau)d\eta_n(\tau) \to \int_a^b \xi(t+\tau)d\eta(\tau) \quad (L^p) \]
uniformly in respect to \( t \in (-\infty, +\infty) \).

Corollary: Let \( \delta \) be any interval, not containing the extended point \( \infty \) and let a function which is continuous on \((-\infty, +\infty)\) and further suppose that \( (\eta_n) \) is a sequence of a.s. non-negative and a.s. non-decreasing random functions, tending \( L^p \)-completely to a random function which is \( L^p \)-bounded.

Then
\[ \int_{\delta} \xi(t+\tau)d\eta_n(\tau) \to \int_{\delta} \xi(t+\tau)d\eta(\tau) \quad (L^p) \]
uniformly in respect to \( t \in (-\infty, +\infty) \).

Remark 1: The corollary holds for any interval \( \delta \), if furthermore \( \xi(-\infty) = 0 \).

Remark 2: If \( \xi \) is a mean-continuous function of bounded variation on \([-\infty, +\infty)\] and \( (\eta_n) \) is a sequence of a.s. non-decreasing random functions of bounded variation in respect to the \( L^p \)-norm tending completely and at all points of \((-\infty, +\infty)\) to an a.s. non-decreasing random function \( \xi \) of bounded variation in respect to the \( L^p \)-norm, then
\[ \int_{\delta} \xi(t)d\eta_n(t) \to \int_{\delta} \xi(t)d\eta(t) \]
for any interval \( \delta \).

Remark 3: The Theorem and its corollaries remain true if \( \xi \) is a random function which is uniformly \( L^{q_1} \)-continuous provided that \( (\eta_n) \) and \( \eta \) are random functions of \( L^{q_2} \)-bounded variation, where
\[ q_1 \geq 1, \quad q_2 \geq 1, \quad 1/p = 1/q_1 + 1/q_2 \leq 1 \]
and the total \( L^p \)-variation of \( \eta_n \) is uniformly bounded in respect to \( n \).

\[ ^1 \text{In LTC, Section 2, it is incorrectly assumed only that } \xi \text{ is uniformly continuous on the largest open interval containing } \delta. \]
function which we denote by $\xi \ast \eta$ ($L^p$) if we want to point out the particular space. When (1) exists as a right-$L^p$-integral we call the corresponding function $\eta$ the right-$L^p$-convolution of $\xi$ and $\eta$ and denote it by $\xi \ast \eta$. In the same way we define the left-$L^p$-convolution $\xi \ast \eta$.

Theorem 1. If $\xi \in V_1^q$, $\eta \in V_2^q$ where $q_1 \geq 1$, $q_2 \geq 1$, $1/p = 1/q_1 + 1/q_2 \leq 1$, then $\xi \ast \eta$ and $\xi \ast \eta$ exist in $L^p$ and belong to $V^p$ and they are equal to $\xi \ast \eta$ if this convolution exists in $L^p$.

Proof: It is sufficient to consider the case when $\xi$ and $\eta$ are a.s. nondecreasing (cf. Theorem 4.3). The left-RS-integral

$$\int_a^\infty \xi(t-\tau)d_1\eta(\tau)$$

exists for every number $a$ (Theorem 6.4).

This integral is an a.s. nondecreasing function in respect to $a$ and a.s. not larger than $\xi(\tau)\eta(\tau)$, where

$$||\xi(\tau)\eta(\tau)||_p \leq ||\xi(\tau)||_q_1||\eta(\tau)||_q_2$$

(cf. Theorem 2.6). By Theorem 2.8 we now find that the $L^p$-RS-integral (2) tends to a $L^p$-limit as $a \to +\infty$.

Clearly $\xi \ast \eta$ is of bounded variation in respect to the $L^p$-norm since it is a.s. nondecreasing. In the same way we find that $\xi \ast \eta$ exists.

If $\xi \ast \eta$ exists then of course we have $\xi \ast \eta = \xi \ast \eta = \xi \ast \eta$.

We shall now show some Theorems for left- and right-$L^p$-convolutions which are also true for convolutions in the narrow sense. Then we shall require that $\xi \in V_1^q$ and $\eta \in V_2^q$ ($V^q$ is the class of $L^q$-mean-continuous random functions belonging to $V^q$). We state these theorems only for
right-$L^p$-convolutions of random functions, belonging to $\overline{V}$, (cf. Section 5).

Theorem 2. If $\xi \in \overline{V}$ then $c \ast \xi = \xi \ast c = \xi$ \quad (L$^p$)

Proof: Follows from the definition of the $L^p$-convolution.

