PROBABILITY INEQUALITIES AND CONVERGENCE PROPERTIES
FOR SUMS OF MULTIPLICATIVE RANDOM VARIABLES

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FSU Statistics Report M151

February, 1969
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Research supported in part by NSF Grant GZ-315.
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0. **Summary.** The class of sequences \( \{X_i\} \) of multiplicative rv's includes sequences of martingale differences and is an important subclass of sequences of mutually orthogonal rv's. This paper gives useful bounds on the moments of \( S_n = \sum_{i=1}^{n} X_i \) and of \( M_n = \max \{|S_1|, \ldots, |S_n|\} \), and on the probability \( P[|S_n| > t] \), and employs these results to investigate convergence properties of \( S_n \) \((n \to \infty)\).

1. **Introduction.** Let \( \{X_i\}_{i=1}^{\infty} \) be a sequence of rv's having finite variances. Assume throughout (without loss of generality) that \( E(X_i) = 0 \). It is not assumed that the \( X_i \)'s are mutually independent or that they are identically distributed. Under certain broad and useful relaxations of these assumptions, the sums

\[
S_{a,n} = \sum_{a+1}^{\infty} X_i
\]

shall be shown to retain some important properties that are true under the more stringent assumptions but are far from holding generally.

The properties will consist of probability inequalities for any sum \( S_{a,n} \) and convergence properties of any sequence \( \{S_{a,n}\}_{a=1}^{\infty} \). It will suffice to prove statements about the sums \( S_{0,n} \) \((n=1, 2, \ldots)\), which shall be denoted simply by \( S_n \) \((n=1, 2, \ldots)\).

It shall be assumed, in lieu of a common distribution, that the \( X_i \)'s have uniformly bounded r-th absolute moments \( (E|X_i|^r \leq M, \text{ all } i) \)

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1 Research supported in part by NSF Grant GZ-315.
for some $r \geq 2$. Instead of mutual independence, it shall be assumed that the $X_i$'s are multiplicative of order $s$,

$$E(X_{i_1} X_{i_2} \cdots X_{i_k}) = 0 \text{ if } 1 \leq k \leq s \text{ and } i_1 < \ldots < i_k,$$

for an integer $s \geq 2$. Further conditions on the values of $r$ and $s$ will be imposed. Provided that the indicated expectations exist (finite), all restrictions will be relaxations of the assumptions of identical distributions and mutual independence.

A collection multiplicative of order $s$ is also multiplicative of orders $\leq s$. A collection multiplicative of all orders is called a multiplicative system. A collection multiplicative of order 2 is better known as a collection of mutually orthogonal rv's.

Thus the condition "multiplicative of order $s"$, where $2 \leq s \leq \infty$, denotes a special case of mutual orthogonality but, on the other hand, is easily seen to be weaker than the requirement that $\{X_i\}$ be a sequence of martingale differences,

$$E(X_i | X_{i-1}, X_{i-2}, \ldots) = 0 \quad (\text{all } i).$$

Moreover, condition (1.1) has the advantage of not involving conditional expectations. Given that the expectations involved exist, we have that the dependence restrictions mentioned are related as follows:

(mutual independence) $\Rightarrow$ (martingale differences) $\Rightarrow$ (multiplicative system) $\Rightarrow$ (multiplicative of order $s$, any $s \geq 2$) $\Rightarrow$ (mutual orthogonality).

Therefore, the results below proved for multiplicative rv's hold also for sequences of martingale differences as well as for special varieties of multiplicative rv's (e.g., "strongly multiplicative" and
"equinormed strongly multiplicative". For background on martingale differences and orthogonal rv's see Doob [3] and on multiplicative rv's and special varieties thereof see Révész [6].

Section 2 deals with moments of \( |S_n| \) and of \( M_n = \max(|S_1|, \ldots, |S_n|) \). It is shown, under appropriate restrictions, that \( E|S_n|^{\nu} = o(n^\frac{\nu}{2}) \), \( n \to \infty \). By results in [7], a similar condition then follows for \( E(M_n^{\nu}) \). The latter condition plays a crucial role in Section 4.

