APPROXIMATE SMALL-SAMPLE DISTRIBUTIONS
FOR MULTIVARIATE TWO-SAMPLE
NONPARAMETRIC TESTS

by

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SUMMARY

There has been much recent interest in multivariate, nonparametric procedures with major emphasis on their asymptotic properties. The multivariate, two-sample problem is considered in this paper with particular interest in small-sample applications. The construction of useful tables seems precluded because of inherent correlations between variates and hence approximate test procedures are developed that are based on exact, small-sample moments. The tests considered are the multivariate randomization test, the multivariate rank test, and the multivariate normal scores test. Examples are given.

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2 Now at the University of Georgia.
1. INTRODUCTION

Consider two $p$-variate populations $\Pi^{(1)}$ and $\Pi^{(2)}$ with corresponding $p$-element variate vectors $\bar{X}^{(1)}$ and $\bar{X}^{(2)}$ having elements $X_i^{(\nu)}$, $\nu = 1, 2$; $i = 1, \ldots, p$. Let $F^{(\nu)}(\bar{X})$ be the cumulative distribution function for $\Pi^{(\nu)}$ where $\bar{X}$ is a $p$-element vector of constants. Consider also two independent samples of sizes $N^{(1)}$ and $N^{(2)}$, $N = N^{(1)} + N^{(2)}$, from $\Pi^{(1)}$ and $\Pi^{(2)}$ of independent observation vectors $\bar{X}^{(\nu)}_\alpha$ on $\bar{X}^{(\nu)}$, $\alpha = 1, \ldots, N^{(\nu)}$; $\nu = 1, 2$.

We shall be concerned with nonparametric tests of the null hypothesis,

$$H_0 : F^{(1)}(\bar{X}) = F^{(2)}(\bar{X}),$$

with the alternative of interest usually being

$$H_a : F^{(1)}(\bar{X}_2) = F^{(2)}(\bar{X}).$$

(1.1) (1.2)

As in the univariate problem, additional generality in (1.1) may be obtained through formulation of $H_0$ as $F^{(1)}(\bar{X} - \bar{\xi}_o) = F^{(2)}(\bar{X})$ given $\bar{\xi}_o$, a vector of $p$ known constants. Three randomization tests will be discussed, each conditional on the observation vectors $X^{(\nu)}_\alpha$: (i) the test based on the observations, (ii) the test based on ranks, and (iii) the test based on normal scores.

The concepts of this paper are not new and both old and recent papers are pertinent. Fisher [7] and Pitman [13] considered univariate, two-sample, randomization tests. Wald and Wolfowitz [18] considered our problems in the bivariate case and implied easy generalization of their results to $p$ variates. Wilcoxon [20] and Mann and Whitney [12] introduced ranks for observations in the univariate problem, being followed by Terry [17] who replaced the ranks by normal scores. Chatterjee and Sen [4], Purî and Sen [14] and Sugiura [16] have considered some large-sample properties of the multivariate rank test as did Bradley and Patel [2].
Other writers have approached the multivariate problem in other ways. These include Lynch and Freund [11], Chung and Fraser [5] and Robson [15].

The contribution of this paper is the development of small-sample moments of test statistics and their use in approximate distribution theory. Examples are given.

2. THE RANDOMIZATION TESTS

Wald and Wolfowitz [18] followed Pitman [13] to suggest use of the statistic,

\[ B^2 = \frac{N(1)N(2)}{N} \left( \frac{1}{N} \frac{2}{N} (\bar{X} - \bar{X})' \bar{S}^{-1} (\bar{X} - \bar{X}) \right), \]  

(2.1)

where

\[ (n-1)\bar{S} = \sum_{\nu=1}^{N} \sum_{\alpha=0}^{(n)} (X_{\nu\alpha} - \bar{X}) (X_{\nu\alpha} - \bar{X}). \]  

(2.2)

with \( \bar{X} = \sum_{\alpha=1}^{N} X_{\nu\alpha} / N(\nu) \), \( \nu = 1, 2 \), and \( \bar{X} = \sum_{\nu=1}^{2} \bar{X} / N(\nu) \).