Theorem 3. If $\xi \in \overline{V}^q_1$, $\eta \in \overline{V}^q_2$, $q_1 \geq 1$, $q_2 \geq 1$, $1/q_1 + 1/q_2 = 1/p \leq 1$ then

$$\xi(\cdot + c) \ast \eta(t) = \xi \ast \eta(\cdot + c)(t) = \xi \ast \eta(t + c)$$

for any number $c$.

Proof: Follows from the definition of the right-$L^p$-convolution.

Theorem 4. Let $\xi \in \overline{V}^q_1$, $\eta \in \overline{V}^q_2$ where $q_1 \geq 1$, $q_2 \geq 1$, $1/p = 1/q_1 + 1/q_2 \leq 1$ and let

$$\xi = \xi_{\infty} + \sum_{\gamma \in \Lambda(\xi)} \alpha_{\gamma} e(\gamma - c_{\gamma})$$

$$\eta = \eta_{\infty} + \sum_{\gamma \in \Lambda(\eta)} \beta_{\gamma} c(\gamma - d_{\gamma})$$

where $\xi_{\infty} \in \overline{V}^q_1$ and $\eta_{\infty} \in \overline{V}^q_2$ are random functions which are $L^q_1$-continuous and $L^q_2$-continuous respectively and $\alpha_{\gamma}$ and $\beta_{\gamma}$ are random variables belonging to $L^q_1$ and $L^q_2$ respectively. Then we have

$$\xi \ast \eta = \xi_{\infty} \ast \eta_{\infty} + \sum_{\gamma \in \Lambda(\xi)} \beta_{\gamma} \xi_{\infty} e(\gamma - d_{\gamma}) + \sum_{\gamma \in \Lambda(\eta)} \alpha_{\gamma} \eta_{\infty} e(\gamma - c_{\gamma}) + \sum_{\gamma \in \Lambda(\xi)} \alpha_{\gamma} \beta_{\gamma} e(\gamma - c_{\gamma}, d_{\gamma}) \quad (L^p) \quad (5)$$

Proof: It is sufficient to consider that case when $\xi$ and $\eta$ are real and a.s. nondecreasing on $[\infty, +\infty]$. Then $\alpha_{\gamma} \geq 0$ a.s. and $\beta_{\gamma} \geq 0$ a.s.

Hence, putting

$$\xi^{(n)} = \xi_{\infty} + \sum_{\gamma = 1}^{n} \alpha_{\gamma} e(\gamma - c_{\gamma})$$

we observe that $\xi - \xi^{(n)} \geq 0$ a.s. and thus

$$0 \leq (\xi - \xi^{(n)}) \ast \eta \leq \sup[\xi^{(n)}(\infty) - \xi^{(n)}(\infty)] \eta(\infty) \quad a.s.
Hence we get
\[
| | (x - x^{(n)}) \ast \eta | |_{p} \leq | | (x^{(+\infty)} - x^{(n)}_{\infty}) | |_{q_{1}} | | \eta^{(+\infty)} | |_{q_{2}}
\]
\[
\leq | | \sum_{v=n+1}^{\infty} \alpha_{v} | |_{q_{1}} | | \eta^{(+\infty)} | |_{q_{2}}
\]  
(6)
and the last quantity tends to 0 as \( n \to +\infty \), since \( \sum \alpha_{v} \) is convergent in respect to the \( L^{p} \)-norm. Similarly we obtain, putting
\[
\eta^{(n)} = \eta^{(+\infty)} + \sum_{v=1}^{n} \alpha_{v} \cdot e(-c_{v})
\]
and observing that \( \eta - \eta^{(n)} \) is an a.s. nondecreasing random function,
\[
0 \leq (x \ast (\eta - \eta^{(n)})) \leq (x^{(+\infty)} \cdot [\eta^{(+\infty)} - \eta^{(n)}]) \text{ a.s. and thus}
\]
\[
| | (x \ast (\eta - \eta^{(n)})) | |_{p} \leq | | (x^{(+\infty)} | |_{q_{1}} | | \sum_{v=n+1}^{\infty} \beta_{v} | |_{q_{2}}
\]
(7)
By (6) and (7) and the theorems 2 and 3 and 4 we obtain
\[
\xi \ast \eta = \xi^{\ast \infty} \ast \eta + \sum_{v=1}^{\infty} \alpha_{v} \cdot \eta(-c_{v}) \quad (L^{p})
\]
(8)
\[
\xi \ast \eta = \xi^{\ast \infty} \ast \eta_{\infty} + \sum_{v=1}^{\infty} \beta_{v} \cdot \xi(-d_{v}) \quad (L^{p})
\]
These relations obviously remain true for any \( \xi \) and \( \eta \) satisfying the conditions of the theorem and combining them we get the relation (5).