Section 3 extends results of Hoeffding [4]. In particular, it is shown that for a uniformly bounded \( (|X_i| \leq B) \) multiplicative system, \( P(|S_n| > nt) \leq 2 e^{-nt^2/2B^2} \) (all \( n \geq 1 \)).

In Section 4 the results of previous sections are utilized to deduce convergence properties of \( S_n \). In particular, a law of the iterated logarithm obtained in Révész [6] for a uniformly bounded equinormed strongly multiplicative system is sharpened and generalized to arbitrary uniformly bounded multiplicative systems.

2. The moments of \( |S_n| \) and of \( M_n = \max(|S_1|, \ldots, |S_n|) \). Given arbitrary rv's \( X_i \), and \( \nu > 1 \), the condition

\[
E|X_i|^{\nu} \leq K \quad (\text{all } i)
\]

implies by Minkowski's inequality that

\[
E|S_n|^{\nu} = O(n^{\nu}) \quad , \quad n \to \infty .
\]

If, in addition to (2.1), it is assumed that the \( X_i \)'s are mutually independent, then (2.2) may be improved to

\[
E|S_n|^{\nu} = O(n^{\frac{\nu}{2}}) \quad , \quad n \to \infty .
\]
Typically (2.3) is sharp. (For example, when the $X_i$'s are mutually independent and have a common variance $\sigma^2$ and satisfy (2.3) for some $\nu > 2$, then it is clear by Hölder's inequality that (2.3) is sharp because $E S^2_n = \sigma^2 n$.) Even though (2.3) is much sharper than (2.2), independence is not necessary for (2.3) to hold. Nevertheless (2.3) does constitute a restriction on the possible dependence in the sequence $\{X_i\}$. Further details and references are available in [7] and [9].

**Theorem 2.1.** Let $\nu$ be an even integer. Suppose that $\{X_i\}$ is multiplicative of order $\nu$ and that $E|X_i|^\nu < \infty$ (all $i$).

Then there exists a constant $A < \infty$ such that

$$E|S_{a,n}^{\nu}| \leq An^{\frac{1}{\nu}} \quad (\text{all } a, \text{ all } n \geq 1).$$

**Proof:** For $\nu = 0$ or $\nu = 2$ the proof is trivial. Assume $\nu > 2$.

We shall demonstrate (2.4) for the case $a = 0$ and it shall be seen that the constant $A$ does not depend upon the choice of $a$, so that (2.4) for all $a$ will follow. Write

$$S_n^{\nu} = \sum_{i_1=1}^{n} \cdots \sum_{i_\nu=1}^{n} X_{i_1} \cdots X_{i_\nu}$$

$$= \sum_{i_1=1}^{n} \cdots \sum_{i_{\nu-1}=1}^{n} X_{i_1} \cdots X_{i_{\nu-1}} (X_{i_1} + \cdots + X_{i_{\nu-1}} + \sum_{i_\nu=1}^{n} X_{i_\nu})$$

$$\sum_{i_1, \ldots, i_{\nu-1}}$$

(2.5) = $(\nu-1) S_n^{\nu-2} \sum_{i=1}^{n} X_i^2 + \sum_{i_1=1}^{n} \cdots \sum_{i_{\nu-1}=1}^{n} \sum_{i_\nu=1}^{n} X_{i_1} \cdots X_{i_{\nu-1}} X_{i_\nu}$.

Denote the right-most term of (2.5) by $A_n$. Then
\[ A_n = \sum_{i_1=1}^{n} \sum_{i_{v-2}=1}^{n} x_{i_1} \cdots x_{i_{v-2}} (x_{i_{1}} + \cdots + x_{i_{v-2}}) \sum_{i_{v-1} \neq \#}^{n} x_{i_{v-1}} \sum_{i_1, \ldots, i_{v-2}}^{i_1, \ldots, i_{v-2}} \]

\[ (2.6) = \sum_{i_1=1}^{n} \sum_{i_{v-2}=1}^{n} x_{i_1} \cdots x_{i_{v-2}} (x_{i_{1}} + \cdots + x_{i_{v-2}}) \sum_{i_1, \ldots, i_{v-2}}^{i_1, \ldots, i_{v-2}} \sum_{i_{v-1} \neq \#}^{n} x_{i_{v-1}} \]

\[ + \sum_{i_1=1}^{n} \sum_{i_{v-2}=1}^{n} \sum_{i_{v-1} \neq \#}^{n} \sum_{i_{v} \neq \#}^{n} x_{i_1} \cdots x_{i_{v}}. \]