They showed that

\[ B^2 = (N-1)T^2 / (n-2 + T^2) \]  

(2.3)

where \( T^2 \) is Hotelling's statistic. \( B^2 \) has the advantage over \( T^2 \) in randomization tests in that \( \bar{S} \) is invariant over separations of the totality of observation vectors into two sets of sizes \( N(1) \) and \( N(2) \) whereas the corresponding matrix in \( T^2 \) is not. The two statistics are monotonically related as seen from (2.3) and would lead to equivalent test procedures.
The randomization test under $H_0$ of (1.1) follows in the usual way
since all observation vectors may be regarded as arising from a single
population. The population designations are regarded as random labels and
all $\binom{N}{n(1)}$ possible separations of the $N$ observation vectors into two sets
of sizes $n(1)$ and $N(2)$ are taken to be equally likely. $B^2$ is computed for
each separation and $B^2(\alpha)$ is the $n(\alpha)$th largest such value with $n(\alpha)$ being
the largest integer in $\binom{N}{n(1)}$ for a specified significance level $\alpha$. If the
observed value of $B^2$ is not exceeded by $B^2(\alpha)$, the null hypothesis $H_0$
is rejected with significance level $\leq \alpha$.

We have described the randomization test (i) in this section.
However, note that for (ii) $x_{i\alpha}^{(v)}$ is replaced by $r_{i\alpha}^{(v)}$, the rank of $x_{i\alpha}^{(v)}$ in
the ordered array of all $N$ observations on the $i$th variate, and for (iii)
$x_{i\alpha}^{(v)}$ is replaced by $e_{i\alpha}^{(v)}$, the expected value of the $r_{i\alpha}^{(v)}$th largest
observation in a sample of $n$ independent observations from a standard normal
population, $i = 1, \ldots, p$; $\alpha = 1, \ldots, n(\alpha)$, $\nu = 1, 2$. With the replacements of
observations noted, the randomization tests (ii) and (iii) proceed just as
described above.

It is computationally difficult to perform one of these randomization
tests. Indeed, tables cannot be developed even for tests (ii) and (iii)
with reasonable ease and utility because of correlations represented in the
fixed associations of elements or transformed elements of the observation
vectors. We shall use the moments of $B^2$ to obtain approximations to $B^2(\alpha)$. 
3. MOMENTS OF $B^2$

The moments of $B^2$ may be obtained after some simplifications. The statistic is rewritten in terms of the means of the first sample. Thus

$$B^2 = \frac{NN(1)}{N(2)} (\bar{X}^{(1)} - \bar{X})' \Sigma^{-1} (\bar{X}^{(1)} - \bar{X}).$$  (3.1)

Since $\Sigma^{-1}$ exists, is symmetric, and is positive definite at least with probability one when $F^{(1)}(X)$ and $F^{(2)}(X)$ are continuous, we factor $\Sigma^{-1} = G G'$ and write $\bar{X}^{(1)} = G' (\bar{X} - \bar{X})$ and

$$B^2 = \frac{NN(1)}{N(2)} (\bar{Z}^{(1)})' \Sigma^{-1} (\bar{Z}^{(1)}) = \frac{NN(1)}{N(2)} \sum_{i=1}^{P} \frac{1}{Z_i} (Z_i)^2.$$  (3.2)

In similar fashion, deviations of observation vectors from the mean vector may be transformed:

$$Z_{\alpha}^{(v)} = G' (Z_{\alpha}^{(v)}) - \bar{X}, \quad \alpha = 1, \ldots, N^{(v)}; \quad v = 1, 2.$$  (3.3)

We shall also use $Z_{v}$ and $X_{v}$, $v = 1, \ldots, N$, to denote the sets of vectors $Z_{\alpha}^{(v)}$ when convenient. Note that the transformation based on $G$ is not dependent on the separations to be considered. Moments of $B^2$ will be computed under $H_0$ through use of (3.2) and in terms of the elements of the $Z_{v}$.

