RANDOM WEAR MODELS IN RELIABILITY THEORY

by

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ABSTRACT

Gaver [1963] and Antelman and Savage [1965] have proposed models for the distribution of the time to failure of a simple device exposed to a randomly varying environment. Each model represents cumulative wear as a specified function of a nonnegative stochastic process with independent increments, and assumes the reliability of the device is conditioned upon realizations of this process. A class $S$ of such processes is introduced, with essential infimum of $X(t)$ zero for all $t \geq 0$, $X \in S$, which excludes any deterministic component. The Lévy representation of the log characteristic function of $X \in S$ at time $t \geq 0$ is

$$\log \mathbb{E} e^{iuX(t)} = \int_{0^+}^{\infty} (e^{iuv} - 1)dN(t,v).$$  \hspace{1cm} (1)

Let $T$ be the random time to failure of the device. The Gaver model is

$$P(T \geq t | X(\cdot)) = e^{-X(t)}$$

for $t \geq 0$ and $X \in S$. The Antelman-Savage model is

$$P(T \geq t | X(\cdot)) = e^{-\int_0^t X(\tau)d\tau}$$

for $t \geq 0$ and $X \in S$. From these models are derived the corresponding unconditional joint distributions for the random failure time vector $T = (T_1, \ldots, T_n)$ of $n$ independent, identical devices exposed simultaneously.
to the same realization of the wear process:

\[ - \sum_{i=1}^{n} X(t_i) \]

\[ P(T \geq t) = \text{Ee} \]

and

\[ - \sum_{i=1}^{n} H(t_i) \]

\[ P(T \geq t) = \text{Ee} \]

where \( t = (t_1, \ldots, t_n) \) is an \( n \)-tuple of nonnegative real numbers, and \( H(t) = \int_{0}^{t} X(\tau) \, d\tau \), \( t \geq 0 \), is a stochastic integral defined in the sense of Riemann integrability of almost all sample functions of \( X \).

Two subclasses of \( S \) are considered: \( S_1 \) consists of those \( X \in S \) such that \( N(t,v) = \alpha(t)N(v) \) in (1), with certain requirements on \( \alpha(t) \) and \( N(v) \) such as monotonicity; \( S_2 \) is a subclass of \( S_1 \), where \( X \in S_2 \) implies \( \alpha(t) = t/v \) for some constant \( v > 0 \). \( S_2 \) includes all \( X \in S \) which have stationary increments. For each \( X \in S_2 \) there exists an \( X^\ast \in S \) such that for \( t \geq 0 \) and real \( u \)

\[ \log \text{Ee}^{iu \int_{0}^{t} X(\tau) d\tau} = \log \text{Ee}^{iuX^\ast(t)} \]

Theorem 1.11. This means the identical failure time distribution for one component can arise from each model.

Joint distributions are developed for \( n \) identical, independent components exposed to a single realization of the wear process under both models. In the Gaver model, sample function discontinuities cause tie failure times to occur with positive probability. The probabilities of specific tie configurations are developed, Theorem 1.14, and for
X ∈ S₁ these probabilities are independent of the function α(t),
Corollary 1.15. The same tie configuration probabilities are also seen to hold for stochastic wear processes with constant jump size
a > 0, regardless of whether they have independent increments,
Section 1.7.

Under the Gaver model with X ∈ S₂ a joint density is developed, depending on the observed tie configuration. The log characteristic function of X ∈ S₂ is of the form
\[
\log \mathbb{E} e^{i u X(t)} = \frac{1}{v} \int_{0^+}^{\infty} (e^{i w v} - 1) dN(v).
\]
The density can be indexed by v > 0 with the tie configuration probability functionally independent of v, so that taking logarithms and differentiating with respect to v is not cumbersome. The resulting maximum likelihood estimate is
\[
\hat{v} = \sum_{j=1}^{k} \frac{N^*(c_j) \Delta t[j]}{k},
\]
where
k is the number of distinct failure times,
\[
\Delta t[j] = t[j] - t[j-1], \quad j = 1, \ldots, k,
\]
t[0] = 0,
t[j] is the j-th smallest distinct failure time, j = 1, \ldots, k,
\[
N^*(u) = \int_{0^+}^{\infty} (1-e^{-w v}) dN(v),
\]
\[
s_j = \sum_{i=j}^{k} c_i, \quad j = 1, \ldots, k \text{ and } c_i \text{ is the number of } t_j \text{ equal to } t[i].
\]
The random variables $\Delta T[i]$ corresponding to the observations $\Delta t[j]$ are independently, exponentially distributed with mean $\frac{\nu}{N^*(s_j)}$, $j = 1, \ldots, k$, Theorem 2.1.

The estimate $\hat{\nu}$ has a conditional distribution depending on the tie configuration only through $k$, and given $k$, $\hat{\nu}$ has a gamma distribution with mean $\nu$ and variance $\nu^2/k$, Theorem 2.2. Among unbiased estimates of $\nu$, $\hat{\nu}$ has minimum variance. Standard techniques of inference are applied to $\hat{\nu}$ given $k$. Unconditionally, the estimate is unbiased and consistent as $n \to \infty$. The variance of $\hat{\nu}$ is

$$\text{Var } \hat{\nu} = \nu^2 E \frac{1}{K_n},$$

in which $K_n$ is the random variable corresponding to $k$, the number of distinct failure times observed among $n$ components exposed to a single environmental realization. For constant jump wear processes, $E \frac{1}{K_n}$ is given explicitly for $n = 2, \ldots, 6$ in Equations (2.17) through (2.21).

For large $n$

$$E \frac{1}{K_n} = \frac{a}{\log n} + o\left(\frac{1}{\log n}\right),$$

where $a$ is the jump size, Theorem 2.9. The problem of minimizing overall cost by sizing $m$, the number of trials, and $n$, the number of components per trial, is considered, using relative squared-error loss, Section 2.4. If $f$ is the fixed cost of a trial, if $b$ is the cost associated with relative squared-error loss, and if components have a cost of one unit, then for large $n$ the size of $m$ and $n$ to minimize the overall expected cost is given by the solution of $n$ to

$$\nu$$
\[ n(\log n - 1) = f \]

and by

\[ m = \frac{1}{\log n} \sqrt{\frac{ba}{n}} , \]

Equations (2.34) and (2.35).

An extension of the estimating procedure is made to \( m \) trials, Section 2.4. The maximum likelihood estimate is

\[ \hat{v} = \sum_{r=1}^{m} \frac{k(r) \hat{v}(r)}{m \sum_{r=1}^{m} k(r)} , \]

where \( \hat{v}(r) \) is the estimate of \( v \) from the \( r^{th} \) trial, and \( k(r) \) is the number of distinct failure times observed in the \( r^{th} \) trial. A second extension concerns estimating \( v \) from failure times of \( M \) different types of components subjected to the same realization of the wear process, Section 2.5. A final extension under the Gaver model relaxes the condition \( X(0) = 0 \) a.s., Section 2.6.

For the Antelman-Savage model the joint distribution of \( n \) failure times \( T_1, \ldots, T_n \) of components exposed to a single realization of the wear process when \( X \in S_2 \), is

\[ \bar{F}(t) = \mathbb{P}\{T_1 > t_1, \ldots, T_n > t_n\} \]

\[ = e^{-\sum_{m=1}^{n} t^{(m)}(m-1)} \int \sum_{j=m}^{n} t^{(j)}(n+1-m-\tau) \, d\tau \]

where \( t = (t_1, \ldots, t_n) \) is an \( n \)-tuple of nonnegative real numbers.
with ordered values $t_{(1)}$; that is, $0 = t(0) \leq t(1) \leq \cdots \leq t(n)$. For $n = 1$, Equation (2) reduces to

$$\bar{F}(t) = e^{-\frac{1}{\nu}z(t)},$$

where

$$z(t) = \int_{0^+}^{\infty} \frac{1}{\nu}(e^{-vt} - 1 + vt)dN(v).$$

For the case of $m$ independent trials with one failure time per trial

the maximum likelihood estimate of $\nu$ is

$$\hat{\nu} = \frac{1}{m} \sum_{j=1}^{m} z(t_j).$$

This estimate has a gamma distribution with mean $\nu$ and variance $\nu^2/m$, Theorem 3.1.

In the Antelman-Savage model with more than one component

per realization, the method of maximum likelihood is cumbersome. The
method of moments is not consistent as $n \to \infty$. A consistent sequence

$\{\hat{\lambda}_n\}$ of estimates of $\lambda = 1/\nu$ is developed, where

$$\hat{\lambda}_n = -\frac{2 \log[1-F_n(u_n)]}{u_n^2};$$

$F_n(u_n)$ is the empirical distribution function, the proportion having
failed by $u_n$; and $u_n = (\log \sqrt{n})^{-1/3}$, $n = 1, 2, \ldots$, Theorem 3.3.

A list of questions for additional research is included. Since
the wear process is in general unobservable, all its parameters may be
unknown, and there is a general problem under both models for developing
joint estimates. Other problems include incorporating a deterministic component in the wear process and using censored or truncated samples.
I. BASIC MODELS AND PROBABILITY CONSIDERATIONS

1.1 Introduction

Two models (Gaver [1963], Antelman and Savage [1965]) have been proposed for the distribution of the time to failure of a simple device exposed to a randomly varying environment. Each model represents environmental cumulative wear as a specified function of a nonnegative stochastic process, and assumes the reliability of the device is conditioned upon realizations of this process. In this paper differences in the random failure time behavior between the two models are examined. For each model joint failure time distributions are derived when n devices, or components, are simultaneously exposed to the same environment. Procedures are given for estimating the time-scale parameter of the underlying stochastic wear process, based on multiple failure times observed from one or more environmental realizations.

Let \( T \) represent the random time to failure of a simple device. A typical model for the distribution of \( T \) is to assume the existence of a nonnegative function \( x(t) \) such that for \( t > 0 \) and \( \delta > 0 \)

\[
P(T \in (t, t+\delta) | T > t) = x(t)\delta + o(\delta).
\]

Throughout, \( o(\cdot) \) denotes terms of smaller order than the expression in parentheses. Similarly, \( O(\cdot) \) denotes terms of at most the same
order, as in Cramér [1946], p. 122. With appropriate regularity conditions on \( \lambda(t) \)

\[
\int_{0}^{t} \lambda(\tau) d\tau \\
P(T > t) = e^{-\int_{0}^{t} \lambda(\tau) d\tau}.
\]

The function \( \lambda(t) \) is variously called the failure rate, the hazard rate, the force of mortality, or the instantaneous wear. The integral \( h(t) = \int_{0}^{t} \lambda(\tau) d\tau \) represents the accumulated wear or damage at time \( t \).

Birnbaum and Saunders [1958] suppose the failure rate \( \lambda(t) \) can be factored into two deterministic functions of time, one representing environmental stress, or loading, and the other, aging independent of loading. They remark that an extension of the theory is to consider random loadings.

Antelman and Savage [1965] assume the hazard rate \( \lambda(t) \) is the realization of a stochastic process \( X(t) \), \( t \geq 0 \), with independent increments. The conditional probability of the device surviving time \( t \) given \( X(\cdot) \) is

\[
\int_{0}^{t} \lambda(\tau) d\tau \\
P(T > t | X(\cdot)) = e^{-\int_{0}^{t} \lambda(\tau) d\tau}, \quad t \geq 0,
\]

(1.1)

and the unconditional probability is

\[
\int_{0}^{t} \lambda(\tau) d\tau \\
P(T > t) = E e^{-\int_{0}^{t} \lambda(\tau) d\tau}, \quad t \geq 0.
\]

(1.2)

The stochastic integral \( H(t) = \int_{0}^{t} X(\tau) d\tau \) is defined in the sense of Riemann integrability of almost all sample functions \( x(\cdot) \) of \( X(\cdot) \).

The function \( H(t) \) represents the random wear accumulated by time \( t \geq 0 \), with \( H(0) = 0 \) almost surely (a.s.).
order, as in Cramér [1946], p. 122. With appropriate regularity conditions on $x(\cdot)$

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P(T > t) = e^{-\int_0^t x(\tau) d\tau}.
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Antelman and Savage [1965] assume the hazard rate $x(t)$ is the realization of a stochastic process $X(t)$, $t \geq 0$, with independent increments. The conditional probability of the device surviving time $t$ given $X(\cdot)$ is

$$
P(T > t|X(\cdot)) = e^{-\int_0^t X(\tau) d\tau}, \quad t \geq 0,
$$

and the unconditional probability is

$$
P(T > t) = E e^{-\int_0^t X(\tau) d\tau}, \quad t \geq 0.
$$

The stochastic integral $H(t) = \int_0^t X(\tau) d\tau$ is defined in the sense of Riemann integrability of almost all sample functions $x(\cdot)$ of $X(\cdot)$. The function $H(t)$ represents the random wear accumulated by time $t \geq 0$, with $H(0) = 0$ almost surely (a.s.).
Gaver [1963] supposes the accumulated wear to be a stochastic process with independent increments, rather than the integral of such a process. The conditional probability corresponding to (1.1) is

\[ P(T > t | X(\cdot)) = e^{-X(t)}, \quad t \geq 0 \]  

(1.3)

and hence

\[ P(T > t) = e^{-X(t)}, \quad t \geq 0. \]  

(1.4)

Gaver considers only processes with stationary increments, an assumption we initially relax.

The model for random wear in Equation (1.1) is referred to as the Antelman-Savage model, and the model underlying (1.3) is called the Gaver model. The basic difference between the two models is in the role played by independent increments. In the Gaver model the independence of increments in the accumulated wear process leads to a failure time distribution that is "memoryless." That is, the probability that an item in service now will survive some future time depends only on the increase in wear between now and then, not on the wear accumulated to date. Antelman-Savage models do not have this property.

As an example of the Gaver model, consider an electronic device subjected to a sequence of shocks of constant intensity, occurring at independent, exponentially distributed intervals. The shocks may be physical jolts, current surges, or some other form of stress of momentary duration. Suppose that regardless of the number of shocks sustained until now, a serviceable device has constant
probability of surviving the next shock. Then the conditional and unconditional failure time distributions are given by (1.3) and (1.4). The accumulated wear process \( X \) is a Poisson process, with points of increase corresponding to the occurrence of shocks, and with the size of each increase, or jump, corresponding to the shock intensity.

For the Antelman-Savage model one might consider an automobile tire as an illustration to fix ideas. Suppose that the more out of line the wheel is, the faster the tread wears out. Because of jolts received, the misalignment tends to get worse with time. Assume for each \( t \geq 0 \) an appropriate measure \( x(t) \) can be placed on the amount of misalignment, so that the accumulated wear is given by

\[
h(t) = \int_0^t x(\tau) d\tau.
\]

Under certain assumptions about driving conditions \( x \) may be considered as a realization of a random process \( X \) with non-negative, independent increments, and Equations (1.1) and (1.2) apply.