The first term in the right-hand side of (2.6) may be reduced to

\[ (2.7) (v-2)S_{\nu-2}^{\nu-2} \sum_{i=1}^{n} x_{i}^{2} - (v-2)S_{\nu-3}^{\nu-3} \sum_{i=1}^{n} x_{i}^{3} - (v-2)(v-3)S_{\nu-4}^{\nu-4} (\sum_{i=1}^{n} x_{i}^{2})^{2}. \]

Hence we have

\[ S_{\nu} = (2\nu-3)S_{\nu-2}^{\nu-2} \left( \sum_{i=1}^{n} x_{i}^{2} \right) - (v-2)S_{\nu-3}^{\nu-3} \left( \sum_{i=1}^{n} x_{i}^{3} \right) - (v-2)(v-3)S_{\nu-4}^{\nu-4} (\sum_{i=1}^{n} x_{i}^{2})^{2} \]

\[ (2.8) + \sum_{i_1=1}^{n} \sum_{i_{v-2}=1}^{n} \sum_{i_{v-1} \neq \#}^{n} \sum_{i_{v} \neq \#}^{n} x_{i_1} \cdots x_{i_{v}}. \]

Continuing this process of reduction, we may express \( S_{\nu}^{\nu} \) as a finite sum of terms, the number of terms depending only on \( \nu \), in which one term is

\[ (2.9) S_{n}^{\nu} = \sum_{i_1}^{\cdots} \sum_{i_{v}}^{\cdots} x_{i_1} \cdots x_{i_{v}^{\nu}}, \]

all distinct
and the other terms are of the form

\[(2.10) \quad + C S_n^{\nu-k} \left( \sum_{i=1}^{n} X_i^{r_{1_i}} \right) \cdots \left( \sum_{i=b}^{n} X_i^{r_{b_i}} \right), \]

where \(C > 0\) is a constant depending only upon \(\nu\), \(b\) is an integer \(\geq 1\), the \(r_{1_i}\) are integers \(\geq 2\), and \(r_{1_i} + \cdots + r_{b_i} = k\). (Thus \(k\) is an integer \(\geq 2\).)

By the assumption that \(\{X_i\}\) is multiplicative of order \(\nu\), we have

\[(2.11) \quad E(S_n^\nu) = 0. \]

Now let \(T\) be a term of the form (2.10). Then, by Hölder's inequality,

\[(2.12) \quad E|T| \leq C(E|S_n^\nu|) \nu \left( E\left| \sum_{i=1}^{n} X_i^{r_{1_i}} \right|^\nu \right)^{1/\nu} \cdots \left( E\left| \sum_{i=1}^{b} X_i^{r_{b_i}} \right|^\nu \right)^{1/\nu}.

By Minkowski's inequality, for \(1 \leq j \leq b\),

\[(2.13) \quad E\left| \sum_{i=1}^{n} X_i^{r_{1_i}} \right|^j \leq \left[ \sum_{i=1}^{n} \left( E|X_i^\nu| \right)^j \right]^{1/j}.

Therefore, by the uniform boundedness of \(E|X_i^\nu|\),

\[(2.14) \quad E|T| \leq C(E|S_n^\nu|) \nu \frac{k}{n^\nu b}.

Since \(r_{1_i} \geq 2\) \((1 \leq i \leq b)\), we have \(b \leq \frac{1}{k} k\) and thus \(n^{b+\frac{2}{\nu}(\nu-k)} \leq n^{\frac{k}{\nu}}\).
It follows that

\[(2.15) \quad n^{-\frac{1}{2} \nu} E|T| \leq C n^{\nu} (n^{-\frac{1}{2} \nu} E|S_n^\nu|)^{\nu-k}.\]

Considering again the expression of \(S_n^\nu\) as a sum of \(S_n^r\) plus terms like (2.10), we see by (2.11) and (2.14) that

\[(2.16) \quad n^{-\frac{1}{2} \nu} E(S_n^\nu) \leq \sum_{k=2}^{\nu} \bar{w}_k \left(n^{-\frac{1}{2} \nu} E|S_n^\nu|\right)^{\nu-k},\]

where each coefficient \(\bar{w}_k\) is non-negative and depends only upon \(\nu\).