According to Theorem 4 the convolution \( \xi \ast \eta \) is essentially reduced to the convolution \( \xi_{\infty} \ast \eta_{\infty} \), where \( \xi_{\infty} \) and \( \eta_{\infty} \) are \( L^{1} \)-continuous and \( L^{q_{2}} \)-continuous respectively.

**Theorem 5.** If \( \xi \in l_{q_{1}}^{1}, \eta \in l_{q_{2}}^{2} \), where \( q_{1} \geq 1, q_{2} \geq 1, 1/q = 1/q_{1} + 1/q_{2} \leq 1 \), then
\[
\xi \ast \eta = \eta \ast \xi \quad (L^{p})
\]
Proof: According to Theorem 4 it is sufficient to consider that case when $\xi$ and $\eta$ are $L^{q_1}$- and $L^{q_2}$-continuous respectively. Then we can approximate $\xi \ast \eta(t)$ by a RS-sum

$$
\sum_{i=1}^{n} \xi(t - t_i)[\eta(t_i) - \eta(t_{i-1})]
$$

(8)

where $t_0 = \infty$, $t_1 < t_2 < \ldots < t_n$. By an Abelian transformation we can write this sum

$$
\sum_{i=1}^{n} \eta(t_i)[\xi(t - t_i) - \xi(t - t_{i+1})]
$$

where $t_{n+1} = \infty$ (hence $t - t_{n+1} = \infty$). Putting $t - t_i = \tau_i$ we can write the sum

$$
\sum_{i=1}^{n} \eta(t - \tau_i)[\xi(\tau_i) - \xi(\tau_{i+1})],
$$

(9)

where $\tau_n < \tau_{n-1} < \tau_{n-2} < \ldots < \tau_1 < \tau_{n+1} = \infty$.

Hence the last sum is a RS-sum also approximating $\eta \ast \xi$. Observing the definition of the right-$L^p$-RS-integral we easily find that the net may be chosen such that the RS-sum (8) (which is equal to (9) ) approximates both $\xi \ast \eta$ and $\eta \ast \xi$ arbitrarily closely in respect to the $L^p$-norm. Thus the theorem follows.

**Theorem 6.** Let $\xi \in V_1$, $\eta \in V_2$ where $q_1 \geq 1$, $q_2 \geq 1$, $1/p = 1/q_1 + 1/q_2 \leq 1$ and let $\xi$ be $L^1$-continuous at every point. Then $\xi \ast \eta$ is $L^p$-continuous to the right and $\xi \ast \eta$ is $L^p$-continuous to the left.

Proof: According to Theorem 4.3 it is sufficient to consider a.s. non-decreasing $\xi$ and $\eta$. Further, observing that

$$
\int_{-\infty}^{+\infty} \xi(t) d_x \eta(t) = \eta(-\infty) \xi(+\infty) + \int_{-\infty}^{+\infty} \xi(t) d_x \eta(t)
$$
and that the right-$L^p$-RS-integral on $(-\infty, +\infty)$ can be approximated arbitrarily closely and uniformly in respect to the $L^p$-norm by a corresponding integral on a finite interval we conclude that it is sufficient to consider the latter one. Now let $t$ be a fixed point. According to the definition of the right-$L^p$-RS-integral we can determine a net

$$N_c : a = t_0 < t_1 < t_2 < \ldots < t_n = b$$

to a given positive number $c$ such that $t_{i+1} - t_i < c$ and

$$0 \leq || \int_a^b \xi(t - \tau) d\eta(\tau) ||_p^p - || \sum_{i=1}^n \xi(t_i - t) [\eta(t_i) - \eta(t_{i-1})] ||_p^p < \frac{1}{2}c^p \quad (10)$$

We observe that the RS-sum is a.s. not larger than the integral since the RS-sums are a.s. nondecreasing in the direction of refinements of nets. Now $\xi$ is $L^p$-continuous at all points and hence we can determine $h > 0$ such that

$$|| \sum_{i=1}^n \xi(t_i - t) [\eta(t_i) - \eta(t_{i-1})] ||_p^p - || \sum_{i=1}^n \xi(t-h-t_i) [\eta(t_i) - \eta(t_{i-1})] ||_p^p \frac{1}{2}c^p \quad (11)$$

But the second sum is a RS-sum belonging to the right $L^p$-RS-integral on the interval $[a, b]$ and hence

$$|| \int_a^b \xi(t-h - \tau) d\eta(\tau) ||_p^p \geq || \sum_{i=1}^n \xi(t-h-t_i) [\eta(t_i) - \eta(t_{i-1})] ||_p^p \quad (12)$$