In Chapter I the basic class \( S \) of stochastic processes to be considered is defined, along with two subclasses \( S_1 \) and \( S_2 \). A simplified form is given for the Lévy canonical representation of the logarithm of the characteristic function (log ch.f.) of members of these classes. One question which arises is whether the Gaver and Antelman-Savage models can give rise to the same unconditional failure time distribution. Theorem 1.11 gives a sufficient condition for this.

For each model a joint failure time distribution is developed for \( n \) identical components subjected simultaneously to the same realization of the wear process. These failure times are dependent. Under the Gaver model, tie failure times have positive probability of occurrence. An expression for the probability of specific tie
configurations is given, and is seen to hold for wear processes with constant jump size, even though they do not have independent increments.

Chapter II is concerned with maximum likelihood estimation under the Gaver model when the wear process has stationary increments and multiple observations are made per realization. The parameter being estimated is the one corresponding to the time scale. For example, a homogeneous Poisson process with size of jump $a > 0$ and mean time $v > 0$ between jumps has log ch. f. $\frac{1}{v} t(e^{iu} - 1)$. A maximum likelihood estimate (MLE) is derived for $v$ when $a$ is known. The distribution of the estimate has a gamma distribution depending only on $K_n$, the number of distinct failure times in the sample of size $n$. Setting confidence intervals and testing hypotheses are done according to standard techniques once the conditioning on $K_n$ is made. Extensions are made to $m$ realizations and to $M$ types of component per realization.

Concerning the design of experiments, the problem of sizing the number of environmental realizations, or trials, versus the number of components per trial, is considered. Finally, estimating $v$ when failures can occur at $t = 0$ is discussed.

In Chapter III the wear process is assumed to follow the Antelman-Savage model with $H(t) = \int_0^t X(x)dx$. For $X$ with stationary increments a MLE for the parameter associated with the time scale is obtained when one device is exposed to each realization. For the particular case when $X(x)$ is a Poisson process with jump size $a > 0$ (known) and rate $\lambda$ of occurrence of jumps, a consistent sequence of estimates is shown to exist for $\lambda$. A list of further research problems concludes Chapter 3.
1.2 Nonnegative processes with independent increments

An underlying probability space \((\Omega, \mathcal{F}, P)\) is assumed for the stochastic processes considered. The time domain \(T\) is taken to be \((0, \infty)\) with closure \(\overline{T} = [0, \infty)\). The general class of processes to be considered is denoted by \(S\):

**Definition 1.1**

Let \(S\) be the class of all separable stochastic processes \(X(t), t \geq 0\), such that

(i) \(X\) has independent increments and \(X(0) = 0\) a.s.

(ii) \(X\) is centered with no fixed points of discontinuity.

(iii) For each \(t \geq 0\), the essential infimum of \(X(t)\), denoted by a.s. \(\inf X(t)\), is zero.

(iv) With probability one the sample functions of \(X\) are right continuous.

See Doob [1953], pp. 357, 408 for precise definitions of fixed points of discontinuity and centering. The purpose of centering is to insure that the sample functions of \(X\) have simple continuity properties. Specifically, almost all sample functions of a centered process are continuous except at most at a countable number of points, and even at these points left- and right-hand finite limits exist, and further, all processes with independent increments can be centered, Doob [1953], pp. 98, 407. In the same reference, p. 418, Doob shows that every process \(X \in S\) has infinitely divisible increments. Then for each \(t > s \geq 0\), the log ch.f. of \(X(t) - X(s)\) can be expressed in the Lévy canonical representation. Requirement (iii)
of Definition 1.1 excludes a deterministic component from the wear process. This exclusion and the nonnegativity of the process make possible a simplification of the Lévy representation, for which Lemma 1.2 is needed. This lemma resembles a theorem of Titchmarsh on convolutions, described by Boehme [1968]. Note that each distribution function (d.f.) is taken to be right-continuous. For a real number \( a \), writing \( \int_a^- \) (or \( \int_a^+ \)) means the lower limit of integration approaches \( a \) from below (or above).

**Lemma 1.2**

Let \( X(t), t \geq 0 \), be a stochastic process with independent increments, and suppose for each \( t \geq 0 \) that a.s. \( \inf X(t) > -\infty \), with \( X(0) = 0 \) a.s.

Then for each \( t > s \geq 0 \)

\[
\text{a.s. } \inf [X(t) - X(s)] = \text{a.s. } \inf X(t) - \text{a.s. } \inf X(s).
\]

**Proof:**

By independent increments

\[
F_{X(t)}(u) = \int_{-\infty}^{\infty} F_{X(t)-X(s)}(u-x) dF_X(s)(x)
\]

where \( F \) is the d.f. of the corresponding subscripted random variable (r.v.). Let \( L(t) = \text{a.s. } \inf X(t) \). Then

\[
F_{X(t)}(u) = \int_{L(s)}^{\infty} F_{X(t)-X(s)}(u-x) dF_X(s)(x).
\]
Integration by parts yields

\[ F_X(t)(u) = F_X(t) - X(s)(u-L(s))F_X(s)(L(s)) \]

\[ + \int_{L(s)}^{\infty} F_X(t) - X(s)(u-x) dF_X(s)(x). \quad (1.5) \]

Let \( \varepsilon > 0 \) be given. If \( F_X(t)(L(t) - \varepsilon) = 0 \), then both terms on the right-hand side of (1.5) must vanish for \( u = L(t) - \varepsilon \). Since \( F_X(s)(L(s) + \varepsilon) > 0 \), then \( F_X(t) - X(s)(L(t) - \varepsilon - L(s)) = 0 \) in order for the integral in (1.5) to vanish. Now suppose \( F_X(t) - X(s)(L(t) + \varepsilon - L(s)) = 0 \).

One has \( F_X(t) - X(s)(L(t) + \varepsilon - x) = 0 \) for \( x \geq L(s) \), and both terms on the right-hand side of (1.5) vanish, contradicting \( F_X(t)(L(t) + \varepsilon) > 0 \).

Thus for each \( \varepsilon > 0 \), \( F_X(t) - X(s)(L(t) - L(s) + \varepsilon) > 0 \) and \( F_X(t) - X(s)(L(t) - L(s) - \varepsilon) = 0 \).

Q.E.D.

Lemma 1.2 states a local property. Under the assumption of right-hand continuity of the sample functions, Theorem 1.3 shows the sample functions are nondecreasing. This guarantees the existence of the Riemann integral \( h(t) = \int_0^t x(\tau) d\tau \), \( t \geq 0 \), for almost all sample functions \( x \) of \( X \in S \). For the Antelma–Savage model this justifies defining the stochastic integral \( H(t) = \int_0^t X(\tau) d\tau \), \( t \geq 0 \), in the sense of Riemann integrability of almost all sample functions.

**Theorem 1.3**

Let \( X \in S \). Then almost all sample functions \( X(\omega, t) \) are nondecreasing in \( t \).
Proof:

Let \( A_{ijk} = \{ \omega: X(\omega, \frac{i}{k}) - X(\omega, \frac{j}{k}) < -\frac{1}{k} \}, i = 0, 1, \ldots, j; j, k = 1, 2, \ldots \). Let \( N = \{ \omega: \text{for some } t_\omega > s_\omega > 0, X(\omega, t_\omega) - X(\omega, s_\omega) < 0 \}. \)

For given \( \omega \in N \) there exists an \( n \) such that \( k \geq n \) implies \( X(\omega, t_\omega) - X(\omega, s_\omega) < -\frac{2}{k} \). By right-continuity for \( k \) sufficiently large there are integers \( i, j \) with \( i < j \) such that \( X(\omega, \frac{i}{k}) - X(\omega, \frac{j}{k}) < -\frac{1}{k} \).

Hence \( \omega \in A_{ijk} \) and \( N = \bigcup_{ijk} A_{ijk} \). If \( P(N) > 0 \) then there is some \( (i, j, k) \) such that \( P(A_{ijk}) > 0 \), contradicting \( a.s. \inf[X(\frac{i}{k}) - X(\frac{j}{k})] = 0 \), which holds by Lemma 1.2. Therefore \( P(N) = 0 \).

Q.E.D.

From Lemma 1.2 for \( t > s \geq 0 \) and \( X \in S \) we have \( a.s. \inf[X(t) - X(s)] = 0 \). The process \( X \) has no normal component, no negative Poisson component, and no deterministic component.

Following Bergström [1966], one obtains the simplified Lévy canonical representation for the log ch. f.

\[
\log E e^{iu[X(t) - X(s)]} = \int_{0^+} (e^{iuv} - 1) d \nu [N(t, v) - N(s, v)]
\]

(1.6)

where \( N(t, v) \) is a function of \( v > 0 \) and \( t \geq 0 \) such that

(i) \( N \) is a nondecreasing function of \( v \) and \( t \) with nonnegative second differences.

(ii) \( N(t, 0) = 0 \) for \( t \geq 0 \).

(iii) \( \int_{0^+} \frac{v}{1 + v^2} d \nu N(t, v) < \infty \) for \( t \geq 0 \).

Next consider the case where \( N(t, v) \) factors into functions \( \alpha(t) \) and \( \nu(v) \).
Definition 1.4

Let $S_1$ be the class of all $X \in S$ with log ch.f. of the form

$$\log E e^{iuX(t)} = \alpha(t) \int_{0^+} (e^{iuv} - 1) dN(v)$$

(1.7)

where

(i) $\alpha(t)$ has a finite, continuous, nonnegative derivative $\alpha'(t)$, $t \geq 0$ ($\alpha'(0)$ may be a right-hand derivative), $\alpha(0) = 0$, and

$$\lim_{t \to \infty} \alpha(t) = \infty.$$  

(ii) $N(v)$ is a nondecreasing function of $v$, $v > 0$, with

$$N(\infty) = 0 \text{ and } \int_{0^+}^{\infty} \frac{v}{1 + v^2} dN(v) < \infty.$$  

For $u \geq 0$ and $X \in S_1$ define $N^*(u)$ by

$$N^*(u) = \int_{0^+}^{\infty} (1 - e^{-uv}) dN(v).$$

(1.8)

It can be shown that $N^*(u) < \infty$ for all $u \geq 0$, and clearly $N^*$ is a positive, continuous, increasing function of $u$, $u > 0$, with $N^*(0) = 0$. Then

$$E e^{-uX(t)} = e^{-\alpha(t)N^*(u)}, \quad u \geq 0.$$  

(1.9)

Suppose for $a > 0$ and $t \geq 0$ that $X \in S$ has the marginal distribution

$$P(X(t) = ka) = \begin{cases} 
\frac{e^{-\lambda(t)} \lambda^k(t)}{k!}, & k = 0, 1, \ldots \\
0, & \text{otherwise}
\end{cases}$$
where \( \lambda(t) \) is a continuous, nondecreasing function of \( t > 0 \) with \( \lambda(0) = 0 \). \( X \in S \) implies \( X \) has independent increments. Such an \( X \) will be called a Poisson process (even though the jump size \( a \) may not be unity) and will be denoted by

\[
X(t) \sim P(\lambda(t);a).
\] (1.10)

If a r.v. \( Y \) has a gamma distribution with mean \( \alpha/\beta \) and variance \( \alpha/\beta^2 \), where \( \alpha > 0 \) and \( \beta > 0 \), write

\[
Y \sim \Gamma(\alpha,\beta).
\] (1.11)

**Example 1.5**

Let \( X(t) \sim P(\lambda(t);a) \) according to (1.10). Then

\[
\log E e^{iuX(t)} = \lambda(t)(e^{iua} - 1), \quad t \geq 0, \quad -\infty < u < \infty
\]

which is obtained from (1.7) by setting

\[
\alpha(t) = \lambda(t)
\]

\[
N(v) = -I_{[0,a)}(v), \quad v > 0,
\]

where \( I \) is the indicator function of the subscripted interval. When \( \lambda(t) \) is differentiable and \( \lambda(t) \to \infty \), as \( t \to \infty \), then \( X \in S_1 \). From (1.8) \( N^*(u) = 1 - e^{-au} \).
Example 1.6

Suppose $X \in S$ has a gamma marginal distribution, $X(t) \sim \Gamma(\alpha(t), \beta)$, where $\beta > 0$ and $\alpha(t)$ is a continuous, nondecreasing function of $t$ vanishing at $t = 0$. The log ch.f. of $X(t)$ is

$$\log E e^{iuX(t)} = -\alpha(t) \log(1 - \frac{iu}{\beta}).$$

$$= -\alpha(t) \sum_{k=1}^{\infty} \frac{-(iu/\beta)^k}{k}$$

$$= \alpha(t) \sum_{k=1}^{\infty} \frac{(iu)^k}{k} \frac{\beta^k}{\Gamma(k)} \int_{0}^{\infty} v^{k-1} e^{-\beta v} dv$$

$$= \alpha(t) \int_{0+}^{\infty} e^{-\beta v} \sum_{k=1}^{\infty} \frac{(iu)^k}{k \Gamma(k) v^k} dv$$

$$= \alpha(t) \int_{0+}^{\infty} (e^{iu} - 1) \frac{1}{v} e^{-\beta v} dv.$$

This is of the form (1.7) with

$$N(v) = -\int_{v}^{\infty} \frac{1}{x} e^{-\beta x} dx, \quad v > 0.$$  

When $\alpha(t)$ is differentiable and $\alpha(t) \to \infty$ as $t \to \infty$, then $X \in S_1$.

Example 1.7

Let $X_1(t) \sim P(t; 1)$ and $X_2(t) \sim P(t^2; 1)$ be independent and define $Y(t) = X_1(t) + 2X_2(t)$. Then

$$\log E e^{iuY(t)} = t(e^{iu} - 1) + t^2(e^{2iu} - 1)$$
which cannot be factored as in (1.7). Thus \( X_1 \in S \) and \( X_2 \in S \) imply \( Y \in S \), but \( Y \notin S_1 \).

Example 1.8

Let \( X \in S \) have stationary increments. Then \( N(t,v) \) in (1.6) is of the form \( ctN(v) \), where \( c \) is a constant (depending on the time scale used) and \( N(v) \) satisfies (ii) of Definition 1.4. Hence \( X \in S_1 \).

Under the Gaver model for random wear with \( X \in S_1 \), let \( F \) be the d.f. of \( T \) and \( \overline{F} = 1 - F \), so that

\[
\overline{F}(t) = E e^{-X(t)} = e^{-\alpha(t)N^*(1)}.
\]

If \( \alpha(t) \) is convex, then \( \log \overline{F}(t) \) is concave. This implies that \( F \) has the property of increasing failure rate (IFR) and all moments of \( T \) exist, Barlow and Proschan [1965], pp. 25-27. This is summarized in the following:

Theorem 1.9

Suppose \( \overline{F}(t) = 1 - F(t) = E e^{-X(t)} \), \( X \in S_1 \), and \( \alpha(t) \) is convex. Then \( F \) is IFR and all moments of \( T \) exist.