Since \(\nu\) is even, the left-hand side of (2.16) is the same as

\[n^{-\frac{1}{2} \nu} E|S_n^\nu|\]

and we have

\[(2.17) \quad 1 \leq \sum_{k=2}^{\nu} \bar{w}_k \left(n^{-\frac{1}{2} \nu} E|S_n^\nu|\right)^{-k/\nu}.\]

Therefore, the quantity \(n^{-\frac{1}{2} \nu} E|S_n^\nu|^{-1}\) is bounded away from 0 as \(n \to \infty\), which is to say

\[(2.18) \quad E|S_n^\nu| = o(n^{\frac{1}{2} \nu}), \quad n \to \infty.\]

Since the relation (2.17) does not depend upon the fact that the sequence \(\{S_n\}\) was considered instead of \(\{S_{a,n}\}\), neither does the right-hand side of (2.18). Hence (2.4) holds.

It should be noted that condition (2.4) implies, by Hölder's inequality, a similar condition for moments of orders \(\leq \nu\).

The hypothesis of Theorem 2.1 also implies useful conclusions about the rv

\[(2.19) \quad M_{a,n} = \max \{|S_{a,1}|, \ldots, |S_{a,n}|\}.\]
When \( \nu > 2 \), a conclusion of form (2.4) holds with \( M_{a,n} \) in place of \( |S_{a,n}| \). Also, for \( \nu \geq 2 \), a modified conclusion holds which is less sharp in asymptotic order of magnitude but gives more specific information for any fixed value of \( n \). These conclusions follow by Theorem 2.1 in conjunction with the following result, which is an immediate consequence of Theorems A and B of [8].

**Lemma 2.1.** Let \( \nu > 2 \). Suppose that, for an \( A < \infty \),

\[
E|S_{a,n}|^\nu \leq An^{\frac{k}{2}\nu} \quad \text{(all } a, \text{ all } n \geq 1) .
\]

Then we have

\[
E(M_{a,n}^{\nu}) \leq A(\log_2 2n)^\nu n^{\frac{k}{2}\nu} \quad \text{(all } a, \text{ all } n \geq 1) .
\]

Also, if \( \nu > 2 \), there exists a constant \( K < \infty \) such that

\[
E(M_{a,n}^{\nu}) \leq Kn^{\frac{k}{2}\nu} \quad \text{(all } a, \text{ all } n \geq 1) .
\]

**Theorem 2.2.** Let \( \nu \) be an even integer \( \geq 2 \). Suppose that \( \{X_i\} \) is multiplicative of order \( \nu \) and that \( E|X_i|^\nu \leq M < \infty \) (all \( i \)). Then (2.21) holds for an \( A < \infty \). Also, if \( \nu > 2 \), then (2.22) holds for a \( K < \infty \).

Note that the implication of Theorem 2.2 in the case \( \nu = 2 \) also follows directly from the Rademacher-Mensov inequality [6].

**Corollary 2.2.1.** Let \( \{X_i\} \) be mutually orthogonal rv's with \( E(X_i^2) \leq M < \infty \) (all \( i \)). Then

\[
E(M_{a,n}^{2}) \leq M(\log_2 2n)^2 n \quad \text{(all } a, \text{ all } n \geq 1) .
\]
3. **Probability inequalities for \( S_n \).** The treatment in this section relies heavily upon Hoeffding [4], to which the reader is referred for background, details, references and other results. The inequalities in the following theorem are given for mutually independent \( X_i \)'s in Theorem 1 of [4], where it is noted that a slight modification of the proof would allow \( \{X_i\} \) to be a sequence of martingale differences. Here we further relax the dependence restrictions.