Combining (10), (11) and (12) we obtain
\[ \left\| \int_a^b \xi(t-\tau) \eta(\tau) \, d\tau \right\|_p \geq \left\| \int_a^b \xi(t-\tau-h) \eta(\tau) \, d\tau \right\|_p \]

\[ \geq \left\| \sum_{i=1}^n \xi(t-h-t_i) \eta(t_i) - \eta(t_{i-1}) \right\|_p \geq \left\| \sum_{i=1}^n \xi(t-t_i) \eta(t_i) - \eta(t_{i-1}) \right\|_p \]

\[ - \frac{1}{2} c^p \geq \left\| \int_a^b \xi(t-\tau) \, d\eta(\tau) \right\|_p - \frac{1}{2} c^p \]

Applying Theorem 2.4 we then get

\[ \left\| \int_a^b \xi(t-\tau) \eta(\tau) - \int_a^b \xi(t-\tau-h) \eta(\tau) \, d\eta(\tau) \right\|_p \leq c \]

Here \( c \) may be chosen arbitrarily and thus we conclude that the right-L^p-convolution of \( \xi \) and \( \eta \) is L^p-continuous to the left at any point.

According to the remarks given above the same holds true for the right-L^p-convolution on \([a, +\infty]\). In the same way we find that \( \xi * \eta \) is \( L^p \)-continuous to the right.

So far we have not shown that \( \xi * \eta \) and \( \xi * \eta \) can be unequal.

We shall give an example of such random functions \( \xi \) and \( \eta \) for which these convolutions are unequal.\(^1\)

Let \( \xi \) be the Markov random function on \(-\infty < t < +\infty\) defined by
\( \xi(t) = 0 \) for \( t < 0 \), \( P[\xi(t) = 0] = e^{-\lambda t} \) for \( t \geq 0 \), \( P[\xi(t) = 1] = 1 - e^{-\lambda t} \) for \( t \geq 0 \), \( P[\xi(t) = 1|\xi(s) = 1] = 1 \) for \( t > s \), \( P[\xi(t) = 0|\xi(s) = 0] = e^{-\lambda(t-s)} \) for \( t > s \), \( P[\xi(t) = 1|\xi(s) = 0] = 1 - e^{-\lambda(t-s)} \) for \( t > s \) where \( \lambda > 0 \).

To these transition probabilities and initial distribution there belongs a regular Markov random function [Loève [1] p. 570]. We have for \( t > s \)

\(^1\) I am indebted to Dr. F. Leysieffer for this example.
E[ξ(t) - ξ(s)]^2 = E[ξ(t)]^2 + E[ξ(s)]^2 - 2Eξ(t) ξ(s)

= 1 - e^{-λt} + 1 - e^{-λs} - 2(1 - e^{-λs}) = e^{-λs} - e^{-λt}

Hence ξ is L^2-continuous. Put η(t) = ξ(-t) and consider the L^k-
convolutions η * ξ(t) and η * ξ(t) at the point t = 0. An RS-sum belonging
to the left L^1-RS-integral

\[ \int_a^b η(-τ)dξ_τ ξ(τ) \]

has the form

\[ \sum_{j=1}^n ξ(t_{j-1})[ξ(t_j) - ξ(t_{j-1})] \]

and this sum takes the value 0 with the probability 1 - e^{-λb} since

ξ(t_j) = ξ(t_{j-1}) with probability 1 if ξ(t_{j-1}) = 1 and P[ξ(b) = 1] = 1 - e^{-λb}.

Hence

\[ η * ξ(-t) = \int_{-∞}^{+∞} η(-τ)dξ_τ ξ(τ) = 0 \text{ a.s.} \]

The RS-sum belonging to the corresponding right integral

\[ \sum_{j=1}^n ξ(t_j) [ξ(t_j) - ξ(t_{j-1})] \]

takes the value 1 with the probability 1 - e^{-λb}, since exactly one term in
the sum is 1 if ξ(t_j) = 1 for some j. Hence

\[ η * ξ(0) = \int_{-∞}^{+∞} η(-τ)dξ_τ ξ(τ) = 1 \text{ a.s.} \]

We further find in the same way that

\[ ξ * η(t) = \begin{cases} 0 & \text{for } t \geq 0 \text{ a.s.} \\ 1 & \text{for } t < 0 \text{ a.s.} \end{cases} \]

\[ ξ * η(t) = \begin{cases} 0 & \text{for } t > 0 \text{ a.s.} \\ 1 & \text{for } t \leq 0 \text{ a.s.} \end{cases} \]
Hence in accordance with the theory $\xi \ast \eta$ is $L^1$-continuous to the right and $\xi \ast \eta$ is $L^1$-continuous to the left. Further these convolutions only differ in one point. We shall show in the next section that $\xi \ast \eta$ and $\xi \ast \eta$ for $\xi \in V^{q_1}$, $\eta \in V^{q_2}$, $1/q_1 + 1/q_2 = 1/p \leq 1$, $q_1 \geq 1$, $q_2 \geq 1$, can at most differ a.s. in a countable set of points.