Example 1.8 motivates the definition of one more subclass \( S_2 \), which includes all \( X \in S \) with stationary increments:

Definition 1.10

Let \( S_2 \) be the class of all \( X \in S_1 \) whose distributions are indexed by a positive real parameter \( \nu \) such that the log ch.f. of \( X(t) \), \( t \geq 0 \), has the representation

\[
\log E e^{iuX(t)} = \frac{1}{\nu} \int_{0+}^{\nu} (e^{iuv} - 1)dN(v). \quad (1.12)
\]
Remark: The wording of this definition is adaptable to the
estimation problem of Chapter 2. For \( X \in S_2 \), the function \( a(t) \) of
Definition 1.4 is assumed to be linear in \( t \). \( N \) is functionally
independent of \( v \). For positive constants \( b \) and \( c \\
\frac{1}{v} \int_0^\infty \frac{e^{ivu} - 1}{v} dN(v) = \frac{1}{v^*} b t \int_0^\infty \frac{e^{ivu} - 1}{v} d[cN(v)]
\]
where \( v^* = bcv \), so assume the desired scales for \( t \) and \( N \) are chosen
and held fixed for all \( v \). For convenience \( v \) and \( \lambda = 1/v \) will be
used interchangeably in (1.12).

1.3 Antelman-Savage and Gaver
models with identical marginal
characteristic functions

The question arises whether the same unconditional failure
time distribution can result from either model. In other words, do
there exist wear processes \( X \) and \( X^* \) in \( S \) with \( H(t) = \int_0^t X(\tau) d\tau \), such
that \( E e^{-H(t)} = E e^{-X^*(t)}, t > 0 \)? Under the Antelman-Savage model,
for \( 0 = t_0 < t_1 < \ldots < t_n \) and with \( H \) as the wear process, the joint
distribution

\[
P\{T_1 > t_1, \ldots, T_n > t_n \} = E e^{-\sum_{i=1}^n H(t_i)} \]

\[
= E e^{-\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X(\tau) d\tau}
\]

is absolutely continuous, so that tied failure times have zero
probability of occurrence. On the other hand, as shown in Section 1.5,
under the Gaver model with \( X^* \in S \) as the wear process, tied failure times occur with positive probability. Thus when several components are exposed to the same environment, the joint failure time distributions under any \( H \) and \( X^* \) will ordinarily differ. Theorem 1.11 shows how the same one-dimensional failure time distribution can arise from both models. Suppose a sample of failure times is obtained, each device having been exposed to a separate realization of the wear process.

One implication of Theorem 1.11 is that it is impossible to say which model is the true underlying one.

**Theorem 1.11**

Let \( X \in S \) with \( H(t) = \int_0^t X(\tau) d\tau \), \( t \geq 0 \).

Then there exists an \( X^* \in S \) such that for every \( t \geq 0 \) and \( u \in (-\infty, \infty) \)

\[
E e^{iuH(t)} = E e^{iuX^*(t)}.
\]

Neither \( H \) nor \( X^* \) has stationary increments, excluding the trivial case \( X(t) \equiv 0 \).

**Proof:**

First consider \( H \). Integrating by parts and using the stationary increments property of \( X \), one has for \( t > s \geq 0 \)
\[ Ee^{iu[H(t)-H(s)]} = Ee^{\int_s^t iuX(\tau)d\tau - \int_s^t (t-\tau)dX(\tau) + (t-s)X(s)} = Ee^{\int_s^t iu(t-s)X(s)} \]

Since \( X \) is not the trivial process, there exist \( t \) and \( s \), \( t > s > 0 \), such that \( Ee^{iu(t-s)X(s)} \neq 1 \). Thus for some real \( u \) and \( t > s > 0 \)

\[ Ee^{iu[H(t)-H(s)]} \neq Ee^{iuH(t-s)} \]

and \( H \) does not have stationary increments.

Following the development of equation (A.5) of Antelman and Savage [1965] but using the simplified representation (1.12), one has for \( t > 0 \)

\[ \log Ee^{iuH(t)} = \log Ee^{\int_0^t iu(t-\tau)dX(\tau)} \]

\[ = \log E \exp {iu \lim_{n \to \infty} \sum_{j=1}^n (t - \frac{jt}{n})[X(\frac{jt}{n}) - X(\frac{j-1}{n})]} \]

\[ = \lim_{n \to \infty} \sum_{j=1}^n \lambda t \int_{0^+} \left[ e^{iu(t-\frac{jt}{n})v} - 1 \right] dN(v) \]

\[ = \lambda \int_{0^+} \int_0^t (e^{iu(t-\tau)v} - 1)d\tau dN(v) \]

\[ = \lambda \int_{0^+} \int_0^t (e^{iuxv} - 1)dx dN(v). \quad (1.13) \]
It remains to be shown that (1.13) is also the log ch.f. of $X^*(t)$ for some $X^* \in S$. Let $t > s \geq 0$. If the desired $X^*$ exists, then by independent increments

$$
\psi(u; s, t) = \log E e^{iuX^*(t)} - \log E e^{iuX^*(s)}
$$

must be the log ch.f. of $X^*(t) - X^*(s)$, which is infinitely divisible with essential infimum zero. Let $\phi(u) = \int_0^\infty (e^{iuv} - 1)dN(v)$ with $N(v)$ as in (1.13). Then $\phi$ is the log ch.f. of an infinitely divisible distribution with essential infimum zero. By the continuity theorem for ch.f.'s and by the closure property of infinitely divisible distributions under passage to the limit, Doob [1953], p. 137, the integral $\lambda \int_s^t \phi(ux)dx$ is also the log ch.f. of an infinitely divisible distribution with essential infimum zero. So set

$$
\psi(u; s, t) = \lambda \int_s^t \phi(ux)dx = \lambda \int_s^\infty (e^{iuxv} - 1)dN(v)dx
$$

(1.14)

and hence

$$
\psi(u; s, t) = \psi(u; 0, t) - \psi(u; 0, s).
$$

Thus $\psi(u; s, t)$ is the log ch.f. of some process $X^*$ with independent increments. Any process with independent increments can be centered. Since $\psi(u; 0, t)$ is also the log ch.f. of $H(t)$ for each $t > 0$, $X^*$ has essential infimum identically zero. Finally, the smoothness property of the integral guarantees no fixed points of discontinuity in $X^*$. Hence $X^* \in S$ and
\[ \log E^{iuX^*(t)} = \psi(u; 0, t) = \log E^{iuH(t)}. \]

Since (1.14) is not of the form
\[ \lambda^*(t-s) \int_0^\infty (e^{iuv} - 1) dN(v) \]
for constant \( \lambda^* > 0 \), \( X^* \) does not have stationary increments.

Q.E.D.

1.4 Multiple failure times from a single trial

1.4.1 Gaver Model

Suppose \( n \) independent, identical devices, or components, are simultaneously exposed to the same realization of the wear process \( X \in S \). If \( T = (T_1, \ldots, T_n) \) denotes a random vector in which \( T_j \) represents the failure time of the \( j^{th} \) component, and \( t = (t_1, \ldots, t_n) \) is an \( n \)-tuple of real numbers, let \([T > t]\) denote the event \([T_1 > t_1, \ldots, T_n > t_n]\) and let
\[ \bar{F}(t) = P(T > t) = EP(T > t | X(\cdot)) = Ee^{-X(t_1) - \cdots - X(t_n)}. \quad (1.15) \]

Note that \( X \in S \) implies \( P(T_j > 0) = 1, j = 1, \ldots, n \). Thus any discussion of failure times can be restricted to positive \( t_j \), i.e., to \( t \in T^n \).

Let \((t_1, \ldots, t_n)\) be a permutation of \( t \) such that
\[ 0 < t_1 \leq \ldots \leq t_n, \]
which need not be unique. Suppose there are \( k \) distinct \( t_j \) in \( t \). Let
\[ D(t) = (t_{[1]}, \ldots, t_{[k]}), \]

where

\[ t_{[j]} = \min \{ t(i) : t(i) > t_{[j-1]} \}, \quad j = 1, \ldots, k \]

and \( t_{[0]} = 0 \). Also let

\[ C(t) = (c_1, \ldots, c_k), \]

where \( c_j \) is the number of \( t_i \) in \( t \) equal to \( t_{[j]} \), \( j = 1, \ldots, k \). The vector \( D(t) \) will be referred to as the distinct ordering of \( t \), and \( C(t) \) as the tie configuration vector of \( t \). A \( k \)-tuple of positive integers \((c_1, \ldots, c_k)\) whose sum is \( n \) may be denoted by \( c_{n,k} \). Also, \( C^{-1}C(t) \) denotes the set \( \{ u \in \mathbb{T}^n : C(u) = C(t) \} \).

Equation (1.15) may now be written

\[ \overline{F}(t) = E \sum_{j=1}^{k} c_j X(t_{[j]}) \]

or

\[ \overline{F}(t) = E \sum_{j=1}^{k} s_j [X(t_{[j]}) - X(t_{[j-1]})], \]

where

\[ s_j = \sum_{i=j}^{k} c_i, \quad j = 1, \ldots, k. \]
By independent increments, for \( t > s \geq 0 \) and real \( u \),

\[
E e^{-uX(t)} = E e^{-u[X(t)-X(s)]} E e^{-uX(s)}
\]

so that an alternative form of (1.19) is

\[
\overline{F}(t) = \sum_{j=1}^{k} \frac{e^{-s_jX(t[j])}}{E e^{-s_{j+1}X(t[j])}}.
\] (1.21)

Throughout, vacuous sums are regarded as zero and vacuous products as unity. Thus in (1.21), \( \alpha_{k+1} = 0 \).

**Example 1.12**

Suppose \( \underline{t} = (7.6, 3.5, 5.0, 8.0, 5.0, 1.9, 3.5, 5.0) \). Then \( n = 8 \), \( k = 5 \), \( C(t) = (1, 2, 3, 1, 1) \) and \( D(t) = (1.9, 3.5, 5.0, 7.6, 8.0) \).

For \( X \in S_1 \), using Equation (1.9), one obtains

\[
\overline{F}(t) = e^{-\sum_{j=1}^{k} \Delta N^*_j \alpha(t[j])}.
\] (1.22)

where

\[
\Delta N^*_j = N^*(s_j) - N^*(s_{j+1}), \ j = 1, \ldots, k.
\] (1.23)

For \( X \in S_2 \), \( \alpha(t) = \frac{1}{v} t \) implies

\[
\overline{F}(t) = e^{-\frac{1}{v} \sum_{j=1}^{k} \Delta N^* t[j]}
\] (1.24)

\[
= e^{-\frac{1}{v} \sum_{j=1}^{k} N^*(s_j) \Delta t[j]}
\] (1.25)
in which, letting $t[0] = 0$, we define

$$\Delta t[j] = t[j] - t[j-1], \quad j = 1, \ldots, k \quad (1.26)$$

1.4.2 Antelman-Savage model

Now let $X \in S$ and $H(t) = \int_0^t X(\tau)d\tau$. For $n$ independent, identical components subjected to the same environment

$$P(T > t) = E e^{-H(t_1) - \ldots - H(t_n)}$$

$$= E e^{-\sum_{j=1}^n \int_0^{t(j)} X(\tau)d\tau}$$

$$\quad = E e^{-\sum_{j=1}^n \left( \sum_{m=1}^n \int_{t(j)-t}^{t(m)} (t(j)-\tau)dX(\tau) \right)}$$

$$\quad = E e^{-\sum_{m=1}^n \sum_{j=m}^n \int_{t(m)-t}^{t(m)} (t(j)-\tau)dX(\tau)}$$

$$\quad = E e^{-\int_{t(n)}^{t(n)} \left[ \sum_{j=m}^n (t(j) - (n+1-m)\tau) \right]dX(\tau)}$$

$$\quad = \prod_{m=1}^n E e^{-t(m-1)} \quad (1.27)$$

Let $u > 0$ and let $W(\tau)$ be a nonnegative, continuous function of $\tau$,

$0 < s \leq \tau \leq t$. For $X \in S_1$, again in a manner similar to the development of (A.5) of Antelman and Savage [1965]
\[
-t \int uW(\tau) dX(\tau) - u \lim_{n \to \infty} \sum_{j=1}^{n} W(s+ \frac{j(t-s)}{n}) [X(s+ \frac{j(t-s)}{n}) - X(s+ \frac{(j-1)(t-s)}{n})] = Ee \\
- u \lim_{n \to \infty} \sum_{j=1}^{n} W(s+ \frac{j(t-s)}{n}) [X(s+ \frac{j(t-s)}{n}) - X(s+ \frac{(j-1)(t-s)}{n})] \\
= \lim_{n \to \infty} \prod_{j=1}^{n} Ee \\
= \exp \left( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ \alpha(s+ \frac{j(t-s)}{n}) - \alpha(s+ \frac{(j-1)(t-s)}{n}) \right] \right) \\
= \exp \left( - \int_{s}^{t} a^{-}(\tau) \int_{0}^{\infty} \left[ 1 - e^{-uW(\tau)v} \right] dN(v) d\tau \right) \\
= \exp \left( - \int_{s}^{t} a^{-}(\tau) N^{*}(uW(\tau)) d\tau \right) \\
= e^{s} . \tag{1.28}
\]

Equation (1.27) becomes

\[
P(T > t) = e^{t} \\
= e^{t} . \tag{1.29}
\]

For \( X \in S_2 \), \( \alpha(t) = t/v \) yields

\[
- \frac{1}{v} \sum_{m=1}^{n} \int_{t}^{(m-1)} \int_{j=m}^{n} t^{(j)} N^{*}(t^{(j)} - (n+1-m)\tau) d\tau \]

\[
P(T > t) = e^{t} . \tag{1.30}
\]
1.5 Probability of specific tie configurations, Gaver model

Each \( X \in S \) is the limit of the sum of a finite number of Poisson processes, each of which has sample function discontinuities, giving rise to such discontinuities in \( X \). Because the sample functions of \( X \) are nondecreasing, these discontinuities will be simple jumps. In a reliability context, this means that under the Gaver model, in a system of \( n \) independent components exposed simultaneously to the same realization of \( X \), two or more components may fail together at one of those instants when the system receives a shock, or sudden increase in stress, corresponding to a jump in \( X \). Thus positive probability is associated with certain hypersurfaces in \( T^n \), the space of failure time outcomes of the \( n \) components.

