A GENERALIZED AGE-DEPENDENT BRANCHING PROCESS

AND ITS LIMIT DISTRIBUTION

by

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1. **Summary.** An age-dependent continuous parameter branching stochastic process \( X_N(t) \) is considered, where \( X_N(t) \) is the number of particles in the population at time \( t \) and \( N \) is the initial size of the population. If the probabilities of splits depend on \( N \), in a way that will be made precise in the text, then, for fixed \( t \), as \( N \) tends to infinity a limiting distribution of the stochastic processes \( X_N(t) - N \) is obtained, and shown to be the distribution of a continuous parameter stochastic process with independent increments \( X(t) \), whose distribution will be determined. To establish this we need to prove that for \( t_1 < t_2 < \ldots < t_n \) the distribution of \( X_N(t_1) - X_N(t_{i-1}) \) converges to the distribution of \( X(t_1) - X(t_{i-1}) \), and \( X_N(t_2) - X_N(t_1) \), \( X_N(t_3) - X_N(t_2), \ldots, X_N(t_n) - X_N(t_{n-1}) \), are independent in the limit. The limiting distribution of \( X_N(t) - N \) gives an approximation to the distribution of \( X_N(t) \) for any fixed \( t \) and large \( N \).

2. **Introduction and Description of the Model.** An integer-valued continuous time branching stochastic process \( X_N(t), t \geq 0 \) is considered, where \( X_N(t) \) is the size of the population at time \( t \), assuming that we start the process with a non-random number \( N > 0 \) of particles. Any particle existing in the population at any time might "split" into \( k \) new particles, where \( k = 0, 1, 2, \ldots \), and we assume that the particle dies when it produces the \( k \) new particles. We also assume that all the particles in the population at any time are stochastically independent between themselves and all other particles that might exist or existed at any time in the population.

Let the conditional probability that any particle splits into \( k \) new particles during the time interval \([t, t + \Delta t]\), given that it exists at time \( t \) and is of age \( x \), \((x \geq 0)\), at time \( t \) be equal to

\[
(1) \quad \lambda_{k,N}(x)\Delta t + o(\Delta t), \quad k = 0, 1, 2, \ldots,
\]
where \( \lambda_{k,N}(x) \geq 0 \) for all \( k \) and not zero for all \( k \), and \( o(\Delta t) \) is uniform in \( t \) and \( x \), and \( \lambda_{k,N}(x) \) is continuous for all \( k \) and \( N \), and let

\[
(2) \quad \sum_{k=0}^{\infty} N \lambda_{k,N} = N \lambda_N(x)
\]

uniformly over every bounded interval, and uniformly in \( N \). Also assume that the conditional probability that a particular particle does not split during the time interval \( [t, \Delta t + t] \), given that it exists at time \( t \) and is of age \( x \) at \( t \) is equal to

\[
(3) \quad 1 - \lambda_N(x) \Delta t + o(\Delta t),
\]

where as before we assume that \( o(\Delta t) \) is uniform in \( t \) and \( x \).

Assuming that members of the sequence of functions \( \{\lambda_{k,N}(x)\} \) depend on \( N \) in such a way that

\[
(4) \quad \lim_{N \to \infty} N \lambda_{k,N}(x) = \lambda_k(x) \geq 0, \text{ for } k = 0, 1, 2, \ldots,
\]

uniformly over every bounded interval, and

\[
(5) \quad \lim_{N \to \infty} N \lambda_N(x) = \lambda(x)
\]

uniformly over every handed interval. Thus

\[
(6) \quad \sum_{k=0}^{\infty} \lambda_k(x) = \lambda(x)
\]

uniformly over every bounded interval. Let

\[
(7) \quad \Lambda(t) = \int_0^t \lambda(x) \, dx.
\]
Given the above assumptions we will be able to find the limiting distribution of the stochastic process \( X_N(t) - N \) as \( N \to \infty \). This limiting distribution will be shown to be the distribution of a stochastic process \( X(t) \) with independent increments.

The process studied in this paper can be considered as a generalization of Kendall's model [8], in which he considered the birth-and-death process where the birth and death rates are given by \( \lambda(t) \) and \( \mu(t) \) respectively. In Kendall's model he obtains, among other things, the mean and variance of the size of the population at time \( t \), and the exact distribution of the size of the population for the Arley model [1], where \( \lambda(t) = \lambda \) and \( \mu(t) = \mu \cdot t \). Also in Kendall's model the process depends on absolute time and not age. Our model not only depends on age but also on the size of the initial population. Also in our model the limiting distribution of \( X_N(t) \) is obtained for \( t \) fixed but \( N \) is allowed to tend to infinity. This type of a limiting procedure is similar to the one considered by Stratton and Tucker [15] and Shimi [14], which differs from the usually studied limit theorems in which the limiting distribution or some limiting probability of \( X_N(t) \) is obtained for fixed \( N \) (usually 1) as \( t \) tends to infinity (see [1]-[13], [16], [17]). Also by the assumption that as \( N \to \infty \) the splitting probabilities decrease so that the number of particles produced in a fixed finite time interval remains bounded makes our model different from the diffusion approximation of Feller [5] and from Sevast'yanov's model [13]. The mathematical methods used in our proofs are similar to the methods used in [14] and [15]. To prove the propositions and main theorems of this paper, we need the following lemmas. Lemma 1, 2, and 4 were proved in [15] and lemma 3 was proved in [14], and no proofs will be given for these lemmas here.
LEMMA 1. Let $f$ be a real-valued function over $[0,1]$ which is Riemann integrable. Let $0 = x_{n,0} < x_{n,1} < \ldots < x_{n,n} = 1$ be such that