**Theorem 3.1.** Let \( \{X_i\} \) be a multiplicative system, with \( a \leq X_i \leq b \) (all \( i \)), where \( a < 0 < b \). Then, for \( 0 < t < b \),

\[
(3.1) \quad P[S_n \geq nt] \leq \left\{ \frac{\exp(-at)}{t-a} \right\} nt \leq e^{-nt^2} g(a,b) \\
(3.2) \quad \leq e^{-nt^2/(b-a)^2}
\]

where

\[
(3.4) \quad g(a,b) = \frac{b-a}{b+a} \ln \frac{b}{-a} \quad \text{if} \quad -a < b
\]

\[
= \frac{(b-a)^2}{-2ab} \quad \text{if} \quad -a \geq b.
\]

**Proof:** For any rv \( Y \) and constant \( h > 0 \),

\[
(3.5) \quad P[Y \geq 0] \leq e^{hY}.
\]

Hence, for \( h > 0 \),

\[
(3.6) \quad P[S_n \geq nt] \leq e^{-ht} e^{hS_n},
\]

without restriction on the dependence among the \( X_i \)'s.

By convexity of the exponential function and since \( a \leq X_i \leq b \), we have
(3.7) \[ e^{\frac{hX_i}{1}} \leq A_h + B_h X_i, \]

where \( A_h = (b e^{h a} - a e^{h b})/(b - a) \) and \( B_h = (e^{h b} - e^{h a})/(b - a) \). Thus

(3.8) \[ e^{\frac{hS}{n}} \leq \prod_{i=1}^{n} (A_h + B_h X_i) \]

and multiplicativity of \( \{X_i\} \) then yields

(3.9) \[ E e^{\frac{hS}{n}} \leq A_h^n. \]

Therefore, for \( h > 0 \),

(3.10) \[ P[S_n \geq nt] \leq (e^{-ht} A_h)^n. \]

The right-hand side of (3.10) is minimum for \( h = h_0 \), where

(3.11) \[ h_0 = \frac{1}{a-b} \ln \frac{a(t-b)}{b(t-a)}, \]

which is positive for \( 0 < t < b \). Putting \( h = h_0 \) in (3.10) we obtain inequality (3.1). Inequalities (3.2) and (3.3) follow as shown in [4].

The following simple corollary is useful.

**Corollary 3.1.1.** Let \( \{X_i\} \) be a multiplicative system, with \( |X_i| \leq b \) (all \( i \)). Then, for \( 0 < t < b \),

(3.12) \[ P[S_n \geq nt] \leq e^{-nt^2/2b^2} \]

and hence also

(3.13) \[ P[|S_n| \geq nt] \leq 2e^{-nt^2/2b^2}. \]
4. Convergence properties of $S_n$. The results of the previous sections shall be applied to obtain laws of the iterated logarithm for sums of multiplicative rv's, to state convergence rates thereof, and to choose constants $a_n$ such that $S_n/a_n$ converges completely to zero.

First, we make use of Theorem 2.2 to sharpen a theorem of Révész [6] pertaining to equinormed strongly multiplicative systems (ESMS).

An ESMS is a (multiplicative) system $\{X_i\}$ satisfying

\begin{align}
(4.1) & \quad E(X_1) = 0, \ E(X_1^2) = 1, \\
(4.2) & \quad E(X_1^{r_1} \cdots X_k^{r_k}) = E(X_1^{r_1}) \cdots E(X_k^{r_k}) \quad (k=1,2,\ldots),
\end{align}

where $r_1,\ldots,r_k$ may equal 1 or 2. Of course, mutually independent rv's satisfying (4.1) form an ESMS. Also, the sequence $\{\sqrt{2} \sin c_n X\}_{n=1}^{\infty}$ is an ESMS if $c_{n+1}/c_n \geq 3$ and $X$ is distributed on the interval $[0,2\pi]$ by the probability measure $P(A) = \lambda(A)/2\pi$ on Borel subsets $A$ of $[0,2\pi]$, where $\lambda$ is the Lebesque measure. The theorem proved by Révész ([6], 90) states that the lim sup in (4.3) below is ≤ 7 and it is remarked by Révész that the constant 7 is surely not the best possible. From the law of the iterated logarithm for mutually independent and identically distributed rv's satisfying (4.1), it is clear that the best constant cannot be less than $\sqrt{2}$. The proof below follows that of Révész except for the use of Theorem 2.2, statement (2.22), at a crucial point and for one other modification.
Theorem 4.1. Let \( \{X_i\} \) be a uniformly bounded ESMS. Then with probability 1

\[
(4.3) \quad \lim_{n \to \infty} \sup \frac{|S_n|}{(n \ln \ln n)^{\frac{1}{2}}} = \sqrt{2}.
\]