Let \( n \) be a fixed positive integer, and for \( k = 1, \ldots, n \) let

\[
C_{n,k} = \{ \text{ordered } k\text{-tuples } (c_1, \ldots, c_k) \text{ of positive integers such that } \sum_{i=1}^{k} c_i = n \}. \tag{1.31}
\]

For \( c_{n,k} \in C_{n,k} \) one has

\[
C^{-1}(c_{n,k}) = \{ \tau \in T^n : C(\tau) = c_{n,k} \}, \tag{1.32}
\]

and \( \{ C^{-1}(c_{n,k}) : c_{n,k} \in C_{n,k}, \ k = 1, \ldots, n \} \) partitions \( T^n \). A given \( c_{n,k} = (c_1, \ldots, c_k) \) corresponds to the union of

\[
\binom{n}{c_{n,k}} \overset{\text{def}}{=} \frac{n!}{c_1! \cdots c_k!} \text{ equally likely, mutually exclusive events, of which}
\]

which
\[ T_1' = \ldots = T_{j'-1} < T_{j'} = \ldots = T_{3'-1} < \ldots < T_{k'-1} = \ldots = T_n \] (1.33)

is typical, where

\[ j' = 1 + \sum_{i=1}^{j-1} c_i, \quad i = 1, \ldots, k. \] (1.34)

Example 1.13

In Example 1.12 the event giving rise to

\[ t = (7.6, 3.5, 5.0, 8.0, 5.0, 1.9, 3.5, 5.0) \]

is \([T_6 < T_2 = T_7 < T_3 = T_5 = T_8 < T_1 < T_4]\). This is one of \((1, 2, 3, 1, 1)\) \(= 3360\) equally likely rearrangements, of which

\[ [T_1 < T_2 = T_3 < T_4 = T_5 = T_6 < T_7 < T_8] \]

corresponds to (1.33).

Now consider \(X \in S_1\) as a wear process under the Gayer model.
Recall that the log ch.f. of \(X\) has the form

\[ \log E e^{iuX(t)} = \alpha(t) \int_{0^+} (e^{iv} - 1)dN(v) \] (1.35)

for real \(u\) and nonnegative \(t\), where \(\alpha(t)\) and \(N(v)\) satisfy properties (i) and (ii), respectively, of Definition 1.4. Theorem 1.14 is stated next, giving the probability of a specific tie configuration among \(n\) independent, identical components exposed to the same realization of \(X\). Corollaries 1.15, 1.16, and 1.17 are then stated and proved, followed by the proof of Theorem 1.14.
Theorem 1.14

Suppose \( n \) independent, identical components are exposed simultaneously to the same realization of a wear process \( X \in S_1 \) under the Gaver model. Then for each \( c_{n,k} \in C_{n,k} \), \( k = 1, \ldots, n \)

\[
P(C^{-1}(c_{n,k})) = \left( \begin{array}{c} n \\ c_{n,k} \end{array} \right) \prod_{j=1}^{k} \int_{0^+}^{\infty} \frac{e^{-s_j t} c_j}{(1-e^{-s_j t})^j} dN(v),
\]

(1.36)

where \( s_j \) is given by (1.20) and \( N \) is as in Definition 1.4.

Corollary 1.15

Under the assumptions of Theorem 1.14, \( P(C^{-1}(c_{n,k})) \) is independent of the function \( \alpha(t) \) in the log ch.f. of \( X \) as represented by (1.35).

Proof:

The proof is clear since (1.36) does not involve \( \alpha(t) \).

Q.E.D.

Corollary 1.16

Suppose the assumptions of Theorem 1.14 hold and further suppose \( X(t) \sim P(\lambda(t); a) \).

Then for each \( c_{n,k} \in C_{n,k} \), \( k = 1, \ldots, n \)

\[
P(C^{-1}(c_{n,k})) = \left( \begin{array}{c} n \\ c_{n,k} \end{array} \right) (e^a-1)^n \prod_{j=1}^{k} (e^{-j-1})^{-1},
\]

(1.37)
Proof:

From Example 1.5 \( N(v) = -1_{(0, a)}(v), v > 0 \). Then it follows upon application of (1.36) that

\[
P(C^{-1}(c_n, k)) = \binom{n}{c_n, k} \prod_{j=1}^{k} \frac{e^{-s_j} a^j}{(1-e^{-a})^j}
\]

\[
= \binom{n}{c_n, k} (1-e^{-c_n}) \prod_{j=1}^{k} \frac{e^{-c_j} a^j}{(1-e^{-a})^j}
\]

\[
= \binom{n}{c_n, k} (c_n-1)^n \prod_{j=1}^{k} (c_n-j)^{-1}
\]

Q.E.D.

**Corollary 1.17**

Let \( X \in S_1, X \neq 0 \), and let \( P \) be the set of sample functions with one or more points of discontinuity.

Then \( P(D) > 0 \).

Proof:

Suppose \( P(D) = 0 \). Then almost all sample functions are continuous and the probability of two or more simultaneous failures among \( n \) components is zero. Since \( X \neq 0 \), the function \( N \) in (1.35) is not identically zero and hence all integrals in (1.36) are positive. Then by Theorem 1.14 \( P(C^{-1}(c_n, k)) > 0 \) for each \( c_n, k \in C_n, k, k = 1, \ldots, n \), which is a contradiction.

Q.E.D.
Note: Corollary 1.17 also follows from a theorem taken from Doob [1953], p. 420, which says: If $X$ is a separable, centered stochastic process with independent increments and no fixed points of discontinuity, then every increment of $X$ is normally distributed if and only if almost all sample functions of $X$ are continuous.

Proof of Theorem 1.14:

Let $c_{n,k} \in C_{n,k}$ for some $k \in \{1, \ldots, n\}$. For each $p = 1, 2, \ldots$ in the notation of (1.33) and (1.34)

$$P\{T_1 < \cdots < T_p \leq \cdots < T_k = \cdots = T_n \leq 2^p\}$$

$$= \lim_{m \to \infty} \sum_{0 \leq i_1 < \cdots < i_k \leq 2^{m+p} - 1} \frac{1}{2^m} \mathbb{P}\{T_{i_1} < \cdots \leq T_{i_{j+1}} - 1 \leq \frac{1}{2^{m}}, j = 1, \ldots, k \mid X(\cdot)\}$$

where the summation is over all integers $i_1, \ldots, i_k$ such that $0 \leq i_1 < \cdots < i_k \leq 2^{m+p} - 1$. For each $m$ and each $0 \leq i_1 < \cdots < i_k \leq 2^{m+p} - 1$ let $u_j = i_j / 2^m$ and $\delta_j = 1 / 2^m$. Then

$$0 \leq u_1 < u_1 + \delta_1 \leq u_2 < u_2 + \delta_2 \leq \cdots \leq u_k < u_k + \delta_k \leq 2^p$$

and

$$\mathbb{P}\{u_j < T_{i_j} \leq \cdots \leq T_{i_{j+1}} - 1 \leq u_j + \delta_j, j = 1, \ldots, k \mid X(\cdot)\}$$

$$= \mathbb{E} \prod_{j=1}^{k} \left[ e^{-X(u_j)} - e^{-X(u_j + \delta_j)} c_j \right].$$

(1.38)

Now

$$\prod_{j=1}^{k} \left[ e^{-X(u_j)} - e^{-X(u_j + \delta_j)} c_j \right] j = \prod_{j=1}^{k} e^{-X(u_j + \delta_j) - X(u_j)} c_j$$
\[
\prod_{j=1}^{k} \left[ e^{-X(u_j)} - e^{-X(u_j+\delta_j)} \right] c_j
\]

where \( u_0 = \delta_0 = 0 \). Equation (1.38) becomes

\[
\prod_{j=1}^{k} e^{-s_j[X(u_j)-X(u_{j-1}+\delta_j)]} \left\{ 1-e^{-[X(u_j+\delta_j)-X(u_j)]} \right\} c_j e^{-s_{j+1}[X(u_j+\delta_j)-X(u_j)]}
\]

Equation (1.39)

\[
r^{-N^k} = e \sum_{j=1}^{k} \left[ \alpha(u_j) - \alpha(u_{j-1}+\delta_j) \right] N^k(s_j) \prod_{j=1}^{k} \sum_{m_j=0}^{c_j} (-1)^m_j
\]

\[
\left\{ 1-\sum_{j=1}^{k} \left[ \alpha(u_j+\delta_j) - \alpha(u_j) \right] N^k(m_j+s_{j+1}) \right\} + o(\sum_{j=1}^{k} \delta_j)
\]

\[
\sum_{j=1}^{k} \left[ \alpha(u_j) - \alpha(u_{j-1}+\delta_j) \right] N^k(s_j) \prod_{j=1}^{k} \sum_{m_j=0}^{c_j} (-1)^m_j
\]

\[
\left[ \alpha(u_j+\delta_j) - \alpha(u_j) \right] N^k(m_j+s_{j+1}) + o(\sum_{j=1}^{k} \delta_j),
\]
where \( N^*(u) = \int_0^\infty (1-e^{-u}) dN(v) \). Then

\[
P\{T_1^- = \ldots = T_{2^k-1}^- < \ldots < T_k^- = \ldots = T_n^-\}
\]

\[
= \lim_{p \to \infty} P\{T_1^- = \ldots = T_{2^k-1}^- < \ldots < T_k^- = \ldots = T_n^- < 2^p\}
\]

\[
= \lim_{p \to \infty} \int \ldots \int_{0<u_1<\ldots<u_k<2^p} \left( -\sum_{j=1}^k \alpha(u_j)-\alpha(u_{j-1}) \right) N^*(s_j)
\]

\[
= \lim_{p \to \infty} \prod_{j=1}^k \sum_{m_j=0}^{c_j-1} \left( m_j \right) (-1)^{m_j} \sum_{m_j=0}^{c_j-1} \left( m_j \right) (-1)^{m_j}
\]

\[
= \left( \prod_{j=1}^k \sum_{m_j=0}^{c_j-1} \left( m_j \right) (-1)^{m_j} \right) \left( \sum_{j=1}^k \alpha(u_j) du_j \right)
\]

(1.40)

where \( \Delta N_j^* = N_j^*(s_j) - N_j^*(s_{j+1}) \), \( j = 1, \ldots, k \). Now

\[
= \left( \prod_{j=1}^k \sum_{m_j=0}^{c_j-1} \left( m_j \right) (-1)^{m_j} \right) \left( \sum_{j=1}^k \alpha(u_j) du_j \right)
\]

(1.41)
Further
\[ \int_{0}^{\infty} \ldots \int_{0}^{\infty} e^{-\sum_{j=1}^{k} \Delta N^*_j(u_j)} \prod_{j=1}^{k} \alpha_j^{u_j} du_j = \prod_{j=1}^{k} \frac{1}{\Delta N^*_j} = \prod_{j=1}^{k} \frac{1}{N^*(s_j)}. \] (1.42)

From (1.41) and (1.42), (1.40) becomes

\[ \mathbb{P} \{ T_1 = \ldots = T_{2^{n-1}} = \ldots = T_n \} = \prod_{j=1}^{k} \frac{\int_{0^+}^{\infty} e^{-s j + 1} v^{j} (1 - e^{-v}) \, dN(v)}{\int_{0^+}^{\infty} (1 - e^{-s j v}) \, dN(v)} \] (1.43)

and hence

\[ \mathbb{P} \{ C^{-1}(x_n, k) \} = (\frac{n}{x_n, k}) \prod_{j=1}^{k} \frac{\int_{0^+}^{\infty} e^{-s j + 1} v^{j} (1 - e^{-v}) \, dN(v)}{\int_{0^+}^{\infty} (1 - e^{-s j v}) \, dN(v)}. \] 

Q.E.D.

1.6 Conditional densities, Gaver model

We have seen that positive probability is associated with those hyperplanes in $T^n$ corresponding to the various possible tie configurations. In Chapter II a MLE of the parameter $\nu$ in (1.12) will be derived from a density conditioned on the tie configuration in the observation $\tau$ when $X \in S_2$ is the Gaver model wear process. This motivates the following.
Let $A$ be an event, or measurable set, in $\mathbb{T}^n$. For each $A$

define the measure $\mu$ by

$$\mu(A) = \sum_{k=1}^{n} \sum_{c_{n,k}} \mu_k(A \cap \mathbb{C}_n^{-1}(c_{n,k}))$$

(1.44)

where the inner summation is over all $c_{n,k} \in \mathbb{C}_n$ and $\mu_k$ is $k$-
dimensional Lebesgue measure. Let $X \in \mathbb{S}_1$, with $\mathbb{F}(t)$ given by (1.22).
The probability measure $P$ is absolutely continuous with respect to $\mu$.
By the Radon-Nikodym Theorem there exists a density $p$ with respect
to $\mu$ such that for each event $A$ in $\mathbb{T}^n$

$$P(A) = \int_A p(t)\,d\mu$$

$$= \sum_{k=1}^{n} \sum_{c_{n,k}} \int_{A \cap \mathbb{C}_n^{-1}(c_{n,k})} p(t)\,d\mu_k.$$

Let $\delta^{(j)}$ be a $j$-tuple of positive numbers. The inequality $D(T) > D(t)$
is defined only if both vectors have the same dimension, which occurs
if $C(T) = C(t)$. Let $k$ be the dimension of $C(t)$. The set
$C^{-1}(C(t))$ is open in $k$-dimensional space, so for the elements of $\delta^{(n)}$
sufficiently small the set $\{u: t < u < t + \delta^{(n)}, C(u) = C(t)\}$
intersects no tie configuration hyperplane of lower dimension than
that of $C(t)$. Then

$$P(t < T < t + \delta^{(n)} | C(T) = C(t))$$

$$= \frac{1}{\binom{n}{C(t)}} P(D(t) < D(T) < D(t) + \delta^{(k)} | C(T) = C(t))$$
where $\binom{n}{C(t)}$ arises because of symmetry. The vector $\vec{\delta}^{(n)}$ becomes $\vec{\delta}^{(k)}$ by letting $\delta_j$, the $j$th element of $\vec{\delta}^{(k)}$, be the minimum of the $c_j$ elements of $\vec{\delta}^{(n)}$ associated with $t[j]$, $j = 1, \ldots, k$. Then

$$\lim_{\vec{\delta}^{(k)} \to 0} \frac{1}{k} \prod_{j=1}^{k} \frac{P\{D(t) < D(T) < D(t) + \vec{\delta}^{(k)} \mid C(T) = C(t)\}}{\Pi \delta_j}$$

$$= \lim_{\vec{\delta}^{(k)} \to 0} \frac{\sum_{\vec{\delta}^{(k)} \mid C(T) = C(t)} P\{C(T) = C(t)\}}{\prod_{j=1}^{k} \delta_j}$$

$$= \frac{1}{\int_{0+}^{\infty} e^{-s} j! (1 - e^{-v})^j dN(v)} \prod_{j=1}^{k} N^*(s_j)$$

$$\binom{n}{C(t)}$$

$$\lim_{\vec{\delta}^{(k)} \to 0} \frac{P\{D(t) < D(T) < D(t) + \vec{\delta}^{(k)} \mid C(T) = C(t)\}}{\prod_{j=1}^{k} \delta_j}$$