$$\max_{1 \leq k \leq n} \{x_{n,k} - x_{n,k-1}\} \to 0 \text{ as } n \to \infty.$$ If $y_{n,k} \in [x_{n,k-1}, x_{n,k}]$, then

$$L_n \equiv \frac{1}{n} \left[ 1 + f(y_{n,k}) (x_{n,k} - x_{n,k-1}) \right] \to \exp \int f(x) \, dx \text{ as } n \to \infty.$$

LEMMA 2. Let $\mu, \mu_1, \mu_2, \ldots$ be a sequence of probability measures over the $n$-dimensional Euclidean space $\mathbb{R}^n$, $f, f_1, f_2, \ldots$ be a sequence of measurable functions over $\mathbb{R}^n$, and assume that

(i) $|\mu_n - \mu| (B) \to 0 \text{ as } n \to \infty$ for every bounded measurable subset $B \subset \mathbb{R}^n$,

(ii) $f, f_1, f_2, \ldots$ are uniformly bounded a.e with respect to $\mu, \mu_1, \mu_2, \ldots$,

(iii) $f_n \to f$ as $n \to \infty$ uniformly over every bounded Borel set except over a subset of $\mu$- and $\mu_n$- measure zero, $m = 1, 2, \ldots$. Then $\int f_n \, d\mu_n \to \int f \, d\mu$ as $n \to \infty$.

LEMMA 3. Let $\mu, \mu_1, \mu_2, \ldots$ be a sequence of probability measures over $n$-dimensional Euclidean space $\mathbb{R}^n$, let $\{A_n\}$ be a sequence of monotonically decreasing measurable subsets such that $\lim_{n \to \infty} A_n = A$, a bounded subset, where $\mu(A) = 1$, $\mu_n(A_n) = 1$, $n = 1, 2, \ldots$, and let $f, f_1, f_2, \ldots$ be a sequence of measurable functions over $\mathbb{R}^n$, assume that

(i) $|\mu_n - \mu| (A) \to 0 \text{ as } n \to \infty$

(ii) $f, f_1, f_2, \ldots$ are uniformly bounded a.e with respect to $\mu, \mu_1, \mu_2, \ldots$,

(iii) $f_n \to f$ as $n \to \infty$ uniformly over $A$. Then $\int_{A_n} f_n \, d\mu_n \to \int_A f \, d\mu$ as $n \to \infty$.

LEMMA 4. Let $\{(Y_{n,1}, \ldots, Y_{n,k})\}$ be a sequence of $k$-dimensional random variables which converges in distribution to $(Y_1, \ldots, Y_k)$. If

$$S_{n,j} = Y_{n,1} + Y_{n,2} + \ldots + Y_{n,j}, \text{ for } 1 \leq j \leq k, \text{ and } n = 1, 2, \ldots,$$

then

$$S_{n,j} \to (Y_1 + \ldots + Y_j) \text{ as } n \to \infty.$$
\[ S_j = Y_1 + Y_2 + \ldots + Y_j, \] then the joint distribution of \( (S_{n,1}, \ldots, S_{n,k}) \) converges to the joint distribution of \( (S_1, S_2, \ldots, S_k) \) as \( n \to \infty \).

3. Propositions. Four propositions, which are used in proving the main theorems of the next section, will be proved in this section.

**PROPOSITION 1.** The conditional probability that a particle splits into \( k \) particles during \( [t, t + \Delta t] \), given that it exists at time \( t \) and at time \( t' < t \), is
\[
\Delta t \int_0^{t'} \lambda_{k,N} (t-t') \, d F_T(t-t') + o(\Delta t),
\]
where \( F_T(t-t') \) is \( \Pr([T < t-t']|A(t)A(t')) \), and \( T \) denote the time a particle came into existence, and \( A(x) \) denote the event that a particle exists at time \( x \).

Also the conditional probability that a particle does not split during \( [t, t + \Delta t] \), given that it exists at \( t \) and \( t' < t \), is
\[
1 - \Delta t \int_0^{t'} \lambda_N (t-t') \, d F_T(t-t') + o(\Delta t).
\]

**PROOF.** Consider a particular particle, let \( S_k \) denote the event that it splits into \( k \) particles during \( [t, t + \Delta t] \). Then by (1)
\[
\Pr(S_k|A(t)A(t')) = \int_0^{t'} \Pr(S_k|A(t'), T = t-t') \, d F_T(t-t')
\]
\[
= \int_0^{t'} \Pr(S_k|A(t), T = t-t') \, d F_T(t-t')
\]
\[
= \int_0^{t'} [\lambda_{k,N}(t-t') \, t + o(\Delta t)] \, d F_T(t-t')
\]
\[
= \Delta t \int_0^{t'} \lambda_{k,N} (t-t') \, d F_T(t-t') + o(\Delta t),
\]
hence the proof of the first part of the proposition. The second part is proved similarly using (3) this time instead of (1).
For $t \geq z \geq 0$ let $Y_N(t; z)$ be the number of particles, from the group that came into existence at time $z$, that split during $[z, t]$, and let $X_N(t; z)$ represent the number of particles, which came into existence at time $z$, that are still in existence at time $t$.