The development of moments of $B^2$ under $H_0$ is equivalent to the consideration of moments of $B^2$ for a simple random sample of size $N^{(1)}$ from a population of vectors $Z_{v}$ of size $N$. Carver [3] considered moments of the sample mean in the univariate problem and Behnken [1] provided means of obtaining mixed moments of the sample means of order up to six in the multivariate problems. We use Behnken's results.
Let us consider \( \mathcal{E}_c(B^2) \) and \( \mathcal{E}_c[(B^2)^2] \) in some detail where \( \mathcal{E}_c \) denotes the required conditional expectation over the randomization distribution under \( H_0 \). The set of \( N \) observation vectors \( \bar{z}_Y \) are special in the sense that

\[
\sum_{\gamma=1}^{N} z_{iY} = 0, \quad \text{if } i \neq j, \; i, j = 1, \ldots, p \tag{3.4}
\]

and

\[
\sum_{\gamma=1}^{N} z_{iY}^2 = N-1, \; i = 1, \ldots, p. \tag{3.6}
\]

(3.4) follows from (3.3) and (3.5) and (3.6) arise since

\[
(N-1)g'\tilde{g} = (N-1)I = \sum_{\gamma=1}^{N} \bar{z}_Y \bar{z}_Y' \quad \text{from (2.2) and } \tilde{g}g' = \bar{g}^{-1}. \]

Ordinary operations with expectations lead us to

\[
\mathcal{E}_c(B^2) = \frac{NN(1)}{N(2)} \sum_{i=1}^{p} \mathcal{E}_c(\bar{z}_i(1)^2) \tag{3.7}
\]

and

\[
\mathcal{E}_c[(B^2)^2] = \frac{NN(1)^2}{N(2)^2} \left[ \sum_{i=1}^{p} \mathcal{E}_c(\bar{z}_i(1)^4) + 2\sum_{i<j} \mathcal{E}_c(\bar{z}_i(1)^2\bar{z}_j(1)^2) \right]. \tag{3.8}
\]

Behnken's general formulas yield, with use of (3.4), (3.5) and (3.6), the special results,

\[
\mathcal{E}_c(\bar{z}_i(1)^2) = (N-1)A_{2i}/N, \quad \mathcal{E}_c(\bar{z}_i(1)^4) = A_{41} \sum_{\gamma=1}^{N} z_{iY}^4 + \frac{3(N-1)^2}{N} A_{43} \tag{3.9}
\]

and

\[
\mathcal{E}_c(\bar{z}_i(1)^2\bar{z}_j(1)^2) = \frac{A_{41}}{N} \sum_{\gamma=1}^{N} z_{iY}^2 z_{jY}^2 + \frac{(N-1)^2}{N^2} A_{43}, \; i \neq j. \tag{3.10}
\]
where
\[
A_{21} = \frac{N}{N(1)^2} \left( N(1) - N(1) \right) \quad A_{41} = \frac{N}{N(1)^4} \left( N(1) - 7N(1)^2 + 12N(2) - 6N(3) \right),
\]
\[
A_{43} = \frac{N}{N(1)^4} \left( N(1) - 2N(1) + N(3) \right)
\]

with \( N(1) = N(1)^{-1} \ldots (N(1)-\alpha)/\bar{N}(N-1) \ldots (N-\alpha), \alpha = 1, \ldots, N. \)

Substitutions in (3.7) and (3.8) yield
\[
\varepsilon_c (B^2) = p \quad (3.9)
\]

and
\[
\varepsilon_c [(B^2)^2] = (k_1/k_0) p (p+2) + \left( \frac{N}{N(1)} \right) (2) \sum_{\gamma=1}^{N} \lambda_\gamma^2 \quad (3.10)
\]

where \( k_\alpha = N(1)^{-1} \ldots (N(1)-\alpha)N(2)^{-1} \ldots (N(2)-\alpha)/N(N-1) \ldots (N-2\alpha-1) \)

and
\[
\lambda_\gamma = \frac{z^\dagger \delta^\dagger}{z^\dagger (-\delta^\dagger)} = \left( \frac{z^\dagger}{z^\dagger} - \frac{\delta^\dagger}{\delta^\dagger} \right) S^{-1} \left( \frac{\delta^\dagger}{\delta^\dagger} - \frac{\delta^\dagger}{\delta^\dagger} \right), \quad (3.11)
\]

the latter form following from (3.3).