**Proof:** Let \( \varepsilon > 0 \). Define

\[
(4.4) \quad \lambda_n = \left( \frac{2 \ln \ln n}{n} \right)^{\frac{1}{2}},
\]

\[
(4.5) \quad \mu_n = 2(1+2\varepsilon) \ln \ln n.
\]

We shall use the inequality

\[
(4.6) \quad e^x \leq 1 + x + \frac{1}{2}(1+\varepsilon) x^2, \quad |x| \text{ sufficiently small.}
\]

For \( n \) sufficiently large, we have by the uniform boundedness of the \( X_i \)'s and the inequality (4.6) that

\[
(4.7) \quad \begin{aligned}
E(e^{\lambda_n S_n - \mu_n}) &= e^{-\mu_n} \prod_{i=1}^{n} E(e^{\lambda_n X_i}) \\
&\leq e^{-\mu_n} E( \prod_{i=1}^{n} [1 + \lambda_n X_i + \frac{1}{2}(1+\varepsilon) \lambda_n^2 X_i^2] )
\end{aligned}
\]

and hence by the ESMS properties (4.1) and (4.2) that

\[
E(e^{\lambda_n S_n - \mu_n}) \leq e^{-\mu_n} [1 + \frac{1}{2}(1+\varepsilon) \lambda_n^2 n^2] \\
\leq e^{-\mu_n} e^{(1+2\varepsilon) \ln \ln n} \\
= (\ln n)^{-(1+2\varepsilon)}.
\]

Putting \( n_k = [e^k] \), where \((1+2\varepsilon)^{-1} < \delta < 1\), it follows that

\[
(4.8) \quad \sum_{k=1}^{\infty} E(e^{\lambda_{n_k} S_{n_k} - \mu_{n_k}}) < \infty
\]
and therefore that, with probability 1,

\[
\sum_{k=1}^{\infty} e^{\frac{\lambda}{n_k} \sum_{n_k}^{n_k} \frac{1}{n_k} < \infty}.
\]

Hence, with probability 1,

\[
\lambda \frac{S_{n_k}}{n_k} - \mu \frac{1}{n_k} \leq 0 \text{ for } k \text{ large enough},
\]

i.e.,

\[
\frac{S_{n_k}}{n_k \ln \ln n_k} \leq \sqrt{2} + 2 \varepsilon \text{ for } k \text{ large enough}.
\]

Likewise, (4.12) holds with probability 1 if \( \frac{S_{n_k}}{n_k} \) is replaced by \( -\frac{S_{n_k}}{n_k} \), so that, with probability 1,

\[
\frac{|S_{n_k}|}{n_k \ln \ln n_k} \leq \sqrt{2} + 2 \varepsilon \text{ for } k \text{ large enough}.
\]

Now, for \( n_k < n \leq n_{k+1} \), we have

\[
\frac{|S_n|}{(n \ln \ln n) \frac{1}{2}} \leq \frac{|S_{n_k}|}{(n_k \ln \ln n_k) \frac{1}{2}} + \varepsilon_k,
\]

where

\[
\varepsilon_k = \max_{n_k < n \leq n_{k+1}} \frac{|S_n - S_{n_k}|}{n_k \ln \ln n_k}.
\]