**PROPOSITION 2.** For every $z \geq 0$, $t \geq 0$ and $I > 0$, where $0 \leq z \leq t < t + I$,

$$\Pr([Y_N(t+I; z) - Y_N(t; z) = 0] | X_N(t; z) = n) = \exp - n \int_t^{t+I} \lambda_N(x-z)dx.$$  

If $t = 0$, then $z = 0$ and $n = N$.

**PROOF.** Consider one of the $n$ particles in existence at time $t$ which came into existence at time $z$. If $t \leq t_1 \leq t_1 + \Delta t_1$, then the conditional probability that it does not split during $[t_1, t_1 + \Delta t_1]$, given that it still exists at time $t_1$, is easily seen to be given by $1 - \lambda_N(t_1 - z)\Delta t_1 + o(\Delta t_1)$.

Thus the conditional probability that it does not split during any of the intervals

$$\{[t + jI/m, t + (j+1)I/m], j = 0, 1, \ldots, m-1\},$$

given that it is in existence at time $t$ is

$$\sum_{j=0}^{m-1} \left(1 - \lambda_N(t + jI/m - z) \right) \frac{I}{m} + o\left(\frac{I}{m}\right),$$

and since $\lambda_N(x)$ is continuous over every bounded interval, thus Riemann integrable over every bounded interval, then, by lemma 1, the above probability converges as $m \to \infty$ to $\exp - \int_t^{t+I} \lambda_N(x-z)dx$. The required result follows since all particles are stochastically independent.

**PROPOSITION 3.** Let $P(t_1, \Delta t_1; t_2, \Delta t_2; k; m)$ denote the conditional probability
that among \( m \) particles, that came into existence from a single split, a split
of size \( k \) occurs during \([t_2, t_2 + \Delta t_2]\), given that they came into existence
during \([t_1, t_1 + \Delta t_1]\) and still exist at time \( t_2 \). Let \( Q(t_1, \Delta t_1; t_2, t_3; m) \)
denote the conditional probability that none of \( m \) of the progenys of one
particle split during \([t_2, t_3]\), given that they came into existence during
\([t_1, t_1 + \Delta t_1]\) and still exist at time \( t_2 \). Then

\[
(9) \quad \lim_{\Delta t_1 \to 0} P(t_1, \Delta t_1; t_2, \Delta t_2; k; m) = m \lambda_{kN}(t_2 - t_1)\Delta t_2 + o(\Delta t_2)
\]
uniformly in \( t_1 \) and \( t_2 \) where \( 0 \leq t_1 \leq t_2 \leq t \) and \( t \) is finite (for fixed
\( N, k, m \)), and

\[
(10) \quad \lim_{\Delta t_1 \to 0} Q(t_1, \Delta t_1; t_2; t_3; m) = \exp - m \int_{t_2}^{t_3} \lambda_N(x - t_1) dx
\]
uniformly in \( t_1, t_2, t_3 \), where \( 0 \leq t_1 < t_2 < t_3 \leq t \) and \( t \) is finite (for
fixed \( N \) and \( m \)).

PROOF. Consider one of these \( m \) particles that came into existence from a
single split during \([t_1, t_1 + \Delta t_1]\) and still exist at time \( t_2 \). Let \( A(x) \) be
the event that such a particle exists at time \( x \), and let \( T \) be the time it
came into existence. Let \( G(v|t_1, \Delta t_1, t_2) = \Pr([T \leq v]|A(t_2), T \in [t_1, t_1 + \Delta t_1]) \).
Let \( S_k[t_2, t_2 + \Delta t_2] \) be the event that it splits into \( k \) particles during
\([t_2, t_2 + \Delta t_2]\), then
\[ \Pr(S_k[t_2, t_2 + \Delta t_2]|A(t_2), T \in [t_1, t_1 + \Delta t_1]) \]

\[ = \int_{t_1}^{t_1 + \Delta t_1} \Pr[S_k[t_2, t_2 + \Delta t_2]|A(t_2), T \in [t_1, t_1 + \Delta t_1], T = v]dG(v|t_1, \Delta t_1, t_2) \]

\[ = \int_{t_1}^{t_1 + \Delta t_1} \Pr[S_k[t_2, t_2 + \Delta t_2]|A(t_2), T = v]dG(v|t_1, \Delta t_1, t_2) \]

\[ = \Delta t_2 \int_{t_1}^{t_1 + \Delta t_1} \lambda_{k,N}(t_2 - v)dG(v|t_1, \Delta t_1, t_2) + o(\Delta t_2). \]

Since the particles are stochastically independent we can see that

\[ \Pr(t_1, \Delta t_1; t_2, \Delta t_2; k;m) = m\Delta t_2 \int_{t_1}^{t_1 + \Delta t_1} \lambda_{k,N}(t_2 - v)dG + o(\Delta t_2). \]