9. Some special convolutions.

We shall consider some cases where the $L^p$-convolutions exist and we shall then only deal with convolutions of a.s. nondecreasing random functions.

In that case it easily follows from the definition of the convolutions that $\xi \ast \eta$ exists if $\xi \ast \eta$ and $\xi \ast \eta$ exist and are equal and then

$$\xi \ast \eta = \xi \ast \eta = \xi \ast \eta$$  \hspace{1cm} (1)

We call a random function a.s. uniformly continuous on $(-\infty, +\infty)$ in respect to a random variable $\xi_0$ if there to any $c > 0$ belongs a number $h(c)$ such that

$$|\xi(t+h) - \xi(t)| \leq c \xi_0 \quad a.s. \hspace{1cm} (2)$$

for any $t \in (-\infty, +\infty)$ and any $h$ such that $|h| < h(c)$.

**Theorem 1.** Let $\xi \in V^{q_1}$ and $\eta \in V^{q_2}$ be a.s. nondecreasing, where $1/q_1 + 1/q_2 = 1/p \leq 1$. Further suppose that $\xi$ is a.s. uniformly continuous in respect to a random variable $\xi_0 \in L^{q_1}$. Then $\xi \ast \eta$ exists ($L^p$) and is a.s. uniformly continuous in respect to $\xi_0 \eta(+\infty)$. 
Corollary: \( \xi \star \eta \) exists (\( L^p \)) if \( \xi \) is a continuous distribution function and \( \eta \in V(\rho) \). Further \( \xi \star \eta \) is then a.s. uniformly continuous in respect to \( \eta(\pm \infty) \).

Proof: According to the definition of the left- and right-\( L^p \)-RS-integrals, the identity (1) surely holds if

\[
\left| \left| \int_{-a}^{a} \xi(t-\tau)d\eta_r(t) - \int_{-a}^{a} \xi(t-\tau)d\eta_l(t) \right| \right|_p = 0 \tag{3}
\]

for any positive number \( a \). The difference on the right side of (3) is the \( L^p \)-limit of differences of sequences of sums of the form

\[
\sum_{j=1}^{n} [\xi(t-t_{j-1}) - \xi(t-t_j)] [\eta(t_j) - \eta(t_{j-1})].
\]

(Since \( \xi \) is \( L^p \)-continuous we have \( \xi(t-) = \xi(t+) = \xi(t) \) \( (L^p) \)). However, when (2) holds this sum is a.s. not larger than \( c(\xi_0) \| \eta(\pm \infty) \|_p \) and thus the \( L^p \)-norm of the difference in (3) is at most equal to

\[
c \| \xi_0 \|_{q_1} \| \eta(\pm \infty) \|_{q_2}.
\]

Since \( c \) is arbitrary we conclude that (3) holds and thus also that (1) is true.

In order to show that \( \xi \star \eta \) is a.s. uniformly continuous in respect to \( \xi_0 \) \( \eta(\pm \infty) \) we observe that

\[
\xi \star \eta(t+h) - \xi \star \eta(t) = \int_{-\infty}^{\infty} [\xi(t+h-\tau) - \xi(t-\tau)] d\eta(\tau) \tag{L^p} \tag{4}
\]

and that this integral is the \( L^p \)-limit of a sequence of sums of the form

\[
\sum_{j=1}^{n} [\xi(t+h-t_j) - \xi(t-t_j)] [\eta(t_j) - \eta(t_{j-1})]
\]

and thus also the a.s. limit of a sequence of such sums (cf. Loève (1) p. 164). The absolute value of such a sum is a.s. not larger than

\[
c \| \xi_0 \| \| \eta(\pm \infty) \|_p \text{ when (1) is satisfied. Hence also the integral in (4) is a.s.}
\]
not larger than this quantity. Hence $\xi \ast \eta$ is a.s. uniformly continuous in respect to $\xi_0 \eta(+\infty)$.

**Lemma 1.** Let $\xi$ and $\eta$ satisfy the conditions in Theorem 1. Then to any positive numbers $a$ and $c$ there exists a net $N_c(a)$ fitted on $[-a,a]$ such that

$$|\int_{-a}^{a} \xi(t-\tau)d\eta(\tau) - \sigma^{(N)}[\xi(t-\cdot), \eta]| \leq c \xi_0 \eta(+\infty) \text{ a.s.} \tag{5}$$

for any refinement $N$ of $N_c(a)$, where $\sigma^{(N)}[\xi(t-\cdot), \eta]$ is an RS-sum belonging to the net $N$. The lemma also remains true if $\eta$ is a continuous distribution function and $\xi$ is any a.s. nondecreasing random function belonging to $V^P$.