Since \( \{X_i\} \) is a uniformly bounded multiplicative system, the hypothesis of Theorem 2.2 is satisfied for a value of \( \nu \) which is greater than \( 2/(1-\delta) \). Therefore, applying (2.22) to each \( \varepsilon_k \),

\[
E(\varepsilon_k^\nu) \leq K \left( \frac{n_{k+1} - n_k}{n_k \ln \ln n_k} \right)^d \nu \quad (k=1,2,\ldots),
\]
for a suitable constant $K$. It is easily seen that $(n_k^{k+1} - n_k)/n_k$ is $O(k^{\delta-1})$ as $k \to \infty$, since $\delta < 1$, and hence the right-hand side of (4.16) is $O(k^{-K\nu(1-\delta)})$. It follows by the choice of $\nu$ that

$$\sum_{k=1}^{\infty} E(\xi_k^\nu) < \infty$$

and hence that, with probability 1,

$$\xi_k \to 0 \quad k \to \infty.$$

Combining (4.13), (4.14) and (4.18), we have with probability 1

$$\limsup_{n \to \infty} \frac{|S_n|}{(n \ln \ln n)^{\frac{1}{2}}} \leq \sqrt{2}(1+3\varepsilon).$$

Since $\varepsilon$ may be arbitrarily small, (4.3) follows, the equality being a consequence of the law of the iterated logarithm for independent and identically distributed variables satisfying (4.1). This completes the proof.

An alternative law is given by the following theorem. Here the constant involved may not be sharp as in the above theorem, but the result is obtained under much broader assumptions. The sequence $\{X_1\}$ need only be multiplicative, not necessarily an ESMS.

**Theorem 4.2.** Let $\{X_1\}$ be a uniformly bounded ($|X_1| \leq 1$) multiplicative system. Then with probability 1

$$\limsup_{n \to \infty} \frac{|S_n|}{(n \ln \ln n)^{\frac{1}{2}}} \leq \sqrt{2}.$$

**Proof:** Let $\theta > 1$. By Theorem 3.1, inequality (3.3), we have

$$P\left[ \frac{|S_n|}{(2\theta n \ln \ln n)^{\frac{1}{2}}} \leq 2(\ln n)^{-\theta} \right].$$
Putting \( n_k = [e^{k^\delta}] \), where \( \theta^{-1} < \delta < 1 \), it follows by the Borel–Cantelli lemma that with probability 1

\[
|S_{n_k}| < (2\theta n_k \ln \ln n_k)^{\frac{1}{2}}, \text{ for all } k \text{ large enough},
\]

(4.22)

The proof may now proceed like that for the previous theorem, leading to the result that with probability 1

\[
\limsup_{n \to \infty} \frac{|S_n|}{(n \ln \ln n)^{\frac{1}{2}}} < \sqrt{2\theta}.
\]

(4.23)

Since \( \theta \) may be chosen arbitrarily close (\( > \)) to 1, (4.20) follows, completing the proof.

Finally, under still milder assumptions, we may conclude by Theorem 2.1 above and by Corollary 3.3 of [9] the following result.

**Theorem 4.3.** Let \( \nu \) by an even integer \( \geq 4 \). Suppose that \( \{X_i\} \) is multiplicative of order \( \nu \) and that \( E|X_i|^{\nu} \leq M < \infty \) (all i). Then with probability 1

\[
|S_n| = o(n^{\frac{1}{2}}(\ln n)^{1/\nu}(\ln \ln n)^{2/\nu}), \quad n \to \infty.
\]

(4.24)

Turning to convergence rates in the law of the iterated logarithm, we have

**Theorem 4.4.** Let \( \{X_i\} \) be a uniformly bounded \((|X_i| < 1)\) multiplicative system. Then, for each \( \theta > 2 \),

\[
\sum_{n} \frac{1}{n \ln n} \mathbb{P} \left[ \sup_{k \geq n} \frac{|S_k|}{(\theta k \ln \ln k)^{\frac{1}{2}}} > 1 \right] < \infty.
\]

(4.25)
Proof: The series in (4.25) is easily found to be no greater than the series

\[
\sum_{j=1}^{\infty} \frac{1}{\ln 2^j} \text{P} \left[ \sup_{k \geq 2^j} \frac{|S_k|}{(\delta k \ln \ln k)^{1/2}} > 1 \right]
\]

and hence also no greater than

\[
\sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=j}^{\infty} \text{P} \left[ |S_k| > (\delta k \ln \ln k)^{1/2} \right],
\]

for a suitable constant C. Applying Theorem 3.1, inequality (3.3), the series in (4.27) is, for a suitable constant C₁, less than

\[
\sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=j}^{\infty} k^{-\delta/2},
\]

which in turn is less than \( C_2 \sum (\ln k) k^{-\delta/2} < \infty \) for some constant \( C_2 \).
Therefore, the series in (4.25) converges.