As \( \Delta t_1 \to 0 \), \( G(v|t_1, \Delta t_1, t_2) \) tends to the degenerate distribution which has a single saltus of magnitude 1 at \( t_1 \). Further, \( \Delta t_2 \lambda_{k,N}(t_2 - v) \) is bounded by 1 and converges to \( \Delta t_2 \lambda_{k,N}(t_2 - t_1) \) as \( \Delta t_1 \to 0 \). Hence lemma 3 applies, yielding the required result (9). We use the continuity of \( \lambda_{k,N}(x) \), thus its uniform continuity on the compact set \([0, t]\), to prove that the above convergence is uniform. To prove (10), consider those \( m \) particles that came into existence from a single split during \([t_1, t_1 + \Delta t_1]\) and still exist at time \( t_2 \). Let \( \mathcal{A}_m(x;[t_1, t_1 + \Delta t_1]) \) be the event that \( m \) of the particles that came into existence from a single split during \([t_1, t_1 + \Delta t_1]\) exist at time \( x \), and let \( T \) be the time they came into existence. Let

\[ H_m(v|t_1, \Delta t_1, t_2) = \Pr([T \leq v]|\mathcal{A}_m(t_2;[t_1, t_1 + \Delta t_1]), T \in [t_1, t_1 + \Delta t_1]). \]

From (8) of proposition 2 we can see that
\[ Q(t_1, \Delta t_1; t_2; t_3; m) = \int_{t_1}^{t_1+\Delta t_1} \left[ \exp - m \int_{t_2}^{t_3} \lambda_N(x-v) \, dx \right] dH_m(v|t_1, \Delta t_1, t_2). \]

Using a similar argument as in the first part we can show that

\[ \lim_{\Delta t_1 \to 0} Q(t_1, \Delta t_1; t_2; t_3; m) = \exp - m \int_{t_2}^{t_3} \lambda_N(x-t_1) \, dx \]

uniformly in \( t_1, t_2, t_3 \) where \( 0 < t_1 < t_2 < t_3 < t \) and \( t \) is finite.

**PROPOSITION 4.** Let \( Q(\Delta t) \) denote the conditional probability that none of the particles produced during \([t, t + \Delta t]\) also split during \([t, t + \Delta t]\), given that one particle in existence at time \( t \) does split during \([t, t + \Delta t]\). Then \( Q(\Delta t) \to 1 \) as \( t \to 0 \) uniformly in \( t \) over every bounded interval for fixed \( N \).

**PROOF.** Let \( p_k(\Delta t) \) denote the conditional probability that a particle existing at time \( t \) splits into \( k \) particles during \([t, t + \Delta t]\), given that it does split during \([t, t + \Delta t]\). Using the results of proposition 1 we can see that

\[
p_k(t) = \left[ \Delta t \int_0^t \lambda_{k,N}(t-t'')dF_T(t'') + o(\Delta t) \right] / \left[ \Delta t \int_0^t \lambda_N(t-t'')dF_T(t'') + o(\Delta t) \right]
\]

\[
= \int_0^t \lambda_{k,N}(t-t'')dF_T(t'') / \left[ \int_0^t \lambda_N(t-t'')dF_T(t'') \right] + o(1),
\]

and since from (2) \( \sum_{k=0}^{\infty} \lambda_{k,N}(x) = \lambda_N(x) \) uniformly over every bounded interval for every \( N \), we can see that \( \sum_{k=0}^{\infty} p_k(\Delta t) = 1 \) for all \( \Delta t > 0 \). Let \( Z \) denote the time of splitting of the particle in existence at time \( t \), and let \( F_z(\Delta t) \) be
the conditional distribution function of Z, given the event $B_k$ that the particle in existence at time $t$ does split into $k$ new particles during $[t, t + \Delta t]$. Let $A$ denote the event that none of the particles produced by a split during $[t, t + \Delta t]$ also split during $[t, t + \Delta t]$. Then

$$Q(\Delta t) = \sum_{k=0}^{\infty} \Pr(A|B_k)\Pr(B_k)$$

$$= \sum_{k=0}^{\infty} \left\{ \int_{t}^{t+\Delta t} P(A|B_k, Z = z) dF(z|\Delta t) \right\} p_k(\Delta t)$$

$$= \sum_{k=0}^{\infty} \left\{ \int_{t}^{t+\Delta t} [1 - \lambda_N(0)(t + \Delta t - z) + o(t + \Delta t - z)]^k dF(z|\Delta t) \right\} p_k(\Delta t).$$

Each integral is absolutely bounded by 1 for every $\Delta t > 0$ and converges to 1 as $\Delta t \to 0$, and

$$P_k(\Delta t) \to \int_{0}^{t} \lambda_k \lambda_N(t-t') dF_k(t') / \int_{0}^{t} \lambda_N(t-t') dF(t')$$

as $\Delta t \to 0$. Thus, using lemma 3, we can see that $Q(\Delta t) \to 1$ as $t \to 0$.