**Proof:** The integral is the $L^P$-limit of a sequence of RS-sums belonging to nets $N'$. Hence the difference on the left-hand side of (5) is the a.s. limit of sums of the form

$$\sum_{j=1}^{n'} [\xi(t-\tau_j) - \xi(t-\tau_j')] \left[\eta(t_j') - \eta(t_{j-1}')\right] \tag{6}$$

where $N'$ is a refinement of $N$. If $\xi$ is a.s. uniformly continuous in respect to $\xi_0$ we can choose $N_c(a)$ to given $c$ and a such that

$$|\xi(t-\tau_j) - \xi(t-\tau_j')| \leq c \xi_0 \text{ a.s.}$$

for $N' > N > N_c(a)$ and all pairs $\tau_j, \tau_j'$ and then the absolute value of the sum in (6) is a.s. not larger than $c \xi_0 \eta(+\infty)$. Then also (5) is satisfied.

If $\xi$ is any a.s. nondecreasing random function belonging to $V^P$ and $\eta$ is a continuous distribution function, then we proceed as follows.
According to Theorem 8.1 the convolutions $\xi \ast \eta$ and $\xi \ast \eta$ exist. Now for any $t$ and any net fitted on $[-a,a]$ we have by definition of the left- and right-integrals a.s.

$$\int_{-a}^{a} \xi(t-t_\tau) d\eta_{\tau} \leq \sum_{j=1}^{n} \xi(t-t_{j-1}) [\eta(t_j) - \eta(t_{j-1})] \quad a.s.$$  

and the sum on the right-hand side tends in the $L^p$-norm to the corresponding integral on the left-hand side as the net is infinitely refined. Hence this convergence also holds a.s. for a certain subsequence of refinements of nets. The difference between the sums is

$$\sum_{j=1}^{n} [\xi(t-t_{j-1}) - \xi(t-t_j)] [\eta(t_j) - \eta(t_{j-1})].$$

Since $\eta$ is a continuous distribution function we can choose nets on $[-a,a]$ such that $|\eta(t_j) - \eta(t_{j-1})| \leq \epsilon$ for any given positive number $\epsilon$ and then the last sum is a.s. at most equal to $\epsilon \xi_0$.

Lemma 2. If $\xi \in V_{q_1}$, $\eta \in V_{q_2}$, $\xi \in V_{q_3}$ with $1/q_1 + 1/q_2 + 1/q_3 = 1/p$, $q_i \geq 1$, $i = 1, 2, 3$, and if $\xi$ is a.s. uniformly continuous in respect to a random variable $\xi_0 \in L_{q_1}$, then

$$(\xi \ast \eta) \ast \xi = (\xi \ast \xi) \ast \eta \quad (L^p)$$

Proof: The convolutions exist according to Theorem 1. Obviously it is sufficient to prove this relation in that case when

$$\xi(-\infty) = \eta(-\infty) = \xi(-\infty) = 0 \quad a.s.$$
Put
\[
(\xi \ast \eta)_a(t) = \int_{-a}^a \xi(t-\tau) \, d\eta(\tau)
\]
\[
[(\xi \ast \zeta) \ast \eta]_a(t) = \int_{-a}^a \xi \ast \zeta(t-\tau) \, d\eta(\tau)
\]

According to Lemma 1 we can approximate the first integral a.s. and uniformly in respect to \( t \) by a sum
\[
\sum_{j=1}^n \xi(t-t_j)[\eta(t_j) - \eta(t_{j-1})]
\]
such that the error is a.s. not larger than \( \varepsilon \xi_0 \cdot \eta(\pm \infty) \) for any \( t \).

Then we get
\[
||(\xi \ast \eta)_a \ast \zeta(t) - \sum_{j=1}^n [\xi \ast \zeta(t-t_j)][\eta(t_j) - \eta(t_{j-1})]||_p \leq \varepsilon ||\xi_0||_q ||\eta(\pm \infty)||_q ||\zeta(\pm \infty)||_q
\]
(7)

Since by Theorem 1 the convolution \( \xi \ast \zeta \) is uniformly continuous in respect to \( \xi_0 \zeta(\pm \infty) \), using Lemma 1 we can also determine the net on \([-a,a]\) such that
\[
||(\xi \ast \xi \ast \eta)_a(t) - \sum_{j=1}^n \xi \ast \zeta(t-t_j)[\eta(t_j) - \eta(t_{j-1})]||_p \leq \varepsilon ||\xi_0||_q ||\eta(\pm \infty)||_q ||\zeta(\pm \infty)||_q
\]