With considerable algebraic manipulations, Behnken's results may be used to obtain
\[
\varepsilon_c [(B^2)^3] = (k_2/k_0^3) p (p+2) (p+4) \sum_{\gamma=1}^{N} \lambda_\gamma^3 + \left( \frac{N}{N(1)} \right) (2) \sum_{\gamma=1}^{N} \lambda_\gamma^3 \quad (3.12)
\]

\[
+ 3(k_1 - 6k_2) (p-1) (p+4) \sum_{\gamma=1}^{N} \lambda_\gamma^2 + 4(k_1 - 4k_2) \sum_{\gamma \neq \delta} \lambda_\gamma^3 \lambda_\delta^3 + \frac{3}{2} \lambda_\gamma \delta \delta' \lambda_\delta' \quad (3.13)
\]
Through analogy with Behnken's methods, rather than direct proof, we suggest that

\[ \mathcal{E}_c[(b^2)^4] = \left(\frac{k_1}{k_0}\right)^4 p(p+2)(p+4)(p+6) + \left(\frac{N}{N(1)}\right)^2 \left(\frac{N}{N(2)}\right)^2 \left(\frac{k_0 - 35k_1 + 364k_2 - 1092k_3}{N} \right) \sum_{\gamma=1}^{N} \lambda^{4}_{\gamma Y} \]

\[ + 4(k_1 - 20k_2 + 80k_3)(N-1)(p+6) \sum_{\gamma=1}^{N} \lambda^{3}_{\gamma Y} \]

\[ + 6(k_2 - 6k_3)(p+4)(p+6)(N-1)^2 \sum_{\gamma=1}^{N} \lambda^{2}_{\gamma Y} \]

\[ + 24(k_1 - 16k_2 + 48k_3)(N-1)^2 \sum_{\gamma \neq \delta} \lambda^{2}_{\gamma Y} \lambda^{2}_{\gamma \delta} + \frac{4}{3} \lambda^{3}_{\gamma Y} \lambda^{3}_{\gamma \delta} \]

\[ + 3(k_1 - 12k_2 + 36k_3)(N-1)^2 \sum_{\gamma \neq \delta} \lambda^{2}_{\gamma Y} \lambda^{2}_{\gamma \delta} + 8\lambda^{2}_{\gamma Y} \lambda^{2}_{\gamma \delta} + \frac{8}{3} \lambda^{4}_{\gamma Y} \lambda^{4}_{\gamma \delta} \]

\[ + 16(k_2 - 4k_3)(p+6)(N-1)^2 \sum_{\gamma \neq \delta} \lambda^{3}_{\gamma Y} + \frac{3}{2} \lambda^{3}_{\gamma Y} \lambda^{3}_{\gamma \delta} \lambda^{3}_{\gamma \delta} \lambda^{3}_{\gamma \delta} \].

(3.13)

If we take averages of the functions of \(\lambda_{\gamma \delta}\) in (3.10), (3.12) and (3.13) to be approximately constant and take \(N(1)\) and \(N(2)\) large, approximate conditional moments are

\[ \mathcal{E}_c[(b^2)^2] = p, \quad \mathcal{E}_c[(b^2)^3] = p(p+2), \]

\[ \mathcal{E}_c[(b^2)^4] = p(p+2)(p+4), \quad \mathcal{E}_c[(b^2)^5] \approx p(p+2)(p+4)(p+6). \]

(3.14)

Thus the approximate first four conditional moments of \(b^2\) correspond with the first four moments of a central chi-square variate with \(p\) degrees of freedom.
4. APPROXIMATE DISTRIBUTIONS OF $B^2$

There has been considerable precedent and success in statistics for the use of moments in the development of approximate distributions. In nonparametric statistics, examples are given by Pitman [13], Durbin [6], Kruskal and Wallis [10] and Wallace [19]. Such methods for tests based on ranks have been good for moderate sample sizes where comparisons with exact tables are possible. We proceed to obtain approximations to the conditional distribution of $B^2$ under $H_o$ even though it appears to be very difficult to check these approximations against any exact distributions even for very special cases.

(i) The Chi-Square Approximation

We have noted in Section 3 that the conditional moments of $B^2$ are approximately those of a central chi-square variate with $p$ degrees of freedom. Under normal theory as developed by Hotelling, $B^2$ would have this distribution with moments given by (3.14) for large $N^{(1)}$ and $N^{(2)}$. The situation suggests that we might take $X^2 = \theta B^2$ to have the chi-square distribution with $\phi$ degrees of freedom with $\theta$ and $\phi$ to be determined from the first two conditional moments of $B^2$.