4. The Limit Process as $N \to \infty$. In this section we shall prove three theorems. The first two, in addition to being of independent interest, are used in proving the last theorem. Theorem 3 shows that the sequence of stochastic processes $\{X_N(t) - N, N = 1, 2, \ldots\}$ converges in distribution as $N \to \infty$ to a stochastic process with independent increments. Also we shall find the limit distribution of the process $X_N(t)$ as $N \to \infty$. The propositions of section 3 are used without being mentioned explicitly. Let $X_n$ denote the change in the size of the population that results from the nth splitting that takes place. Also let $Y_N(t)$ denote the number of particles which split during $[0, t]$. 
THEOREM 1. For every set of integers \( \{n_i\} \), where \( n_i \geq -1 \) for \( 1 \leq i \leq k \), where \( k \) is any non-negative integer, and \( t \) any positive real number

\[
\lim_{N \to \infty} P(Y_N(t) = k) = \exp[-\Lambda(t)] \cdot \frac{[\Lambda(t)]^k}{k!},
\]

and

\[
\lim_{N \to \infty} P\left( \bigcap_{i=1}^{k} [X_i = n_i] \bigg| [Y_N(t) = k]\right)
= \frac{k!}{[\Lambda(t)]^k} \int_0^t \int_{t_1}^t \cdots \int_{t_{k-1}}^t \lambda_{n_1+1}(t_1) dt_1 \cdots dt_{k-1} dt_k,
\]

where \( \Lambda(t) \) is given by (7).

PROOF. To prove this theorem we first need to establish that

\[
\lim_{N \to \infty} P\left( [Y_N(t) = k] \bigcap \bigcap_{i=1}^{k} [X_i = n_i]\right)
= \left[\exp - \Lambda(t)\right] \int_0^t \int_{t_1}^t \cdots \int_{t_{k-1}}^t \lambda_{n_1+1}(t_1) dt_1 \cdots dt_{k-1} dt_k.
\]

We shall prove this result in detail only in the case \( k = 2 \). The proof for \( k > 2 \) is the same and only involves more complicated notation. Let

\( 0 = x_0 < x_1 < x_2 < \ldots < x_m = t \) be a partition of \([0,t]\), and let \( \Delta(i_r) \) denote the difference \( x_{i_r + 1} - x_{i_r} \). We can see that

\[
P\left( (Y_N(t) = 2) \bigcap (X_i = n_i) \right) = \lim_{m \to \infty} \sum_{i=1}^{m} \sum_{i_2 = i_1 + 1}^{m} P[\text{(no split in } [0,x_{i_1}]) \cap \text{(a split in } [x_{i_1},x_{i_1+1}]) \text{ of size } n_1 + 1) \cap \text{(no split in}}
\]

\[
[x_{i_1+1},x_{i_2}] \cap \text{(a split, of one of the } N - i \text{ original particles left after}}
\]
the first split, in $[x_{i_2}, x_{i_2+1}]$ of size $n_2 + 1) \cap \{\mathrm{no \ split \ in \ } [x_{i_2+1}, t]\} \cup \{\mathrm{a \ split, \ one \ of \ the \ } n_1 + 1 \mathrm{ \ particles \ that \ came \ into \ existence \ during} \ [x_{i_1}, x_{i_1+1}], \ in \ [x_{i_2}, x_{i_2+1}] \mathrm{ \ of \ size \ } n_2 + 1) \cap \{\mathrm{no \ split \ in \ } [x_{i_2+1}, t]\}\right\}.

Using propositions 2, 3 and 4 we conclude that

$$P\left[\left(\bigcap_{i=1}^{2} (X_i = n_1)\right)\right] = \lim_{m \to \infty} \sum_{i_1 = 0}^{m} \sum_{i_2 = i_1 + 1}^{m} \left[\exp - \int_{0}^{x_{i_1}} \lambda_N(x)dx\right][N\lambda_{n_1+1, n_1}^N(x_{i_1})\Delta(i_1) + (\Delta(i_1))]

Q(\Delta(i_1))\left[\exp - (N-1)\int_{x_{i_1}}^{x_{i_1+1}} \lambda_N(x)dx\right] \int_{x_{i_1}}^{x_{i_2}} \left[\exp - (n_1+1) \int_{x_{i_1}}^{x_{i_1+1}} \lambda_N(x-t_1)dx\right] \left[\exp - (n_2+1) \int_{x_{i_2}}^{x_{i_2+1}} \lambda_N(x-t_2)dx\right] \left[\exp - (N-1)\int_{x_{i_2}}^{x_{i_2+1}} \lambda_N(x-v_1)dx\right].

As $m \to \infty$ and $\max(\left\{\Delta(i_1)\right\}) \to 0$ using propositions 3 and 4 we obtain

$$P\left[\left(\bigcap_{i=1}^{2} (X_i = n_1)\right)\right] = \int_{t_1}^{t_2} \left[\exp - \int_{0}^{t_1} \lambda_N(x)dx\right][N\lambda_{n_1+1, n_1}^N(t_1)] \left[\exp - \int_{t_1}^{t_2} \lambda_N(x-t_1)dx\right] \left[\exp - (N-1)\int_{t_1}^{t_2} \lambda_N(x-t_2)dx\right].$$
\[
\left( \exp \left[ - (N-2) \int_{t_2}^{t} \lambda_N(x) \, dx - (n_1+1) \int_{t_2}^{t} \lambda_N(x-t_1) \, dx - (n_2+1) \int_{t_2}^{t} \lambda_N(x-t_2) \, dx \right] \right)^{13}
+ \left[ (n_1+1) \lambda_{n_2+1,N}(t_2-t_1) \right] \left( \exp \left[ - (N-1) \int_{t_2}^{t} \lambda_N(x) \, dx - n_1 \int_{t_2}^{t} \lambda_N(x-t_1) \, dx - (n_2+1) \int_{t_2}^{t} \lambda_N(x-t_2) \, dx \right] \right) \right) \, dt_2 \, dt_1.
\]