(1.45)

from Theorem 1.14 with $C(t) = (c_1, \ldots, c_k)$. From (1.38) and (1.39) with $\delta_0 = t[0] = 0$ it follows that

$$\lim_{\vec{\delta}^{(k)} \to 0} \frac{P\{D(t) < D(T) < D(t) + \vec{\delta}^{(k)} \mid C(T) = C(t)\}}{\prod_{j=1}^{k} \delta_j}$$

$$= \binom{n}{C(t)} \lim_{\vec{\delta}^{(k)} \to 0} \prod_{j=1}^{k} \frac{E(t[j] \leq T \leq t[j+1] \leq \cdots \leq t[j] + \delta_j, j=1, \ldots, k})}{\prod_{j=1}^{k} \delta_j}$$
\[
\begin{align*}
&= \binom{n}{C(t)} \lim_{\delta(k) \to 0} \frac{1}{\prod_{j=1}^{k} \delta_j} \left[ \prod_{j=1}^{k} \left[ e^{-X(t_{[j]})} - e^{-X(t_{[j]} + \delta_j)} \right] c_j \right] \\
&= \binom{n}{C(t)} e^{\sum_{j=1}^{k} \alpha(t_{[j]})} \left[ \prod_{j=1}^{k} \alpha^-(t_{[j]}) \right] \\
&= \binom{n}{C(t)} e^{\sum_{j=1}^{k} \alpha(t_{[j]})} \left[ \prod_{j=1}^{k} \alpha^-(t_{[j]}) \right] e^{-s_{j+1} \nu} \int_{0^+} e^{-v} c_{j} dN(v)
\end{align*}
\]

with the notation of (1.22). Equation (1.45) now becomes

\[
\lim_{\delta(k) \to 0} \frac{1}{\prod_{j=1}^{k} \delta_j} P(D(t) \leq D(T) < D(t) + \delta(k) | C(T) = C(t))
\]

\[
= \frac{1}{k} \prod_{j=1}^{k} \Delta_N^{s_j} e^{\sum_{j=1}^{k} \alpha(t_{[j]})} \left[ \prod_{j=1}^{k} \alpha^-(t_{[j]}) \right] \Delta_N^{*}(s_j) .
\]

(1.46)

Define for \( t \in \mathcal{T}^n \), and for the \( \binom{n}{C(t)} - 1 \) other \( u \in C^{-1}C(t) \) such that \( C(u) = C(t) \),

\[
p(D(t) | C(t)) = \lim_{\delta(k) \to 0} \frac{1}{\prod_{j=1}^{k} \delta_j} P(D(t) \leq D(T) < D(t) + \delta(k) | C(T) = C(t))
\]
so that from (1.46)

\[
p(D(t)|C(t)) = \frac{1}{k} \cdot \frac{\alpha(t[j])^{\Delta N_j^*}}{\prod_{j=1}^{k} N_j^*(s_j)} \cdot e^{-\sum_{j=1}^{k} \alpha^-(t[j])^{\Delta N_j^*}}. \tag{1.47}
\]

Cancelling the $\Delta N_j^*$'s in (1.47) yields

\[
p(D(t)|C(t)) = e^{-\sum_{j=1}^{k} \alpha(t[j])^{\Delta N_j^*}} \cdot \prod_{j=1}^{k} \alpha^-(t[j])^{N_j^*(s_j)}. \tag{1.48}
\]

In the light of (1.18) and (1.22) $p(D(t)|C(t))$ may also be expressed as

\[
p(D(t)|C(t)) = \frac{(-1)^k}{k} \cdot \frac{\beta(t[1]) \cdots \beta(t[k])}{\prod_{j=1}^{k} N_j^*(s_j)} \cdot e^{-\sum_{j=1}^{k} c_j X(t[j])}. \tag{1.49}
\]

That $p(D(t)|C(t))$ is a density with respect to $k$-dimensional Lebesgue measure may be seen from its nonnegativity and from integrating (1.47) over $0 < t[1] < \ldots < t[k] < \infty$, obtaining unity. This integration includes all $u \in C^{-1}C(t)$. Thus for $t \in T^n$ the density $p(t)$ can be expressed conditionally upon $C(t)$ by

\[
p(t) = \frac{P(C^{-1}C(t))}{\binom{n}{C(t)}} \cdot p(D(t)|C(t)). \tag{1.50}
\]

To see that $p(t)$ integrates to unity over $T^n$, observe that
\[ \int p(t) d\mu = \sum_{k=1}^{n} \sum_{n, k} \frac{p(t) d\mu_k}{C_{n, k}^{-1}(c_{n, k})} \]

\[ = \sum_{k=1}^{n} \sum_{n, k} \frac{p(C_{n, k}^{-1}(c_{n, k}))}{C_{n, k}^{-1}(c_{n, k})} \]

\[ = \sum_{k=1}^{n} \sum_{n, k} p(C_{n, k}^{-1}(c_{n, k})) \]

\[ = 1. \]

For \( X \in S_2 \), \( \alpha(t) = \frac{1}{\nu} t \), which when substituted in (1.48)
yields

\[ p(D(t)|C(t)) = e^{-\frac{1}{\nu} \sum_{j=1}^{k} \Delta N^*(t_{[j]})} \frac{1}{\nu} \prod_{j=1}^{k} N^*(s_j) \]  

(1.51)

and equivalently

\[ p(D(t)|C(t)) = e^{-\frac{1}{\nu} \sum_{j=1}^{k} N^*(s_j) \Delta t_{[j]}} \frac{1}{\nu} \prod_{j=1}^{k} N^*(s_j). \]  

(1.52)

**Example 1.18**

Let \( X(t) \sim P(\frac{1}{\nu} t; 1) \) and \( n = 2 \). Then \( C_{2, 1} = \{(2)\} \) and \( C_{2, 2} = \{(1,1)\} \). For \( c_{2, 1} = (2) \) it is seen that \( s_1 = c_1 = 2 \),

\[ [D(t)|(2)] = (t_{[1]}), \ N^*(s_1) = 1-e^{-2} \text{ and } \Delta N^*_1 = N^*(s_1). \]

By (1.52)

\[ p(D(t)|(2)) = \frac{1}{\nu} (1-e^{-2}) e^{-\frac{1}{\nu} (1-e^{-2}) t_{[1]}}. \]
Substituting in (1.50)

\[ p(t) = \frac{P(C^{-1}(2))}{\binom{2}{2}} p(t^2 | (2)) . \]

From Corollary 1.16

\[ P(C^{-1}(2)) = P(T_1 = T_2) = \frac{1 - e^{-1}}{1 + e^{-1}} \]

so for \( t = (t, t), t > 0 \)

\[ p(t, t) = \frac{1}{v} \left( 1 - e^{-1} \right)^2 e^{-\frac{1}{v}(1 - e^{-2})t} . \] \hspace{1cm} (1.53)

For \( c_2, 2 = (1, 1) \), one has \( s_1 = 2, s_2 = 1, [D(t) | (1, 1)] = (t[1], t[2]), \)
\[ N^*(s_1) = 1 - e^{-2}, N^*(s_2) = 1 - e^{-1}, \Delta N^*_1 = e^{-1}(1 - e^{-1}), \text{ and} \]
\[ \Delta N^*_2 = (1 - e^{-1}). \] Thus

\[ p(D(t) | (1, 1)) = \frac{1}{v^2} (1 - e^{-1})(1 - e^{-2})e^{-\frac{1}{v}(1 - e^{-1})(e^{-1}t[1] + t[2])}. \]

Further \( P(C^{-1}(1, 1)) = \frac{2}{1 + e^{-1}} \) and for \( t = (t, u), 0 < t < u \)

\[ p(t, u) = p(u, t) = \frac{P(C^{-1}(1, 1))}{\binom{2}{2}} p(D(t) | 2) \]

\[ = \frac{1}{v^2} (1 - e^{-1})^2 e^{-\frac{1}{v}(1 - e^{-1})(e^{-1}t[1] + t[2])}. \] \hspace{1cm} (1.54)

Equations (1.53) and (1.54) show this distribution is bivariate exponential, as expounded by Marshall and Olkin [1967]; in fact, for
X ∈ S_2 (1.24) shows that \( F(t) \) is multivariate exponential. Marshall and Olkin state that they do not attempt a decomposition into parts absolutely continuous with respect to Lebesgue measure because "it is cumbersome and apparently of little importance." In our case, symmetry reduces the cumbersome aspect, and the decomposition is important because it gives a conditional likelihood for MLE procedures.

1.7 Constant jump processes

This section considers as models for accumulated wear, stochastic processes beginning at zero and increasing by jumps of constant size, described more fully in Definition 1.19. The Poisson processes are examples. It will be seen that such wear processes need not have independent increments for the results of Corollary 1.16 to hold. These processes may also have fixed points of discontinuity, thus serving as models for shocks of constant intensity which may occur with positive probability at specific times.

Definition 1.19

The stochastic process \( Y(t) \), \( t ≥ 0 \), will be called a constant jump process if

(i) \( Y(0) = 0 \) a.s.

(ii) \( Y \) increases by jumps of constant size \( a > 0 \) and never decreases.

(iii) The probability of two or more jumps in an interval of length \( δ > 0 \) is of order \( o(δ) \) as \( δ → 0 \).
(iv) \( Y(t) \to \infty \mathrm{a.s.\ as\ } t \to \infty. \)

Now follows an alternate derivation of the probability of ties and tie configurations, giving fresh insight to Section 1.5.

Suppose a system of \( n \) components is exposed to an environment such that the accumulated wear is a constant jump process \( Y(t), \)
\( t \geq 0, \) with jump size \( a > 0, \) and

\[
P(T_j > t | Y(\cdot)) = e^{-Y(t)}, \quad t \geq 0, \ j = 1, \ldots, n.
\]

For each \( \omega \in \Omega \) let \( \tau_0(\omega) = 0 \) and \( \tau_m(\omega), \ m = 1, 2, \ldots, \) be the time of the \( m^{th} \) jump of the sample function \( Y(\omega, \cdot). \) Then for almost all \( \omega \)

\[
P(T_j = \tau_m(\omega)) = P(T_j > \tau_{m-1}(\omega)) - P(T_j > \tau_m(\omega))
\]

\[
= e^{-a(m-1)} - e^{-am}, \quad m = 1, 2, \ldots.
\]

It follows that

\[
P(T_j > \tau_m(\omega) | T_j > \tau_{m-1}(\omega)) = e^{-a},
\]  \hspace{1cm} (1.55)

which gives a probabilistic interpretation to the size \( a \) of the jump in the wear process: the probability of a live component surviving the next shock is \( e^{-a}. \) Because \( a = \infty \) has not been admitted, the fatal shock model is excluded.

Then for \( 1 \leq k \leq n \)
\[ P\{T_1 = \ldots = T_k\} = \mathbb{E}P\{T_1 = \ldots = T_k | Y(\omega, \cdot)\} \]
\[ = \sum_{m=1}^{\infty} \mathbb{E}P\{T_1 = \ldots = T_k | \tau_m(\omega) > Y(\omega, \cdot)\} \]
\[ = \sum_{m=1}^{\infty} \left[ e^{-a(m-1)} - e^{-am}\right]^k \]
\[ = \frac{(1-e^{-a})^k}{1-e^{-ak}}. \quad (1.56) \]

Letting \( q = e^{-a} \), one further has
\[ P\{T_1 = \ldots = T_k\} = \frac{(1-q)^k}{1-q^k}. \quad (1.57) \]

A tie configuration \( c_{n,k} = (c_1, \ldots, c_k) \) occurs if and only if one of the \( \binom{n}{c_{n,k}} \) events like
\[ [T_1^- = \ldots = T_{2^-1} < T_{2^-} = \ldots = T_{3^-1} < \ldots < T_{k^-} = \ldots = T_n] \]
occurs, where \( j^- = 1 + \sum_{i=1}^{j-1} c_i, \ j = 1, \ldots, k \). Let \( \tau(m), m = 1, 2, \ldots \) be the random time of the \( m \)th jump.

\[ P\{T_1^- = \ldots = T_{2^-1} < \ldots < T_{k^-} = \ldots = T_n\} \]
\[ = \sum_{1 < m_1 < \ldots < m_k} P\{T_{j^-} = \ldots = T_{(j+1)^-1} = \tau(m_j), j = 1, \ldots, k\} \]
\[ = \sum_{m_1=1}^{\infty} c_1\{T_{1^-} = \tau(m_1)\} \sum_{m_2=m_1+1}^{\infty} c_2\{T_{2^-} = \tau(m_2)\} \ldots \sum_{m_k=m_{k-1}+1}^{\infty} c_k\{T_{k^-} = \tau(m_k)\} \]
\[ \sum_{m_1=1}^{\infty} \left( q^{m_1-1} - q^{-1} \right) c_1 \sum_{m_2=m_1+1}^{\infty} \left( q^{m_2-1} - q^{-1} \right) c_2 \ldots \sum_{m_k=m_{k-1}+1}^{\infty} \left( q^{m_k-1} - q^{-1} \right) c_k \]

\[ = (1-q)^n \sum_{m_1=0}^{\infty} q^{m_1} \sum_{m_2=m_1+1}^{\infty} q^{m_2} \ldots \sum_{m_k=m_{k-1}+1}^{\infty} q^{m_k} \]

Repeatedly using the relation

\[ \sum_{m=m^{(k+1)+1}}^{\infty} q^{cm} = \frac{q^{c \cdot (m^{(k+1)+1})}}{1-q^c} \sum_{m=0}^{\infty} q^{cm} \]

one obtains, using \( c_j = \sum_{i=j}^{k} c_i \), \( j = 1, \ldots, k \) as before,

\[ P(T_1 = \ldots = T_{k-1} < T_k = \ldots = T_n) = (1-q)^n \left( \frac{q^{s_k}}{1-q^k}, \ldots, \frac{q^{s_2}}{1-q^2} \right) \sum_{m=0}^{\infty} q^{s_1m} \]

and ultimately

\[ P(C^{-1} \{ c_{n,k} \}) = \left( \frac{n}{\lambda_n} \right) (q^{-1}-1)^n \prod_{j=1}^{k} (q^{-j}-1)^{-1}. \] (1.58)

Substitution of \( q = e^{-\lambda} \) in (1.37), Corollary 1.16, in the case when \( X \) is a Poisson process, shows agreement with (1.58).

Note the following independence relation, which is seen to hold for constant jump processes:

\[ P[T_{j=1} = \ldots = T_{(j+1)-1}, j=1, \ldots, k] = \prod_{j=1}^{k} P[T_{j=1} = \ldots = T_{(j+1)-1}]. \] (1.59)

If we define for \( Y \) the function \( N^*(u), u \geq 0 \), by

\[ N^*(u) = 1-q^u, \]

then (1.37) and (1.43) hold for constant jump wear processes as well as for \( X \in S_1 \).
II. INFERENCE, GAVER MODEL

2.1 Maximum likelihood estimation, multiple observations from a single trial

In this chapter the MLE of the parameter \( \nu \) in (1.12) is obtained when the wear process follows a Gaver model with stationary increments. In this section this is done when failure times of \( n \) identical, independent components are observed from the same environmental realization, called a trial. The conditional distribution of the estimate is derived, along with the properties of consistency, minimum variance, and unbiasedness. Confidence intervals and tests of hypothesis are discussed in Section 2.3. The estimation procedure is extended in Section 2.4 to \( m \) independent trials, including a discussion of experimental design considerations. In Section 2.5 an extension is made to \( M \) types of components per trial. In Section 2.6 the condition \( X(0) = 0 \) a.s. is relaxed.