To complete the details, we equate moments: 

$$\mathcal{E}_c(\theta B^2) = \theta p = \phi,$$

$$\mathcal{E}_c[(\theta B^2)^2] = \theta^2 \mathcal{E}_c[(B^2)^2] = \phi(\phi+2).$$

It follows that

$$\theta = 2p/v \quad \text{and} \quad \phi = 2p^2/v \tag{4.1}$$

where

$$v = \mathcal{E}_c[(B^2)^2] - [\mathcal{E}_c(B^2)]^2 \tag{4.2}$$

with the conditional expectations determined from (3.9) and (3.10).
To use the approximation, compute $B^2$ for the observed separation and also
\[ \theta \text{ and } \phi \] and compare $\theta B^2$ with $X^2_\phi(\alpha)$ for the approximate randomization test
of significance. $X^2_\phi(\alpha)$ is the 100(1-\alpha) percentile of the central chi-square
distribution obtained from interpolation in tables of the cumulative distribu-
tion of chi-square, the interpolation being necessary since \phi will not be
an integer in most applications. Alternatively, $B^2$ may be compared with
$X^2_\phi(\alpha)/\theta$ so that the latter approximates to $B^2(\alpha)$ of Section 2.

(ii) The Variance-Ratio Approximation

It follows from (2.3) that $W = B^2/(N-1)$ ranges over the unit interval
and we know that $W$ has a discrete conditional distribution with values,
$0 \leq W < 1$. Normal theory gives the unit beta distribution for $W$ under $H_0$
with parameters $\nu_1 = p$ and $\nu_2 = N-p-1$ in the density function
\[ f(w) = \frac{\frac{1}{2}\nu_1-1}{(1-w)^{\frac{1}{2}\nu_1-1}} / B\left(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2\right), \quad 0 \leq w \leq 1, \] and then the first and second
moments of $W$ are $\nu_1/(\nu_1+\nu_2)$ and $\nu_1/(\nu_1+\nu_2)(\nu_1+\nu_2+2)$ respectively.

These facts suggest use of the unit beta distribution as an approximation to
the randomization distribution of $W$. We equate first and second conditional
moments of $B^2/(N-1)$ to those of the unit beta and solve for $\nu_1$ and $\nu_2$. Then
\[ \nu_1 = \Delta p \quad \text{and} \quad \nu_2 = \Delta(N-p-1) \] (4.3)
where
\[ \Delta = 2[p(N-p-1) - v]/(N-1)v \] (4.4)
with $v$ in (4.2)

If $W$ has the unit beta distribution with parameters $\nu_1$ and $\nu_2$,
$F = \nu_2W/\nu_1(1-W)$ has the variance-ratio or $F$-distribution with $\nu_1$ and $\nu_2$ degrees
of freedom. Thus we define

$$F = \frac{(N-p-1)B^2}{p(N-1-B^2)} \tag{4.5}$$

for the randomization test and, under $H_o$, take it to have the variance-ratio distribution with $\nu_1$ and $\nu_2$ degrees of freedom as given in (4.3). For the test, compute $B^2$ for the observed separation and then the corresponding $F$ of (4.5) and compare that $F$ with $F_{\nu_1, \nu_2}(\alpha)$, the $100(1-\alpha)$ percentile of the $F$-distribution obtained from tables of that distribution after interpolation necessary because $\nu_1$ and $\nu_2$ may not be integers.

Analogy with other test procedures suggests that the variance-ratio approximation will be better than the chi-square approximation.

5. THE APPROXIMATE RANDOMIZATION TESTS

Three randomization tests were set forth in Section 2. Test (i), the randomization test on the observations, follows directly from Section 4. The test of $H_o$ is judged significant if, for the observed separation,

$$B^2 \geq B^2(\alpha) = X^2_\phi(\alpha)/\theta$$

for the chi-square approximation, or, from (4.5), if

$$F \geq F_{\nu_1, \nu_2}(\alpha)$$

for the variance-ratio approximation.