Using the Lebesgue dominated convergence theorem and noticing that \( \lambda_N(x) \) and \( \lambda_m,N(x) \to 0 \) as \( N \to \infty \) uniformly over every bounded interval, and \( N \lambda_N(x) \to \lambda(x) \) and \( N \lambda_m,N(x) \to \lambda_m(x), m = 0,1,2, \ldots \), uniformly over every bounded interval, we thus obtain

\[
\lim_{N \to \infty} \mathbb{P} \left[ \bigcap_{i=1}^{2} (X_i = n_i) \right] = \int_{t_1=0}^{t} \int_{t_2=t_1}^{t} \left\{ \exp \left[ - \int_{t_1}^{t} \lambda(x) \, dx - \int_{t_1}^{t} \lambda(x) \, dx - \int_{t_2}^{t} \lambda(x) \, dx \right] \right\} \lambda_{n_1+1}(t_1) \lambda_{n_2+1}(t_2) \, dt_2 \, dt_1,
\]

\[
= (\exp - \Lambda(t)) \int_{t_1=0}^{t} \int_{t_2=t_1}^{t} \lambda_{n_1+1}(t_1) \lambda_{n_2+1}(t_2) \, dt_2 \, dt_1.
\]

Therefore we can see that for any integer \( k \geq 0 \)

\[
\lim_{N \to \infty} \mathbb{P} \left[ \bigcap_{i=1}^{k} (X_i = n_i) \right] = (\exp - \Lambda(t)) \int_{t_1=0}^{t} \int_{t_2=t_1}^{t} \cdots \int_{t_k=t_{k-1}}^{t} \prod_{i=1}^{k} \lambda_{n_i+1}(t_i) \, dt_k \, dt_{k-1} \cdots dt_1.
\]

To prove (11) we use the fact that
\[ P(Y_N(t) = k) = \sum_{n_1 = -1}^{\infty} \sum_{n_2 = -1}^{\infty} \sum_{n_k = -1}^{\infty} P \left[ (Y_N(t) = k) \cap (X_i = n_i) \right] . \]

Using the properties of double series of non-negative terms and an Abelian argument we can very easily prove that

\[ \lim_{N \to \infty} P(Y_N(t) = k) = \sum_{n_1 = -1}^{\infty} \sum_{n_k = -1}^{\infty} \lim_{N \to \infty} P \left[ (Y_N(t) = k) \cap (X_i = n_i) \right] \]

\[ = \sum_{n_1 = -1}^{\infty} \sum_{n_k = -1}^{\infty} (\exp - \Lambda(t)) \int_{t_1=0}^{t} \int_{t_2=t_1}^{t} \cdots \int_{t_k=t_{k-1}}^{t} \prod_{i=1}^{k} \lambda_{n_i+1}(t_i) dt_{k} \cdots dt_1. \]

Using (6) we can see that

\[ \lim_{N \to \infty} P(Y_N(t) = k) = (\exp - \Lambda(t)) \int_{t_1=0}^{t} \int_{t_2=t_1}^{t} \cdots \int_{t_k=t_{k-1}}^{t} \prod_{i=1}^{k} \lambda(t_i) dt_{k} \cdots dt_1 \]

\[ = [\exp - \Lambda(t)][\Lambda(t)]^k/k! . \]

From this and (13) we get (12) in an obvious way.

The next theorem is a generalization of theorem 1. Let

0 = t_0 < t_1 < t_2 < \ldots < t_m be any increasing sequence of non-negative numbers,

let \( X_{ij} \) denote the change in the size of the population that results from

the jth splitting that takes place during \([t_{i-1}, t_i]\).

**THEOREM 2.** Given any set of non-negative integers \( \{k_i\} \), \( 1 \leq i \leq m \), and any

set of integers \( \{n_{ij}\} \) where \( n_{ij} \geq -1 \), \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, k_i \), we

have
\[
\lim_{N \to \infty} P\left( \bigcap_{i=1}^{m} \left[ Y_N(t_i) - Y_N(t_{i-1}) = k_i \right] \right)
= \exp \left( -\Lambda(t_m) \prod_{i=1}^{m} \left[ \Lambda(t_i) - \Lambda(t_{i-1}) \right] \right) \frac{k_i}{k_1}
\]

and

\[
\lim_{N \to \infty} P\left( \bigcap_{i=1}^{m} \bigcap_{j=1}^{k_i} \left[ X_{ij} = n_{ij} \right] \bigcap_{i=1}^{m} \left[ Y_N(t_i) - Y_N(t_{i-1}) = k_i \right] \right)
= \prod_{i=1}^{k_i} \left[ \Lambda(t_i) - \Lambda(t_{i-1}) \right] \int_{0}^{t_1} dx_{11} \int_{x_{11}}^{t_1} dx_{12} \cdots \int_{x_{1k_i-1}}^{t_1} dx_{1k_i} \int_{t_1}^{t_2} dx_{21} \cdots \int_{x_{2k_2-1}}^{t_2} dx_{2k_2}
\]

\[
\cdots \int_{t_m}^{t_m} dx_{m1} \int_{x_{m1}}^{t_m} dx_{m2} \cdots \int_{x_{mk_m-1}}^{t_m} dx_{mk_m} \prod_{i=1}^{k_i} \prod_{j=1}^{n_{ij}+1} (x_{ij}) dx_{mk_m}
\]