First let \( X \in S_1 \) and suppose the function \( \alpha(t) \) in Definition 1.4 is also a function of a real parameter \( \nu \) in some interval \( \Theta \), while \( N(\nu) \) in the same definition does not involve \( \nu \). Assume \( \frac{\partial \alpha}{\partial t} > 0 \) for \( t > 0 \) and assume \( \frac{\partial^2 \alpha}{\partial \nu \partial t} \) exists. Indexing the distribution of \( T \) by the parameter \( \nu \), we have according to (1.50)

\[
P_{\nu}(t) = \frac{P(C^{-1}C(t))}{\int_{C(t)}^R P_{\nu}(D(t)|C(t))},
\]

\[ (2.1) \]
where by Corollary 1.15 $P(C^{-1}C(t))$ does not involve $\nu$ and where

$$p_{\nu}(D(t)|C(t)) = e^{\sum_{j=1}^{k} \Delta N^{*}_{j}(t_{[j]};\nu)} \prod_{j=1}^{k} \frac{\partial N^{*}_{j}(s_{j})}{\partial t} (t_{[j]};\nu)$$

(2.2)

as in (1.48).

Once a sample $t$ of $n$ failure times from a single trial is obtained, all the pertinent information is in $D(t)$ and $C(t)$, and the likelihood $L(t)$ is the conditional density $p_{\nu}(D(t)|C(t))$. Then

$$\log L(t) = - \sum_{j=1}^{k} \Delta N^{*}_{j}(t_{[j]};\nu) + \sum_{j=1}^{k} \log N^{*}(s_{j}) + \sum_{j=1}^{k} \log \frac{\partial N^{*}_{j}}{\partial t} (t_{[j]};\nu)$$

(2.3)

and

$$\frac{\partial}{\partial \nu} \log L(t) = - \sum_{j=1}^{k} \Delta N^{*}_{j} \frac{\partial N^{*}}{\partial \nu} (t_{[j]};\nu) + \sum_{j=1}^{k} \frac{\partial^2 N^{*}}{\partial \nu \partial t} (t_{[j]};\nu) = 0.$$  

(2.4)

A solution $\hat{\nu}$ to (2.4) is not available in closed form without specifying $\alpha(t;\nu)$. In $S_{2}$, $\alpha(t;\nu) = t/\nu$, a rather simple form. It might seem that more generality could be achieved by letting $\alpha(t;\nu) = f(t)g(\nu)$, where $f(t)$ is nonnegative, increasing, and differentiable and $g(\nu)$ is positive, strictly monotonic, and differentiable. However, the transformations

$$t^{*} = f^{-1}(t), \quad g^{*} = g^{-1}(1/\nu)$$

lead to

$$\alpha(t^{*};\nu^{*}) = f(f^{-1}(t))g(g^{-1}(1/\nu)) = t/\nu.$$
Thus knowledge of $f$ and $g$ means the problem can be transformed into one involving the simple form $t/v$. The reason for working with $1/v$ rather than $\lambda$, say, as the constant multiplier of $t$ is that the MLE $\hat{v}$, given $k$, has a convenient distribution (gamma, Theorem 2.2), which has an easily obtained mean and variance in contrast to the corresponding MLE $\hat{\lambda}$.

For $X \in S_2$ the likelihood equation (2.4) becomes

$$\sum_{j=1}^{k} \frac{\Delta N^*_j t[j]}{v^2} - \frac{k}{v} = 0$$

with the solution

$$\hat{v} = \sum_{j=1}^{k} \frac{\Delta N^*_j t[j]}{k}$$

(2.5)

or equivalently

$$\hat{v} = \sum_{j=1}^{k} \frac{N^*_j(s_j)\Delta t[j]}{k}$$

(2.6)

where $\Delta t[j] = t[j] - t[j-1]$, $j = 1, \ldots, k$ with $t[0] = 0$. (See (1.8), (1.20), and (1.23) for the notation $N^*$ and $s_j$.) Equation (2.5) expresses $\hat{v}$ directly in terms of the distinct failure times; equation (2.6) is more convenient in investigating the conditional distribution of $\hat{v}$. Theorem 2.1 is useful to this end, and is of interest itself in providing a decomposition of the distinct failure times.

Let $T$ represent the random vector of failure times of $n$ components subjected to the same realization of the wear process $X$ under the Gaver model. Define for each $c_{n,k} \in C_{n,k}$, $k = 1, \ldots, n$
\[(D(T)|C(T) = c_n,k) = (T[1], \ldots, T[k]),\]

where \(D(T)\) and \(C(T)\) are the random vector analogs of \(D(t)\) and \(C(t)\), (1.16) and (1.17) and \(\mathcal{N},k\) is as in (1.31). Let \(\Delta T[j] = T[j] - T[j-1], j = 1, \ldots, k\) with \(T[0] = 0\).

**Theorem 2.1**

Let \(X \in S_2\). Then given \(C(T) = c_n,k\), the r.v.'s \(\Delta T[j], j = 1, \ldots, k\) are independently, exponentially distributed with mean \(\frac{\nu}{\mathcal{N}(s_j)}, j = 1, \ldots, k\).

**Proof:**

According to the development of (1.47), the joint density of \((D(T)|C(T) = c_n,k)\) is given for \(t \in C^{-1}(c_n,k)\) by (1.52): 

\[p_{\nu}(D(T)|c_n,k) = e^{-\frac{1}{\nu} \sum_{j=1}^{k} \mathcal{N}(s_j) \Delta T[j]} \left( \prod_{j=1}^{k} \frac{1}{\nu^k} \mathcal{N}(s_j), 0 < \Delta T[j] < \infty, j = 1, \ldots, k \right).\]

This is the product of \(k\) exponential densities. It follows that the \(\Delta T[j]\)'s are independently, exponentially distributed with mean \(\frac{\nu}{\mathcal{N}(s_j)}, j = 1, \ldots, k\).

**Q.E.D.**

**Note:** When \(k = n\), then \(s_j = n - j + 1, j = 1, \ldots, n\) and the \(\Delta T[j]\)'s are independently, exponentially distributed with mean

\[E\Delta T[j] = \frac{\nu}{\mathcal{N}(n-j+1)}, j = 1, \ldots, n.\]

This is similar to the Epstein-Sobel-Renyi decomposition of order
statistics, which says (Epstein and Sobel, [1953], Appendix): If 
\( X(j) \) is the \( j^{\text{th}} \) order statistic of a random sample of size \( n \) drawn 
from an exponential distribution with mean \( \theta \), and \( X(0) = 0 \), then the 
random variables \( X(j) - X(j-1) \) are independently, exponentially distributed with 
\[
E[X(j) - X(j-1)] = \frac{\theta}{n-j+1}, \quad j = 1, \ldots, n. 
\]
Suppose one failure time is observed from each of \( n \) independent realizations of the wear process 
given in the above theorem. Then 
\[
P(T_j > t) = e^{- (1/\nu) \Lambda(1)t, \quad t \geq 0, \quad j = 1, \ldots, n.} 
\]
According to the Epstein-Sobel-Renyi decomposition, the 
\( \Delta T[j] \)'s are independently, exponentially distributed with 
\[
E[\Delta T[j]] = \frac{\nu}{(n-j+1)\Lambda(1)}, \quad j = 1, \ldots, n. \tag{2.8} 
\]
Equations (2.7) and (2.8) differ because the former is based on \( n \) 
failure times from a single trial, while the latter assumes each failure 
time comes from an independent trial.

Theorem 2.2 summarizes some properties of the MLE \( \hat{\nu} \).

**Theorem 2.2**

The MLE \( \hat{\nu} \) as given by (2.6), conditioned on \( k \) distinct failure 
times, has a gamma distribution with mean \( \nu \) and variance \( \nu^2/k \) and 
is a minimum variance unbiased estimate of \( \nu \).

**Proof:**

Recall (2.6):

\[
\hat{\nu} = \sum_{j=1}^{k} \frac{N^*(s_j) \Delta T[j]}{k}, \tag{2.10} 
\]

where the \( \Delta T[j] \)'s are defined above Theorem 2.1. It follows from
Theorem 2.1 that \( \hat{\nu} \) has, given \( c_{n,k} \), a gamma distribution with mean \( \nu \) and variance \( \nu^2/k \). Thus the distribution of \( \hat{\nu} \) depends on \( c_{n,k} \) only through \( k \), and given \( c_{n,k} \), \( \hat{\nu} \) is unbiased.

By the Cramér–Rao Inequality, for each \( c_{n,k} \)

\[
\text{Var}(\hat{\nu} | c_{n,k}) \geq \frac{1}{\mathbb{E} \left[ \frac{\partial}{\partial \nu} \log p_{\nu}(D(T)|C(T)=c_{n,k}) \right]^2}.
\]

For \( X \in S_2 \), in accord with (1.51)

\[
p_{\nu}(D(T)|C(T)=c_{n,k}) = e^{-\frac{1}{\nu} \sum_{j=1}^{k} \frac{\Delta N^*_{T}[j]}{\nu}} \frac{1}{\nu^k} \prod_{j=1}^{k} N^*(s_j),
\]

whence

\[
\frac{\partial}{\partial \nu} \log p_{\nu}(D(T)|C(T)=c_{n,k}) = \sum_{j=1}^{k} \frac{\Delta N^*_{T}[j]}{\nu^2} - \frac{k}{\nu}
\]

\[
= \frac{k}{\nu^2} \left( \sum_{j=1}^{k} \frac{\Delta N^*_{T}[j]}{k} - \nu \right)
\]

and

\[
\mathbb{E} \left[ \frac{\partial}{\partial \nu} \log p_{\nu}(D(T)|C(T)=c_{n,k}) \right]^2 = \frac{1}{\nu^4} \text{Var}(\hat{\nu} | c_{n,k})
\]

\[
= \frac{\nu^2}{k} = \text{Var}(\hat{\nu} | c_{n,k}).
\]

Hence \( \hat{\nu} \) has minimum variance, \( k = 1, \ldots, n \).

Q.E.D.
In considering unconditionally the estimate $\hat{\nu}$, let $K_n$ be the r.v. representing the number of distinct failure times among the $T_i$, $i = 1, \ldots, n$ from a single trial. The distribution of $\hat{\nu}$ depends on $C(T)$ only through $K_n$. In particular $\mathbb{E}[\hat{\nu} | C(T) = c_{n,k}] = \nu$ for each $c_{n,k}$ implies $\mathbb{E}\hat{\nu} = \nu$, and then

$$\text{Var} \hat{\nu} = \mathbb{E}[\text{Var}(\hat{\nu} | C(T))] = \nu^2 \mathbb{E} \frac{1}{K_n}.$$ 

For each positive integer $n$ the distribution of $K_n$ is given by

$$P(K_n = k) = \begin{cases} \sum_{c_{n,k}} P(C^{-1}(c_{n,k})), & k = 1, \ldots, n \\ \frac{1}{C_{n,k}} & 0, \text{otherwise} \end{cases} \quad (2.13)$$

where the summation is over all $c_{n,k} \in C_{n,k}$ (cf. (1.31)). From Theorem 1.14 for $X \in S_1$

$$P(C^{-1}(c_{n,k})) = \binom{n}{c_{n,k}} \prod_{j=1}^{\infty} \frac{(-s_j^+)^{j} \nu \epsilon_{j} \text{dN}(\nu)}{0^+ \int (1-e^{-j}) \text{dN}(\nu)} \quad (2.14)$$

No simplification is evident for $X \in S_2$ except for constant jump processes discussed in Sections 1.7 and 2.2. Lemmas 2.3 and 2.4 show that $\mathbb{E} \frac{1}{K_n} \to \infty$ as $n \to \infty$, which means Var $\hat{\nu} \to 0$ as $n \to \infty$ and hence $\hat{\nu}$ is consistent.

**Lemma 2.3**

For $X \in S_1$, $K_n \to \infty$ in probability as $n \to \infty$. 
Proof:

Let $k$ be a fixed positive integer and let $\epsilon \in (0,1)$. We wish to show $\Pr\left( K > k \right) > 1 - \epsilon$ for large $n$.

In Definition 1.4 almost all sample functions of $X \in S_1$ get arbitrarily large with $t$. Thus for each $\omega \in \Omega$ except possibly for those in a set $Z$ of probability zero, there exists a partition

$$0 = t_0 < t_1 < \ldots < t_{k+1} < \infty,$$

depending on $\omega$, such that

$$0 = X(\omega, t_0) < X(\omega, t_1) < \ldots < X(\omega, t_{k+1}) < \infty,$$

where $X(\omega, t)$ is a sample function of $X$.

For each integer $n > k+1$ let the random vector of failure times $T = (T_1, \ldots, T_n)$ be partitioned into $T = (T_1, \ldots, T_{k+1})$, where the $i$th partition contains $n(i)$ elements, $\sum_{i=1}^n n(i) = n$. The idea is to show that as each $n(i)$ gets large, at least one failure time from the $i$th partition falls into the interval $(t_{i-1}, t_i]$ for each $i = 1, \ldots, k+1$ with probability greater than $1 - \epsilon$, so that for large $n$ there are more than $k$ distinct failure times with probability tending to one.