We have noted that $r^{(v)}_{i\alpha}$ replace the observations $x^{(v)}_{i\alpha}$ for Test (ii), the generalization of the univariate Wilcoxon or Mann-Whitney test. The statistic, from the form (3.1), with simple modification to most familiar terms, becomes

$$B^2(r) = \frac{12N(1)}{(N+1)N(2)} \left( \bar{r}(1) - \frac{1}{2} \frac{1+1}{2} r^{-1} \bar{r}(1) - \frac{1}{2} \frac{1+1}{2} \right) \tag{5.1}$$

where $1$ is the $p$-element column vector of unities and $R$ is the matrix of rank
correlations with elements

\[ R_{ij} = \frac{12}{N^3 - N^2} \sum_{\nu=1}^{2} \sum_{\alpha=1}^{N} (\xi_{i\alpha}^{(\nu)} - \frac{N+1}{2})(\xi_{j\alpha}^{(\nu)} - \frac{N+1}{2}). \]  \hspace{1cm} (5.2)

\( i, j = 1, \ldots, p. \) The Mann-Whitney form of \( B^2(r) \) follows from direct analogy with the univariate case. We let \( U \) be the number of observations from \( \Pi \) that precede observations from \( \Pi \) on the \( i \)th variate. Then

\[ \bar{r}_i = \frac{1}{p} \frac{U}{N+1} = \frac{1}{p} \frac{U_{(1)}(1)}{N(2)}/N(1) \text{ and, if } \Psi \text{ is the } p \text{-element column vector of values } U, \ i = 1, \ldots, p, \]

\[ B^2(r) = \frac{12}{N(1)N(2)(N+1)} (U - \frac{1}{p} \Psi_{(1)}(1))_N(2)(U - \frac{1}{p} \Psi_{(1)}(1))_N(2). \] \hspace{1cm} (5.3)

The conditional moments of \( B^2(r) \) follow from the substitution of ranks for observations. For ranks, we rewrite (3.11) so that

\[ \lambda_{\gamma\delta}(r) = \frac{12}{N(N+1)} (r_{\gamma} - \frac{N+1}{2})_N^{-1} (r_{\delta} - \frac{N+1}{2}), \] \hspace{1cm} (5.4)

\( \gamma, \delta = 1, \ldots, N, \) where \( r_{\gamma} \) is the column vector of ranks on the \( p \) variates for one of the observation vectors \( \gamma \) of the total set of \( N \) such vectors. The approximate chi-square and variance-ratio tests are conducted as for Test (i) with the use of \( B^2(r) \) and \( v(r) \), the latter computed from (4.2) with use of \( \lambda_{\gamma\delta}(r) \) instead of \( \lambda_{\gamma\delta} \), to obtain \( \theta(r), \phi(r) \) in (4.1) and \( v_1(r) \) and \( v_2(r) \) in (4.3).

Test (iii) is the generalization of Terry's normal scores test. The test could proceed through replacement of the \( x_{i\alpha}^{(\nu)} \) by the corresponding scores \( e_{i\alpha}^{(\nu)} \) as noted in Section 2. Minor simplifications arise since
\[ 2 \sum_{\nu=1}^{N} \sum_{i=1}^{N} e_{i\nu} = 0, \ i = 1, \ldots, p. \quad \text{Now} \]

\[ (N-1) S(e) = \sum_{\nu=1}^{N} \sum_{i=1}^{N} \xi_{i\nu}^{(\nu)} \xi_{i\nu}^{(\nu)'}, \quad (5.5) \]

from (2.2) and

\[ B^2(e) = \frac{N^{(1)}}{N^{(2)}} \xi^{(1)}' S^{-1}(e) \xi^{(1)} \quad (5.6) \]

from (3.1). In parallel with (5.4),

\[ \lambda_{\gamma\delta}(e) = \xi_{\gamma}^{(\nu)} S^{-1}(e) \xi_{\delta}, \quad (5.7) \]

\( \gamma, \delta = 1, \ldots, N. \) The approximate tests follow with \( B^2(e) \) and \( v(e) \) computed and used as for Test (ii). Table XXI of Fisher and Yates [9] yields minor assistance in the computation of \( S(e). \)

6. A NUMERICAL EXAMPLE

To illustrate the approximate variance-ratio tests, we use the data of Table 1 selected from Fisher's article [8] on the use of multiple measurements in taxonomic problems. We give details on the calculations for Test (i) and summarize the calculations for Tests (ii) and (iii). In Table 1, ranks \( r_{i\alpha}^{(\nu)} \) and normal scores \( c_{i\alpha}^{(\nu)} \) are given with the observations \( x_{i\alpha}^{(\nu)} \). For tied observations the average rank or the average score has been used.