**PROOF.** As we did in Theorem 1, to prove (14) we first need to prove that

\[
\lim_{N \to \infty} P\left( \bigcap_{i=1}^{m} \left[ Y_N(t_i) - Y_N(t_{i-1}) = k_i \right] \bigcap_{j=1}^{k_i} \left[ X_{ij} = n_{ij} \right] \right)
= \left( \exp \left( -\Lambda(t_m) \right) \right) \int_{0}^{t_1} dx_{11} \int_{x_{11}}^{t_1} dx_{12} \cdots \int_{x_{1k_i-1}}^{t_1} dx_{1k_i} \int_{t_1}^{t_2} dx_{21} \cdots \int_{x_{2k_2-1}}^{t_2} dx_{2k_2}
\]

\[
\cdots \int_{t_m}^{t_m} dx_{m1} \int_{x_{m1}}^{t_m} dx_{m2} \cdots \int_{x_{mk_m-1}}^{t_m} dx_{mk_m} \prod_{i=1}^{k_i} \prod_{j=1}^{n_{ij}+1} (x_{ij}) dx_{mk_m}
\]

We shall prove this result only in the case \( m = 2 \) and \( k_i = 2 \) for \( i = 1, 2 \).

The proof follows the same arguments for \( m > 2 \) and general values for the set
of non-negative integers \( \{k_i\} \) and only involves more complicated notations. By an exactly similar argument to the one used in Theorem 1 we will be able to show that

\[
\lim_{N \to \infty} P \left( \bigcap_{i=1}^{2} \left( [Y_N(t_i) - Y_N(t_{i-1}) = 2] \cap \bigcap_{j=1}^{2} [X_{ij} = n_{ij}] \right) \right)
\]

\[
= \left[ \exp - \int_0^{t_2} \lambda(x)dx \right] \int_0^{t_2} dx_{11} \int_0^{t_1} dx_{12} \int_0^{t_2} dx_{21} \int_0^{t_2} dx_{22}
\]

\[
\lambda_{n_{11}+1}(x_{11}) \lambda_{n_{12}+1}(x_{12}) \lambda_{n_{21}+1}(x_{21}) \lambda_{n_{22}+1}(x_{22}) dx_{22}
\]

\[
= \left[ \exp - \Lambda(t_2) \right] \int_0^{t_1} dx_{11} \int_0^{t_1} dx_{12} \int_0^{t_2} dx_{21} \int_0^{t_2} dx_{22} \int_{i=1}^{2} \int_{j=1}^{2} \lambda_{n_{ij}+1}(x_{ij}) dx_{22}
\]

Therefore (16) is true for \( m = 2 \) and \( k_i = 2 \) for \( i = 1, 2 \). For any \( m \geq 2 \) and a general set of non-negative integers \( \{k_i\}, i = 1, 2, \ldots, m \), the proof of (16) is now obvious.

To prove (14) from (16) we apply the same argument used in Theorem 1 to prove (11) from (13). Finally (15) follows from (14) and (16) in an obvious way.

THEOREM 3. Assuming the model defined in section 2, the sequence of stochastic processes \( \{X_N(t) - N\} \) converges in distribution to a stochastic process with independent increments \( X(t) \), whose characteristic function is given by

\[
\phi(u) = \exp \int_0^t \sum_{n=0}^{\infty} (e^{iu(n-1)} - 1) \lambda_n(x) dx,
\]

where \( \lambda_n(x) \) is given by (4).
PROOF. Let $0 = t_0 < t_1 < t_2 < \ldots < t_m < \ldots$ be any increasing sequence of non-negative numbers, and let $U_{N,r} = X_N(t_r) - X_N(t_{r-1})$. If $f_N(u_1, u_2, \ldots, u_m)$ is the joint characteristic function of $U_{N,1}, U_{N,2}, \ldots, U_{N,m}$, then

$$f_N(u_1, u_2, \ldots, u_m) = E \exp i \sum_{r=1}^{m} u_r U_{N,r}$$

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \left\{ \left( \sum_{r=1}^{m} u_r U_{N,r} \right)^{k_r} \right\} \mathbb{E} \left[ \left( \exp i \sum_{r=1}^{m} u_r U_{N,r} \right)^{k_r} \right]$$

$$\cap_{r=1}^{m} \left[ Y_N(t_r) - Y_N(t_{r-1}) = k_r \right] \cap_{j=1}^{m} \left[ X_{rj} = n_{rj} \right]$$

$$= P \left( \cap_{r=1}^{m} \left[ Y_N(t_r) - Y_N(t_{r-1}) = k_r \right] \cap_{j=1}^{m} \left[ X_{rj} = n_{rj} \right] \right)$$

where the sum $\sum^\infty_{k_1=0} \cdots \sum^\infty_{k_m=0}$ is taken over all the integers $n_{rj} \leq -1$, $1 \leq j \leq k_r$, and $1 \leq r \leq m$.