For each $\omega \in \Omega \setminus Z$, the event $\left[ K > k+1 \mid X(\omega, \cdot) \right]$ includes events $\bigcap_{i=1}^{k+1} B_i$, where for each $i$, $B_i = \left[ \text{at least one } T_j \text{ from } T_i \text{ falls in } (t_{i-1}, t_i] \mid X(\omega, \cdot) \right]$. From (1.3) and the independence of the $T_j$ within a realization it follows that

$$\Pr\{ B_i \} = 1 - \left[ 1 - e^{-X(\omega, t_i)} - e^{-X(\omega, t_{i-1})} \right] n(i).$$

Since $X(\omega, t_i) > X(\omega, t_{i-1}) > 0$, then

$$0 < 1 - e^{-X(\omega, t_{i-1})} + e^{-X(\omega, t_i)} < 1$$

and

$$-X(\omega, t_{i-1}) < -X(\omega, t_i) < 0$$

thus

$$1 - e^{-X(\omega, t_i)} < 1 - e^{-X(\omega, t_{i-1})}.$$
and a positive integer $N_{\epsilon, X(\omega, \cdot)}$ exists such that $n(i) \geq N_{\epsilon, X(\omega, \cdot)}$ implies

$$[1 - e^{-X(\omega, t_{i-1})} + e^{-X(\omega, t_i)}]n(i) < \frac{\epsilon}{2(k+1)}, \ i = 1, \ldots, k+1.$$ 

Then for each $n = \sum_{i=1}^{k+1} n(i)$ with each $n(i) \geq N_{\epsilon, X(\omega, \cdot)}$

$$P(K_n \geq k+1 | X(\omega, \cdot)) \geq P\left( \bigcap_{i=1}^{k+1} B_i \right)$$

$$= \prod_{i=1}^{k+1} \left( 1 - [1 - e^{-X(\omega, t_{i-1})} + e^{-X(\omega, t_i)}]n(i) \right)$$

$$> [1 - \frac{\epsilon}{2(k+1)}]^{k+1}$$

$$> 1 - \frac{\epsilon}{2}. \quad (2.15)$$

For each $m = 1, 2, \ldots$ let $A_m = \{\omega \in \Omega \setminus \mathbb{Z}: N_{\epsilon, X(\omega, \cdot)} = m\}$. Clearly

$$\bigcup_{m=1}^{\infty} A_m = \Omega \setminus \mathbb{Z}$$

and $P(\Omega \setminus \mathbb{Z}) = 1$, so there exists a positive integer $M_{\epsilon}$ such that

$$P(\bigcup_{m=1}^{M_{\epsilon}} A_m) > 1 - \frac{\epsilon}{2}.$$ 

From (2.15) for $\omega \in \bigcup_{m=1}^{M_{\epsilon}} A_m$ and $n(i) \geq M_{\epsilon}$, $i = 1, \ldots, k+1$

$$P(K_n \geq k+1 | X(\omega, \cdot)) > 1 - \frac{\epsilon}{2},$$

where $n = \sum_{i=1}^{k+1} n(i)$, and
\[ P(K_n \geq k+1) = \mathbb{E} P(K_n \geq k+1 | X(\omega, \cdot)) \]

\[ \geq \int_M P(K_n \geq k+1 | X(\omega, \cdot)) dP \]

\[ \bigcup_{m=1}^{\infty} A_m \]

\[ > (1 - \frac{\epsilon}{2})^2 \]

\[ > 1 - \epsilon. \]

Thus \( K_n \to \infty \) in probability as \( n \to \infty \).

Q.E.D.

Lemma 2.4

Let \( \{X_n\} \) be a sequence of r.v.'s concentrated on the positive integers. For each \( k = 1, 2, \ldots \) suppose \( \lim_{n \to \infty} P(X_n = k) = 0 \).

Then \( E \frac{1}{X_n} \to 0 \) as \( n \to \infty \).

Proof:

Let \( \epsilon > 0 \) be given and choose an integer \( m > 2/\epsilon \). There exists an \( N_{\epsilon, m} < \infty \) such that \( n > N_{\epsilon, m} \) implies \( \sum_{k=1}^{m} P(X_n = k) < \frac{\epsilon}{2} \). Then

\[ E \frac{1}{X_n} = \sum_{k=1}^{m} \frac{1}{k} P(K_n = k) + \sum_{k=m+1}^{\infty} \frac{1}{k} P(K_n = k) \]

\[ \leq \sum_{k=1}^{m} P(K_n = k) + \frac{1}{m} \sum_{k=m+1}^{\infty} P(K_n = k) \]

\[ < \epsilon. \]

Q.E.D.

We have thus proved:
Theorem 2.5

The estimate $\hat{v}$ is unbiased and consistent as $n \to \infty$.

Remark: Although the estimate $\hat{v}$ has conditionally minimum variance for each $c_{n,k}$, this may not be unconditionally true for $\hat{v}$ when $n > 1$.

From (2.1)

$$E\left[\frac{3}{\nu} \log p_{\nu}(T)\right]^2 = E\{E\left[\frac{3}{\nu} \log p_{\nu}(T)\right]^2 | C(T)\}\}
$$

$$= \sum_{k=1}^{n} \sum_{C_{n,k}} P\{C(T)=c_{n,k}\} E\left[\frac{3}{\nu} \log p_{\nu}(D(T)|C(T)=c_{n,k})\right]^2.$$

From (2.12)

$$E\left[\frac{3}{\nu} \log p_{\nu}(D(T)|C(T)=c_{n,k})\right]^2 = \frac{k}{\nu^2}$$

and it then follows by (2.13) that

$$E\left[\frac{3}{\nu} \log p_{\nu}(T)\right] = \frac{1}{\nu^2} EK_n.$$

Thus $\nu^2/EK_n$ is the Cramér-Rao lower bound for the unbiased estimate of $\nu$. It follows from Jensen's inequality that for $n > 1$

$$\text{Var} \hat{v} = \nu^2 E \frac{1}{K_n} > \nu^2 \frac{1}{EK_n}.$$

However, $\hat{v}$ is asymptotically efficient whenever $E \frac{1}{K_n} = \frac{1}{EK_n} + o\left(\frac{1}{EK_n}\right)$.

This condition is fulfilled by constant jump wear processes, as shown in Section 2.2.
2.2 \( E K_n^{-1} \) for constant jump wear processes

Let \( Y(t), t \geq 0 \), be a constant jump wear process as given in Definition 1.19, which includes all Poisson processes. Let \( q \) be the probability of a live component surviving the next shock; \( q = e^{-a} \), where \( a \) is the size of the jump in the accumulated wear process. Let \( K_n \) be the random number of distinct failure times among \( n \) components exposed to a single realization of \( Y \). From (1.58) and (2.13) the distribution of \( K_n \) is

\[
P(K_n = k) = \sum_{c_{n,k}} \left( \begin{array}{c} n \\ c_{n,k} \end{array} \right) (q^{-1} - 1)^n \prod_{j=1}^{k} (q^{-1} - j)^{-1}, k = 1, \ldots, n. \tag{2.16}
\]

From this \( E \frac{1}{K_n} \) may be calculated. Following are explicit formulae for \( E \frac{1}{K_n} \) in terms of \( q \) for \( n = 2, \ldots, 6 \):

\[
E \frac{1}{K_2} = \frac{1}{1+q} \tag{2.17}
\]

\[
E \frac{1}{K_3} = \frac{2+q+q^2}{2(1+q)(1+q+q^2)} \tag{2.18}
\]

\[
E \frac{1}{K_4} = \frac{1}{(1+q)^2} \tag{2.19}
\]

\[
E \frac{1}{K_5} = \frac{6 + 9q + 24q^2 + 25q^3 + 30q^4 + 22q^5 + 20q^6 + 5q^7 + 4q^8 - q^9}{6(1+q)(1+q+q^2)(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)} \tag{2.20}
\]

\[
E \frac{1}{K_6} = \frac{(1-q+q^2)(2+4q+11q^2+13q^3+12q^4+4q^5+3q^6-q^7)}{2(1+q)(1+q+q^2)(1+q+q^3)(1+q+q^3+q^4+q^5)} \tag{2.21}
\]

Explicit representations of \( E \frac{1}{K_n} \) for larger \( n \) are not attempted here. Instead, bounds on \( E \frac{1}{K_n} \) as \( n \to \infty \) are developed, culminating in Theorem 2.9.
Lemma 2.6

Let $X \geq 1$ a.s., $EX = \mu$ and $\text{Var} X = \sigma^2$. Let $\lambda$ be positive and satisfy $\mu - \lambda \sigma > 0$.

Then \[ \frac{1}{\mu} \leq \frac{1}{X} \leq \frac{1}{\mu - \lambda \sigma} + \frac{1}{\lambda^2}. \]

Proof:

The left-hand inequality is an application of Jensen's inequality.

Consider

\[ \frac{1}{EX} = \int_{-\infty}^{\infty} x^{-1} dF(x). \]

If $\mu - \lambda \sigma \leq 1$, then $\frac{1}{EX} \leq 1 \leq \frac{1}{\mu - \lambda \sigma} < \frac{1}{\mu - \lambda \sigma} + \frac{1}{\lambda^2}$. If $\mu - \lambda \sigma > 1$, then

\[ \frac{1}{EX} = \int_{1}^{\infty} x^{-1} dF(x) + \int_{\mu - \lambda \sigma}^{\infty} x^{-1} dF(x) \]

\[ \leq P(X \leq \mu - \lambda \sigma) + \frac{1}{\mu - \lambda \sigma} P(X \geq \mu - \lambda \sigma). \]

By Chebyshev's inequality

\[ P(X \leq \mu - \lambda \sigma) \leq P(|X - \mu| \geq \lambda \sigma) \leq \frac{1}{\lambda^2}. \]

Hence

\[ \frac{1}{EX} \leq \frac{1}{\mu - \lambda \sigma} + \frac{1}{\lambda^2}. \]

Q.E.D.
Lemma 2.7

The random number \( K_n \) of distinct failure times among \( n \) components subjected to the same realization of a constant jump wear process with jump size \( a > 0 \) satisfies

\[
E K_n \approx \frac{1}{a} \log n + O(1). \tag{2.22}
\]

Proof:

Let \( q = e^{-a} \), so that \( q \) is the probability of a live component surviving the next jump, or shock, and let \( p = 1 - q \). Let \( X_i \) be the index of the shock at which the \( i \)th component fails. \( X_i \) has a geometric distribution,

\[
P(X_i = r) = q^{r-1}p, \quad r = 1, 2, \ldots.
\]

Let

\[
Y_r = \begin{cases} 
1, & \text{if some component fails on the } r \text{th shock} \\
0, & \text{otherwise.}
\end{cases}
\]

Then

\[
P(Y_r = 1) = 1 - (1-q^{r-1}p)^n.
\]

Set \( z = \frac{1}{a} \log p < 0 \). Since \( K_n = \sum_{r=1}^{\infty} Y_r \),

\[
E K_n = \sum_{r=1}^{\infty} P(Y_r = 1) = \sum_{r=1}^{\infty} \{1 - [1 - e^{-a}(r-1-z)]^n\}.
\]

Let \( B \) be the unique integer such that \( B-1 \leq z < B \leq 0 \). Then
\[
\sum_{r=1}^{\infty} \left(1-e^{-a(r-B)}\right)^n < EK_n < \sum_{r=1}^{\infty} \left(1-e^{-a(r-1-B)}\right)^n.
\] (2.23)

Considering the left-hand side of (2.23),
\[
\sum_{r=1}^{\infty} \left(1-e^{-a(r-B)}\right)^n = \sum_{s=0}^{\infty} \left(1-e^{-as}\right)^n - \sum_{s=0}^{-B} \left[1-(1-e^{-as})^n\right].
\]

Then
\[
\sum_{s=0}^{\infty} \left[1-(1-e^{-as})^n\right] > \int_{0}^{\infty} [1-(1-e^{-ax})^n]dx = \frac{1}{a} \sum_{i=1}^{n} \frac{1}{i}.
\] (2.24)

(For a proof of the last equality, see Epstein and Sobel [1953]

Appendix, noting that the integral in (2.24) gives the expectation of the largest order statistic of a random sample of size \(n\) from an exponential distribution with mean \(\frac{1}{a}\)) Hence
\[
EK_n > \frac{1}{a} \sum_{i=1}^{n} \frac{1}{i} - 1 - \sum_{s=1}^{-B} \left[1-(1-e^{-as})^n\right]
\]

with the last summation vacuous for \(B = 0\). Considering the right-hand side of (2.23) in a similar manner,
\[
\sum_{r=1}^{\infty} \left(1-e^{-a(r-1-B)}\right)^n = 1 + \sum_{s=1}^{\infty} \left[1-(1-e^{-as})^n\right] - \sum_{s=1}^{-B} \left[1-(1-e^{-as})^n\right]
\]

and
\[
\sum_{s=1}^{\infty} \left[1-(1-e^{-as})^n\right] < \int_{0}^{\infty} [1-(1-e^{-ax})^n]dx = \frac{1}{a} \sum_{i=1}^{n} \frac{1}{i}.
\]

Therefore
\[
\frac{1}{a} \sum_{i=1}^{n} \frac{1}{i} - 1 - \sum_{s=1}^{n} [1-(1-e^{-as})^{n}] < \frac{1}{a} \sum_{i=1}^{n} \frac{1}{i} + 1 - \sum_{s=1}^{n} [1-(1-e^{-as})^{n}].
\]

Since \( \log n < \sum_{i=1}^{n} \frac{1}{i} < \log n + \gamma \), where \( \gamma \) is Euler's constant, it follows that

\[
\frac{E_{K}}{n} = \frac{1}{a} \log n + O(1).
\]

Q.E.D.

**Lemma 2.8**

The random number \( K_n \) of distinct failure times among \( n \) components subjected to the same realization of a constant jump wear process with jump size \( a > 0 \) has a variance which remains bounded as \( n \to \infty \).

**Proof:**

Using the same notation as in Lemma 2.7,

\[
E_{K_n} = \sum_{r=1}^{\infty} [1-(1-q^{-r-1}p)^n] = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} E_{Y_{r,s}}^2.
\]

For \( r \neq s \),

\[
E_{Y_{r,s}} = P(Y_{r,s}=1) = 1 - P(Y_{r}=0) - P(Y_{s}=0) + P(Y_{r,Y_{s}}=0)
= 1 - (1-q^{-r-1}p)^n - (1-q^{-s-1}p)^n + (1-q^{-r-1}p - q^{-s-1}p)^n.
\]
Thus (2.26) becomes

\[
E_n^2 = \sum_{r=1}^{\infty} [1-(1-q^{r-1}p)^n] + \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} [1-(1-q^{r-1}p)^n-(1-q^{s-1}p)^n+(1-q^{r-1}p-q^{s-1}p)^n],
\]

\[
= \sum_{r=1}^{\infty} [(1-q^{r-1}p)^n-(1-2q^{r-1}p)^n]
\]

\[
+ \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} [1-(1-q^{r-1}p)^n-(1-q^{s-1}p)^n+(1-q^{r-1}p-q^{s-1}p)^n].
\]

Squaring both sides of (2.25) yields

\[
E_n^2 = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} [1-(1-q^{r-1}p)^n-(1-q^{s-1}p)^n+(1-q^{r-1}p-q^{s-1}p)^n+(1-q^{r-1}p-q^{s-1}p+q^{r+s-2}p^2)^n].
\]

Then

\[
\operatorname{Var} K_n = A_n + B_n,
\]

where

\[
A_n = \sum_{r=1}^{\infty} [(1-q^{r-1}p)^n-(1-2q^{r-1}p)^n]
\]

\[
B_n = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} [(1-q^{r-1}p-q^{s-1}p)^n-(1-q^{r-1}p-q^{s-1}p+q^{r+s-2}p^2)^n].
\]

Consider the behavior of \(A_n\) and \(B_n\) as \(n \rightarrow \infty\).