Here \( \varepsilon \) is arbitrary. Hence
\[
(\xi \ast \eta)_a \ast \xi(t) = [(\xi \ast \xi) \ast \eta]_a(t)
\]
(\text{LP})

Letting \( a \to \pm \infty \), we get \( (\xi \ast \eta) \ast \xi = (\xi \ast \zeta) \ast \eta \) in that case when \( \eta(\pm \infty) = \xi(\pm \infty) = 0 \) and thus the theorem is proved.
Lemma 3. Let $\xi \in V_{q_1}^{q_1}$, $\eta \in V_{q_2}^{q_2}$ be a.s. nondecreasing where $1/q_1 + 1/q_2 = 1/p \leq 1$, $q_1 \geq 1$, $q_2 \geq 1$. Further let $G$ be a continuous distribution function. Then

$$(i) \quad (\xi * \eta) * G = (\xi * G) * \eta = (\xi * \eta) * G.$$ 

Proof: According to Lemma 1 we can approximate $(\xi * G)_{a}(t)$ a.s. and uniformly in respect to $t$ by an RS-sum.

$$\sum_{j=1}^{n} \xi(t-\tau_j)[G(t_j) - G(t_{j-1})]$$

with an error a.s. not larger than $c \xi^{(+\infty)}$ for any given positive number $c$. Then

$$\begin{align*} 
\| (\xi * G)_{a} * \eta(t) & - \sum_{j=1}^{n} \xi * \eta(t-\tau_j)[G(t_j) - G(t_{j-1})] \|_p \\
& \leq c \| \xi^{(+\infty)} \|_{q_1} \| \eta^{(+\infty)} \|_{q_2}. 
\end{align*}$$

(9)

However, by Lemma 1 we can also approximate $((\xi * \eta) * G)_{a}$ by this sum in the norm such that the error is arbitrarily small.

Hence

$$(\xi * G)_{a} * \eta(t) = [(\xi * \eta) * G]_{a} \quad (L^P)$$

Since this relation holds true for every $a > 0$ we get

$$(\xi * G) * \eta = (\xi * \eta) * G \quad (L^P)$$

By Theorem 1, however, we have

$$(\xi * G) * \eta = (\xi * G) * \eta. \quad (L^P)$$

Thus the first relation (i) follows, and the second relation is obtained in the same way.
Let $\xi \in L^{q_1}$, $\eta \in L^{q_2}$, where $1/q_1 + 1/q_2 = 1/p \leq 1$, $q_1 \geq 1$, $q_2 \geq 1$, and define the generalized $L^p$-convolution by

$$\xi \boxplus \eta = 1/2[\xi \otimes \eta + \xi \ast \eta].$$

Using the theorem and the lemmas in section 9 we shall now examine the connections between $\xi \otimes \eta$, $\xi \ast \eta$ and $\xi \ast \eta$ more closely and particularly we shall show that $\xi \otimes \eta \in L^p$.

For the following we let $G$ be a continuous symmetric distribution function and $\alpha$ any positive number. Now consider $\xi \ast \eta$ and $\xi \ast \eta$ and assume that $\xi$ and $\eta$ are a.s. nondecreasing. According to Theorem 8.6 the first convolution is $L^p$-continuous to the right and the second one $L^p$-continuous to the left at every point. Hence (c.f. Theorem 5.1)

$\xi \ast \eta$ has a representation

$$\xi \ast \eta = \xi_{\infty} + \sum \xi_{\nu} e_{\nu}(\cdot - c_{\nu}).$$

where $c_{\nu}(x)$ is 0 or 1 according as $x < 0$ or $x \geq 0$, $\xi_{\infty}$ is an a.s. nondecreasing and a.s. $L^p$-continuous random function, $\xi_{\nu}$ are random variables belonging to $L^p$, and $c_{\nu}$, $\nu = 1, 2, \ldots$, are the $L^p$-discontinuity points of $\xi \ast \eta$. From (10) we get

$$(\xi \ast \eta) \ast G(\frac{\cdot}{\alpha}) = \xi_{\infty} \ast G(\frac{\cdot}{\alpha}) + \sum \xi_{\nu} G\left(\frac{\cdot - c_{\nu}}{\alpha}\right)$$

In the same way we obtain

$$(\xi \ast \eta) \ast G(\frac{\cdot}{\alpha}) = \xi_{\infty}^{\prime} \ast G(\frac{\cdot}{\alpha}) + \sum \xi_{\nu}^{\prime} G\left(\frac{\cdot - c_{\nu}^{\prime}}{\alpha}\right)$$
in a corresponding notation. According to Lemma 9.3 we have
\[(\xi \ast \eta) \ast G(\frac{c}{\alpha}) = (\xi \ast \eta) \ast G(\frac{c}{\alpha}) \quad (L^p)\]
\[
\text{Letting } \alpha \to 0^+ \text{ and applying Theorem 7.2 we get}
\xi_{\infty} + \sum_{\nu} \xi_{\nu} e(\cdot - c_{\nu}) = \xi'_{\infty} + \sum_{\nu} \xi'_{\nu} e(\cdot - c'_{\nu}) \quad (L^p)
\]
(where \(e(x) = 0, 1/2 \) or 1 according as \(x < 0, x = 0\) or \(x > 0\)).