Thus

$$f_N(u_1, \ldots, u_m) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \left\{ \left( \sum_{r=1}^{m} u_r \sum_{j=1}^{k_r} n_{rj} \right)^{k_r} \right\} \mathbb{E} \left[ \left( \exp i \sum_{r=1}^{m} u_r \sum_{j=1}^{k_r} n_{rj} \right)^{k_r} \right]$$

$$\cap_{r=1}^{m} \left[ Y_N(t_r) - Y_N(t_{r-1}) = k_r \right] \cap_{j=1}^{m} \left[ X_{rj} = n_{rj} \right]$$

$$\cap_{r=1}^{m} \left[ Y_N(t_r) - Y_N(t_{r-1}) = k_r \right]$$

By (15) of Theorem 2, the Helly-Bray theorem and by (6) we have
\begin{align*}
\lim_{N \to \infty} & \sum_{r=1}^{m} \exp \left[ i \sum_{r=1}^{m} \left( \sum_{j=1}^{k_r} u_{r,j} \binom{k_r}{n_{r,j}} \right) \prod_{r=1}^{m} \left[ \sum_{j=1}^{k_r} X_{r,j} = n_{r,j} \right] \prod_{r=1}^{m} \left[ Y_r(t_r) - Y_r(t_{r-1}) = k_r \right] \right] \\
& = \prod_{r=1}^{m} \frac{k_r!}{[\Lambda(t_r) - \Lambda(t_{r-1})]} \prod_{r=1}^{m} \left[ \varnothing(u_r, t_r) - \varnothing(u_r, t_{r-1}) \right]^{k_r/k_r!} \\
& = \prod_{r=1}^{m} \frac{k_r!}{[\Lambda(t_r) - \Lambda(t_{r-1})]} \prod_{r=1}^{m} \left[ \varnothing(u_r, t_r) - \varnothing(u_r, t_{r-1}) \right]^{k_r/k_r!},
\end{align*}

where

\[ \varnothing(u, t) = \int_{0}^{t} \sum_{n=-\infty}^{\infty} e^{iun} \lambda_{n+1}(x) dx. \]

Thus by lemma 2 and theorem 2 we obtain

\[ \lim_{N \to \infty} f_N(u_1, u_2, \ldots, u_m) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \exp - \Lambda(t) \prod_{r=1}^{m} \left[ \varnothing(u_r, t_r) - \varnothing(u_r, t_{r-1}) \right]^{k_r/k_r!} \]

\[ = \exp - \Lambda(t) \prod_{r=1}^{m} \exp \left[ \varnothing(u_r, t_r) - \varnothing(u_r, t_{r-1}) \right] \]

\[ = \prod_{r=1}^{m} \exp \left\{ \left( \varnothing(u_r, t_r) - \Lambda(t_r) \right) - \left( \varnothing(u_r, t_{r-1}) - \Lambda(t_{r-1}) \right) \right\} \]

\[ = \prod_{r=1}^{m} f_r(u_r) \]

where

\[ f_r(u_r) = \exp \left\{ \left( \varnothing(u_r, t_r) - \Lambda(t_r) \right) - \left( \varnothing(u_r, t_{r-1}) - \Lambda(t_{r-1}) \right) \right\}. \]
Let \( X(t) \) be a stochastic process with independent increments whose characteristic function is given by
\[
\phi(u) = \exp[\theta(u, t) - \Lambda(t)] = \exp \int_0^t \sum_{n=-1}^\infty (e^{iu_n} - 1)\lambda_{n+1}(x) dx.
\]

If \( U_r = X(t_r) - X(t_{r-1}) \), and if \( f(u_1, u_2, \ldots, u_m) \) denotes the characteristic function of \( U_1, U_2, \ldots, U_m \), then \( f(u_1, u_2, \ldots, u_m) = \prod_{r=1}^m f_r(u_r) \).

Thus
\[
\lim_{N \to \infty} f_N(u_1, \ldots, u_m) = f(u_1, u_2, \ldots, u_m),
\]
thus establishing the result that the joint distribution of the sequence \( \{U_N, 1, \ldots, U_N, 2, \ldots, U_N, m\} \) of \( m \)-dimensional random variables converges to the distribution of \( U_1, U_2, \ldots, U_m \). Lemma (4) establishes the required result that the joint distribution of \( \{X_{N_1}(t_1), \ldots, X_{N_m}(t_m)\} \) converges to the joint distribution of \( \{X(t_1), \ldots, X(t_m)\} \).

Remarks. If we consider the model in which the conditional probabilities given by (1) and (3) depend only on the absolute time and are independent of the time the particle came into existence, we can show that the stochastic process \( X(t) - N \) converges in distribution to the same stochastic process as in our model. Also if we assume that \( \lambda_{k,N}(x) = \lambda_k \phi_N(x) \) for \( k = 0, 1, \ldots \), and \( \lambda_N(x) = \lambda \phi_N(x) \), and \( \lim_{N \to \infty} \lambda_k = \nu_k \) for \( k = 0, 1, 2, \ldots \), and
\[
\lim_{N \to \infty} \nu_N = \nu = \sum_{k=0}^\infty \nu_k, \quad \text{and} \quad \lim_{N \to \infty} \phi_N(x) = \phi(x) \quad \text{uniformly over every bounded interval,}
\]
in this special case the model considered in this paper reduces to the model considered in [14]. Also if we assume that \( \phi_N(x) \) depends on
absolute time only and $\lambda_k$ are independent of the time the particle came into existence, we can see again that our general model reduces to the one considered in [15]. In this respect our proof for theorem 3 gives a correct proof to theorem 2 in [15].
REFERENCES