Basic sample parameters for the tests are \( N^{(1)} = 7, \ N^{(2)} = 9, \ N = 16, \ p = 2. \) Sample means are: \( \bar{x}_1^{(1)} = 6.0143, \ \bar{x}_2^{(1)} = 4.3429, \ \bar{x}_1^{(2)} = 6.5222, \)
\( \bar{x}_2^{(2)} = 5.5778, \ \bar{x}_1 = 6.3000, \ \bar{x}_2 = 5.0375. \) The matrix of (2.2) and its inverse
\[ \Sigma = \begin{pmatrix} 0.6640 & 0.5820 \\ 0.5820 & 0.8038 \end{pmatrix} \quad \text{and} \quad \Sigma^{-1} = \begin{pmatrix} 4.1221 & -2.9846 \\ -2.9846 & 3.4051 \end{pmatrix}. \]

It follows from (2.1) that \( B^2 = 9.893 \) and from (4.5) that \( F = 12.590. \)

Table 1

Measurements on the Flowers *Iris Versicolor* and *Iris Virginica*
for Selected Samples of Sizes Seven and Nine

<table>
<thead>
<tr>
<th>( n^{(1)}: ) IRIS VERSICOLOR</th>
<th>( n^{(2)}: ) IRIS VIRGINICA</th>
</tr>
</thead>
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<td><strong>Petal Length</strong></td>
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<td><strong>Sepal Length</strong></td>
<td><strong>Petal Length</strong></td>
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<td>( r_{1\alpha} )</td>
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<tr>
<td>6.4</td>
<td>9.5</td>
</tr>
<tr>
<td>7.6</td>
<td>15</td>
</tr>
<tr>
<td>7.1</td>
<td>14</td>
</tr>
<tr>
<td>7.9</td>
<td>16</td>
</tr>
<tr>
<td>5.8</td>
<td>4</td>
</tr>
</tbody>
</table>
The approximate degrees of freedom \( v_1 \) and \( v_2 \) of (4.3) are computed after calculation of values of \( \lambda_{\gamma\gamma} \) of (3.11) in Table 2, which also contains the corresponding values of \( \lambda_{\gamma\gamma}(r) \) and \( \lambda_{\gamma\gamma}(e) \). Note that a check on the computations is possible since \( \sum_{\gamma=1}^{N} \lambda_{\gamma\gamma} = p(N-1) \), a result which follows since

\[
\sum_{\gamma=1}^{N} \lambda_{\gamma\gamma} = \frac{2N}{N} \sum_{\nu=1}^{N} \left( \bar{x}_{\alpha}^{(\nu)} - \bar{x}^{(\nu)} \right) \right]^{-1} \left( \bar{x}_{\alpha}^{(\nu)} - \bar{x}^{(\nu)} \right) = \frac{2N}{N} \sum_{\nu=1}^{N} \text{tr} \left[ \left( \bar{x}_{\alpha}^{(\nu)} - \bar{x}^{(\nu)} \right) \right]^{-1} \left( \bar{x}_{\alpha}^{(\nu)} - \bar{x}^{(\nu)} \right),
\]

\[= \text{tr} \left( N-1 \right) I = p(N-1).\]

The calculations may be completed now: \( \sum_{\gamma=1}^{N} \lambda_{\gamma\gamma}^{2} = 90.3905 \); from (3.9) and (3.10),

\[k_{0} = 0.2625, \quad k_{1} = 0.069231, \quad \xi_{c} (B_{c}^{2}) = 2, \quad \xi_{c} \left[ (B_{c}^{2})^{2} \right] = 7.1464; \quad \text{from (4.2)},
\]

\[v = 3.1464; \quad \text{and, from (4.4),} \quad \Delta = 0.9685. \quad \text{The approximate degrees of freedom are} \quad v_{1} = 1.94 \quad \text{and} \quad v_{2} = 12.59.
\]