Hence
\[
\xi_{\infty} = \xi'_{\infty}, \ c_{\nu} = c'_{\nu}, \ \xi_{\nu} = \xi'_{\nu} \quad (L^p)
\]
Thus we have proved that \(\xi \ast \eta\) and \(\xi \ast \eta\) only can differ at the
\(L^p\)-discontinuity points of \(\xi \ast \eta\) and \(\xi \ast \eta\) and that the \(L^p\)-discontinuity
points of these convolutions coincide. From (10) and the corresponding
expansion for \(\xi \ast \eta\), it follows that \(\xi \otimes \eta\) belongs to \(\overline{V^p}\).

Since the right- and left-\(L^p\)-convolutions are commutative
according to Theorem 8.6 we get

**Theorem 1.** If \(\xi \in V_{q_1}, \ \eta \in V_{q_2}\), where \(1/q_1 + 1/q_2 = 1/p \leq 1, q_1 \geq 1, q_2 \geq 1,\)
then
\[
\xi \otimes \eta = \xi \otimes \eta \quad (L^p)
\]

The generalized convolution is an associative operation in the following
sense.

**Theorem 2.** If \(\xi \in V_{q_i}, \ \eta \in V_{q_i}, \ \zeta \in V_{q_3}\) where \(1/q_1 + 1/q_2 + 1/q_3 = 1/p \leq 1,\)
\(q_i \geq 1, \ i = 1, 2, 3,\) then
\[
(\xi \otimes \eta) \otimes \zeta = \xi \otimes (\eta \otimes \zeta) \quad (L^p)
\]

**Proof:** It is sufficient to prove the theorem under the additional assump-
tion that \(\xi, \eta\) and \(\zeta\) are a.s. nondecreasing.
By Lemma 9.3 we get, also using the commutative law of the operation
\[ \otimes \] (and \( \ast \))
\[ G(\frac{\cdot}{\alpha}) \ast [(\eta \otimes \xi) \otimes \xi] = [G(\frac{\cdot}{\alpha}) \ast (\eta \otimes \xi)] \ast \xi = [(G(\frac{\cdot}{\alpha}) \ast \eta) \ast \xi] \ast \xi. \]

It follows from Theorem 9.1 that \( G(\frac{\cdot}{\alpha}) \ast \eta \) is a.s. uniformly continuous in respect to \( \eta(\cdot)^{+\infty} \) and hence we find by Lemma 9.2 that the last convolution may be given in the form
\[ [(G(\frac{\cdot}{\alpha}) \ast \eta) \ast \xi] \ast \xi. \]

Using the same argument as above we find that it is then also equal (L^p) to
\[ G(\frac{\cdot}{\alpha}) \ast [(\eta \otimes \xi) \otimes \xi]. \]

Hence we get
\[ [(\eta \otimes \xi) \otimes \xi] \ast G(\frac{\cdot}{\alpha}) = [(\eta \otimes \xi) \otimes \xi] \ast G(\frac{\cdot}{\alpha}). \]

Letting \( \alpha \to 0^+ \) and observing that \( (\eta \otimes \xi) \otimes \xi \) and \( (\eta \otimes \xi) \otimes \xi \) are mean-continuous, we obtain, applying Theorem 7.2,
\[ (\eta \ast \xi) \otimes \xi = (\eta \otimes \xi) \otimes \xi. \]

Since the operation \( \ast \) is commutative this relation can also be written.
\[ (\xi \otimes \eta) \otimes \xi = \xi \otimes (\eta \otimes \xi). \]

Clearly \( \overline{V^p} \) is a linear space. The convolution \( \xi \otimes \eta \), however, need not belong to \( \overline{V^p} \) and need not ever by defined if \( \xi \in \overline{V^p} \) and \( \eta \in \overline{V^p} \).

Hence we cannot make \( \overline{V^p} \) an algebra by introducing the convolution as a multiplication rule. Not even the union of all \( \overline{V} \), \( 1 \leq p \leq +\infty \) can be made an algebra in this way.
REFERENCES