The result (4.25) has been proved by Davis [2] for \( \{X_i\} \) a sequence of independent and identically distributed rv's satisfying

\[ E(X_i^2) = 1 \text{ and } E(X_i^2 \ln \ln |X_i|) < \infty. \]

The following result characterizes the rate of convergence corresponding to the law given by Theorem 4.3. The proof of the result is analogous to that of Theorem 4.4 and appears in [9].

**Theorem 4.5.** Let \( \nu \) be an even integer \( \geq 4 \). Suppose that \( \{X_i\} \) is multiplicative of order \( \nu \) and that \( E|X_i|^\nu \leq M < \infty \) (all \( i \)). Then, for each \( \alpha > 1/\nu \),

\[
\sum_{n=1}^{\infty} \frac{1}{\ln n} \text{P} \left[ \sup_{k \geq n} \frac{|S_k|}{k^{1/2}(\ln k)^{1/2}} > 1 \right] < \infty.
\]
A consequence of the law of the iterated logarithm is the strong
law of large numbers,

\[(4.30) \quad P\left(\frac{S_n}{n} \to 0\right) = 1,\]

and accordingly a rate of convergence for (4.30) is of interest.

**Theorem 4.6.** Let \(\{X_i\}\) be a uniformly bounded \((|X_i| < 1)\) multipli-
cative system. Then, for each \(\varepsilon > 0\),

\[(4.31) \quad P\left(\sup_{k \geq n} \left| \frac{S_k}{k} \right| > \varepsilon \right) \leq C_\varepsilon \rho^n_{\varepsilon} \quad \text{(all } n \geq 1),\]

where \(\rho_{\varepsilon} = e^{-\varepsilon^2/2}\) and \(C_\varepsilon = 2/(1 - \rho_{\varepsilon}).\)

**Theorem 4.7.** Let \(\nu\) be an even integer \(\geq 4\). Suppose that \(\{X_i\}\)
is multiplicative of order \(\nu\) and that \(E|X_i|^{\nu} \leq M < \infty\). Then for each
\(\varepsilon > 0\) there exists a constant \(C_\varepsilon < \infty\) such that

\[(4.32) \quad P\left(\sup_{k \geq n} \left| \frac{S_k}{k} \right| > \varepsilon \right) \leq Cn^{-\frac{k}{2}\nu} \quad \text{(all } n \geq 1).\]

Theorem 4.6 follows by a trivial application of inequality (3.3).
The argument may be seen in [9], as well as the proof of Theorem 4.7
and related results for \(X_i\)'s not necessarily multiplicative.

In conclusion, we consider the complete convergence of suitably
normed \(S_n\). Following Hsu and Robbins [5], we say that the sequence
\(\{\xi_n\}\) of rv's converges completely to zero if

\[(4.33) \quad \sum P(|\xi_n| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.\]
Theorem 4.8. Let \( \{X_i\} \) be a uniformly bounded \((|X_i| < B)\) multiplicative system. Then, if \( g(n) \to \infty \),

\[
\frac{S_n}{n \ln n} \left( \frac{1}{2} \right) g(n) \quad \text{converges completely to 0.}
\]

(4.34) \[
\frac{S_n}{n \ln n} \left( \frac{1}{2} \right) g(n) \quad \text{converges completely to 0.}
\]

The result follows from inequality (3.3). Condition (4.34) is also true of a sequence \( \{X_i\} \) of independent and identically distributed rv's with \( E|X_i|^4 < \infty \). This is implied by results of Chow [1] and Stout [10]. Stout's results also imply (4.34) for a sequence of uniformly bounded martingale differences. As discussed in [10], condition (4.34) is sharp.
REFERENCES