Table 2

Values of \( \lambda_{\gamma\gamma} \) for the Two Samples

<table>
<thead>
<tr>
<th>( \Pi^{(1)}: ) IRIS VERSICOLOR</th>
<th>( \Pi^{(2)}: ) IRIS VIRGINICA</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_{\gamma\gamma} )</td>
<td>( \lambda_{\gamma\gamma}(r) )</td>
</tr>
<tr>
<td>2.8967</td>
<td>3.9474</td>
</tr>
<tr>
<td>0.1498</td>
<td>0.1213</td>
</tr>
<tr>
<td>0.4965</td>
<td>1.0380</td>
</tr>
<tr>
<td>1.3490</td>
<td>2.0015</td>
</tr>
<tr>
<td>2.3338</td>
<td>2.8686</td>
</tr>
<tr>
<td>1.3389</td>
<td>1.5369</td>
</tr>
<tr>
<td>3.7628</td>
<td>2.7022</td>
</tr>
<tr>
<td>3.8609</td>
<td>2.7038</td>
</tr>
</tbody>
</table>
We interpolate in the table of the variance-ratio distribution to obtain $F_{1,94} = 12.59(.05) = 3.89$ and note that $F$ for the observed separation at 12.59 is considerably in excess of the tabular value and judge the separation to be significant.

The calculations discussed above and the corresponding ones for Tests (ii) and (iii) are summarized in Table 3. It is seen that values for the three tests are quite comparable.

Table 3

Summary of Calculations for the Three Tests

<table>
<thead>
<tr>
<th>Quantities Calculated</th>
<th>Test (i)</th>
<th>Test (ii)</th>
<th>Test (iii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B^2$, $B^2(r)$, $B^2(c)$</td>
<td>9.8926</td>
<td>9.3323</td>
<td>9.3008</td>
</tr>
<tr>
<td>$N$, $N$</td>
<td>$\sum \lambda_{YY}$, $\sum \lambda_{YY}^r$, $\sum \lambda_{YY}^c$</td>
<td>30.0014</td>
<td>30.0001</td>
</tr>
<tr>
<td>$\gamma=1$ $YY$, $\gamma=1$ $YY$</td>
<td>$\sum \lambda_{YY}^2$, $\sum \lambda_{YY}^{2r}$, $\sum \lambda_{YY}^{2c}$</td>
<td>90.3905</td>
<td>73.8676</td>
</tr>
<tr>
<td>$\gamma=1$ $YY$, $\gamma=1$ $YY$</td>
<td>$v$, $v(r)$, $v(c)$</td>
<td>3.1464</td>
<td>3.3093</td>
</tr>
<tr>
<td>$\Delta$, $\Delta(r)$, $\Delta(c)$</td>
<td>0.9685</td>
<td>0.9142</td>
<td>0.9456</td>
</tr>
<tr>
<td>$v_1$, $v_1(r)$, $v_1(c)$</td>
<td>1.94</td>
<td>1.83</td>
<td>1.89</td>
</tr>
<tr>
<td>$v_2$, $v_2(r)$, $v_2(c)$</td>
<td>12.59</td>
<td>11.88</td>
<td>12.29</td>
</tr>
<tr>
<td>$F$, $F(r)$, $F(c)$</td>
<td>12.59</td>
<td>10.70</td>
<td>10.61</td>
</tr>
<tr>
<td>$F_{\nu_1, \nu_2}$, $F_{\nu_1, \nu_2}(.05)$, $F_{\nu_1}(c), v_2(c)(.05)$</td>
<td>3.89</td>
<td>4.05</td>
<td>3.96</td>
</tr>
</tbody>
</table>
REFERENCES


There has been much recent interest in multivariate, nonparametric procedures with major emphasis on their asymptotic properties. The multivariate, two-sample problem is considered in this paper with particular interest in small-sample applications. The construction of useful tables seems precluded because of inherent correlations between variates and hence approximate test procedures are developed that are based on exact, small-sample moments. The tests considered are the multivariate randomization test, the multivariate rank test, and the multivariate normal scores test. Examples are given.