First consider \(A_n\): Let \(q = e^{-a}\) and \(z = \frac{1}{a} \log p\). Let \(B\) be the unique nonpositive integer such that \(B-1 < z < B \leq 0\). There exists a positive integer \(R\) such that \(r \geq R + B > 0\) implies
\[1 - q^{r-1} p \geq 1 - 2q^{r-1} p \geq 0. \text{ Write} \]

\[ A_n = \sum_{r=1}^{R+B} [(1-q^{r-1} p)^n - (1-2q^{r-1} p)^n] \]

\[ + \sum_{r=R+B+1}^{\infty} q^{r-1} p^n \sum_{k=0}^{n-1} (1-q^{r-1} p)^k (1-2q^{r-1} p)^{n-1-k}. \]

Since \((1-q^{r-1} p)^n \to 0\) and \((1-2q^{r-1} p)^n \to 0\) as \(n \to \infty\),

\[ \lim_{n \to \infty} \sum_{r=1}^{R+B} [(1-q^{r-1} p)^n - (1-2q^{r-1} p)^n] = 0. \]

Also

\[ \sum_{r=R+B+1}^{\infty} q^{r-1} p^n \sum_{k=0}^{n-1} (1-q^{r-1} p)^k (1-2q^{r-1} p)^{n-1-k} \]

\[ \leq \sum_{r=R+B+1}^{\infty} q^{r-1} p^n (1-q^{r-1} p)^{n-1} \]

\[ = \sum_{r=R+B+1}^{\infty} ne^{-a(r-1-z)} [1-e^{-a(r-1-z)}]^{n-1} \]

\[ \leq \sum_{r=R+B+1}^{\infty} ne^{-a(r-1-B)} [1-e^{-a(r-B)}]^{n-1} \]

\[ \leq e^a \sum_{s=R+1}^{\infty} \int_{s-1}^{s} ne^{-ax} [1-e^{-a(x+1)}]^{n-1} dx \]

\[ \leq \frac{1}{a} e^{2a} \int_{0}^{\infty} ne^{-ax} (1-e^{-ax})^{n-1} dx \]

\[ = \frac{1}{a} e^{2a} \int_{0}^{1} n y^{n-1} dy \]

upon substituting \(y = 1-e^{-ax}\). Since \(\int_{0}^{1} n y^{n-1} dy = 1\) for each \(n = 1, 2, \ldots\),

it follows that \(A_n\) is bounded.
Now consider $B_n$: By inspection all but a finite number of terms in the double summation defining $B_n$ are negative, and the positive ones are bounded by one. $\text{Var } K_n = A_n + B_n \geq 0$ and $A_n$ is bounded. Hence $\text{Var } K_n$ is bounded as $n \to \infty$.

Q.E.D.

**Theorem 2.9**

Suppose $n$ components are subjected to a single realization of a constant jump wear process with known jump size $a > 0$, under the Gaver model. Let $K_n$ be the number of distinct failure times.

Then

$$E \frac{1}{K_n} = \frac{a}{\log n} + o\left(\frac{1}{\log n}\right). \tag{2.27}$$

Proof:

Let $\mu_n = E_k$ and $\sigma_n = \sqrt{\text{Var } K_n}$. By Lemma 2.7 $\mu_n = \frac{1}{a} \log n + O(1)$ and by Lemma 2.8, $\sigma_n = O(1)$. For large $n$, $\frac{\mu_n}{\sigma_n} \geq 0$. Applying Lemma 2.6 with $\lambda_n = \mu_n^{2/3}$

$$\frac{1}{\mu_n} \leq E \frac{1}{K_n} \leq \frac{1}{\mu_n^{2/3} O(1)} + \frac{1}{\mu_n^{4/3}}$$

and hence

$$E \frac{1}{K_n} = \frac{1}{\mu_n} + o\left(\frac{1}{\mu_n}\right) = \frac{a}{\log n} + o\left(\frac{1}{\log n}\right).$$

Q.E.D.
2.3 Confidence intervals and hypothesis testing

This section contains some methods of setting confidence intervals and testing hypotheses with regard to the MLE of \( \nu \). It is not intended to give a complete coverage of inferential techniques. Recall the distribution of \( \hat{\nu} \) given \( c_{n,k} \) depends on \( c_{n,k} \) only through \( k \), the number of distinct failures in the sample of size \( n \) from a single realization. By Theorem 2.2 in the notation of (1.11) given \( k \), \( \hat{\nu} \sim \Gamma(k, k/\nu) \). An exact 100(1-\( \alpha \))% confidence interval can be set about an observed value \( \hat{\nu} = \hat{\nu}_0 \) with the help of \( \chi^2 \) tables. Since, given \( k \), the estimate \( \hat{\nu} \sim \Gamma(k, k/\nu) \), then \( \frac{2k\hat{\nu}_0}{\nu} \) is distributed as \( \chi^2_{2k} \), a \( \chi^2 \) distribution with 2\( k \) degrees of freedom. For a given \( \alpha \)-level choose \( x_{2k} \) and \( y_{2k} \) from the tables such that

\[
P(\chi^2_{2k} > x_{2k}) = \frac{\alpha}{2}, \quad P(\chi^2_{2k} > y_{2k}) = 1 - \frac{\alpha}{2}\]

(2.28)

and set

\[
\nu_L = \frac{2k\hat{\nu}_0}{x_{2k}}, \quad \nu_U = \frac{2k\hat{\nu}_0}{y_{2k}}.
\]

(2.29)

Then \([\nu_L, \nu_U]\) is a 100(1-\( \alpha \))% confidence interval. One-sided confidence intervals can be set in a similar manner. Given large \( k \), \( \hat{\nu} \) is asymptotically normally distributed with mean \( \nu \) and variance \( \nu^2/k \). Then from tables of the normal distribution an approximate 100(1-\( \alpha \))% confidence interval \( \hat{\nu}_0 \pm z_{\alpha/2} \frac{\hat{\nu}_0}{\sqrt{k}} \) can be set, where \( z_{\alpha/2} \) is the standard normal deviate corresponding to \( \alpha/2 \).
Example 2.10

Suppose the underlying wear process giving rise to
t = (7.6, 3.5, 5.0, 8.0, 5.0, 1.9, 3.5, 5.0) is X(t) ∼ P(\frac{1}{\nu} t; a), as
in (1.10). Suppose a = - \log 0.9 is known, which is equivalent to a
probability of 0.9 of a live component surviving the next shock. The
parameter ν is to be estimated. As in Example 1.12

C(t) = (1, 2, 3, 1, 1), k = 5,

D(t) = (1.9, 3.5, 5.0, 7.6, 8.0).

From (1.19)

(S_1, S_2, S_3, S_4, S_5) = (8, 7, 5, 2, 1)

and N*(u) = 1 - .9^u, u ≥ 0. Then using (2.6),

\hat{\nu} = \frac{1}{k} \sum_{j=1}^{k} \frac{N*(s_j)\Delta t}{l_j} = 0.613.

For a two-sided 95% confidence interval select x_{10} = 20.4831,
y_{10} = 3.24697 according to (2.28), from a table of the χ^2 distribution
with 10 degrees of freedom. Equation (2.29) gives the 95% confidence
interval conditioned on k = 5:

[\nu_L, \nu_U] = [.299, 1.888].

To test the simple hypothesis H_0: \nu = \nu_0 against the simple
alternative H_1: \nu = \nu_1 > \nu_0 on the basis of a sample of n failures.
with \( k \) distinct failure times from a single trial, consider the ratio
\[
\frac{p_{\nu_1} (D(t) | C(t))}{p_{\nu_0} (D(t) | C(t))} = \left( \frac{\nu_0}{\nu_1} \right)^k e^{-\left( \frac{1}{\nu_1} - \frac{1}{\nu_0} \right) \sum_{j=1}^{k} \Delta N^*_t[j]}.
\]

By the Neyman-Pearson lemma given \( k \) a most powerful size-\( \alpha \) test has the form:

Reject \( H_0 \) if \( \sum_{j=1}^{k} \Delta N^*_t[j] \geq b_{\alpha,k} \)

Accept \( H_0 \) if \( \sum_{j=1}^{k} \Delta N^*_t[j] < b_{\alpha,k} \)

where \( b_{\alpha,k} \) is a constant depending on \( \alpha \) and \( k \) but not on \( \nu_1 \) as long as \( \nu_1 > \nu_0 \). This test, conditioned on \( k \), is uniformly most powerful for \( H_0 \) vs. \( H_1 \).

Suppose \( t_1 \) and \( t_2 \) represent \( n_1 \) and \( n_2 \) failure times, respectively, from two independent trials, and say that one wants to test the hypothesis \( H_0: \nu_1 = \nu_2 \) vs. \( H_1: \nu_1 \neq \nu_2 \). Suppose \( C(t_1) = c_{n_1,k_1} \) and \( C(t_2) = c_{n_2,k_2} \). Since the r.v.'s \( \hat{\nu}_1 \) and \( \hat{\nu}_2 \) are independently distributed as \( \Gamma(k_1, k_1/\nu_1) \) and \( \Gamma(k_2, k_2/\nu_2) \) respectively, given \( k_1 \) and \( k_2 \), under \( H_0 \) the ratio
\[
R = \frac{\hat{\nu}_1}{\hat{\nu}_2} = \frac{\frac{2k_1\hat{\nu}_1}{\nu}}{\frac{2k_2\hat{\nu}_2}{\nu}} = \frac{2k_1\nu_1}{2k_2\nu_2}
\]
is the ratio of two independent \( \chi^2 \) r.v.s divided by their respective degrees of freedom, and therefore \( R \) has an \( F \)-distribution with \( 2k_1 \) degrees of freedom in the numerator and \( 2k_2 \) degrees of freedom in the
denominator. Critical values for rejecting $H_0$ with probability $\alpha$ (conditioned on $k_1$ and $k_2$) when $H_0$ is true are obtained from a table of the $F$-distribution. In Section 2.4 it is shown (cf. (2.30)) that under $H_0$ the pooled MLE $\hat{\nu}$ of $\nu$ is

$$\hat{\nu} = \frac{k_1 \hat{\nu}_1 + k_2 \hat{\nu}_2}{k_1 + k_2}.$$ 

It should be noted that the properties of the procedures given in this section are stated conditionally upon the observed number $k$ of distinct failure times, which is an ancillary statistic. Unconditional methods may be preferable, as in the case of testing hypotheses when power is the only consideration. For a treatment of the subject see Wallace [1959].

2.4 Extension to $m$ independent trials

Suppose $m$ independent realizations of the wear process $X \in S_2$ are made, with $n(r)$ identical components exposed to the $r$th realization, $r = 1, \ldots, m$. Let $t_{r}$ be the vector of $n(r)$ observed failure times. The likelihood function is

$$L(t_1, \ldots, t_m) = \prod_{r=1}^{m} p_{\nu}(D(t_{r})|C(t_{r})),$$

where for $r = 1, \ldots, m$

$$C(t_{r}) = (c_{r,1}, \ldots, c_{r,k(r)})$$

$$D(t_{r}) = (t_{r,[1]}, \ldots, t_{r,[k(r)]}).$$
Then, using (2.4),

\[
\frac{\partial}{\partial \nu} \log L = \sum_{r=1}^{m} \frac{\partial}{\partial \nu} \log p_{\nu}(D(t_r)|C(t_r))
\]

\[
= \sum_{r=1}^{m} \frac{\partial}{\partial \nu} \left[ -\frac{1}{\nu} \sum_{j=1}^{k(r)} \Delta N^*_{r,j} tr[j] - k(r) \log \nu \right],
\]

where for each \(r\)

\[
\Delta N^*_r, j = N^*_r(s_{r,j}) - N^*_r(s_{r,j+1}), \quad j = 1, \ldots, k
\]

\[
s_{r,j} = \sum_{i=j}^{k(r)} c_{r,i}, \quad j = 1, \ldots, k+1.
\]

Finally

\[
\hat{\nu} = \frac{\sum_{r=1}^{m} k(r) \frac{\sum_{j=1}^{k(r)} \Delta N^*_r, j \ t_r, [j]}{k(r)}}{\sum_{r=1}^{m} k(r)}.
\]  \hspace{1cm} (2.30)

An application of Theorem 2.2 shows this combined estimate given the vector \((k(1), \ldots, k(m))\) is distributed as \(\Gamma(\sum_{r=1}^{m} k(r), \sum_{r=1}^{m} k(r)/\nu)\) and is a minimum variance unbiased estimate of \(\nu\). The unconditional estimate \(\hat{\nu}\) is unbiased and consistent as \(m \to \infty\) or as one or more of the \(n(r)\) approach infinity.

This knowledge is helpful in designing experiments, where the number of trials and the number of components per trial must be decided. Minimum variance of the estimate is achieved by running one component per trial. This is the thing to do if the components
are rare or expensive and if the cost of running a trial is relatively small. On the other hand, when components are inexpensive but the trials are costly, then several, or many, components per trial may be desirable. More specifically, suppose the number of trials \( m \) and the number of components \( n \) per trial are to be sized to minimize expected cost. (For convenience take \( n \) to be the same for each trial.) Assume a unit cost per component, a fixed cost \( f \) per trial, and relative squared error loss \( L(\hat{v}) = \frac{b(\hat{v} - v)^2}{v^2} \) so that

\[
EL(\hat{v}) = \frac{bE(\hat{v} - v)^2}{v^2} = bE \frac{1}{\sum_{r=1}^{m} K_{n,r}}
\]

where \( b \) is a known constant and \( K_{n,r} \) is the random number of distinct failure times on the \( r \)-th trial. The expectation of the overall cost \( A \) of performing the experiment and using the MLE \( \hat{v} \) is

\[
EA = mn + mf + bE \frac{1}{\sum_{r=1}^{m} K_{n,r}}.
\] (2.31)

The problem is to minimize \( EA \) subject to \( m \) and \( n \) being positive integers.

Some prior knowledge of the wear process is necessary to establish

\[
E \frac{1}{\sum_{r=1}^{m} K_{n,r}}. \text{ If } E \frac{1}{K_{n,r}} = \frac{1}{EK_{n,r}} + o \left( \frac{1}{EK_{n,r}} \right), \text{ then it can be shown that}
\]

\[
E \frac{1}{\sum_{r=1}^{m} K_{n,r}} = \frac{1}{E \sum_{r=1}^{m} K_{n,r}} + o \left( \frac{1}{E \sum_{r=1}^{m} K_{n,r}} \right). \text{ The condition holds whenever the wear process has constant jump size } \alpha > 0 \text{ (Lemma 2.7 and Theorem 2.9)}
\]

and using (2.27) one obtains
\[ E \frac{1}{m} = \frac{a}{m \log n} + o \left( \frac{1}{\log n} \right). \]

Then

\[ EA = mn + mf + ba \frac{1}{m \log n} + o \left( \frac{1}{\log n} \right) \]

\[ = mn + mf + ba \frac{1}{m \log n}. \]  \hspace{1cm} (2.32)

Then setting partial derivatives equal to zero

\[ \frac{\partial EA}{\partial m} = n + f - ba \frac{1}{m^2 \log n} = 0 \]

\[ \frac{\partial EA}{\partial n} = m - ba \frac{1}{mn \log^2 n} = 0 \]

yields

\[ m^2 n \log n = \frac{ba}{\log n} = m^2 (n+f). \]

Thus

\[ n \log(n) - 1 = f \]  \hspace{1cm} (2.34)

and

\[ m = \frac{1}{\log n} \sqrt{\frac{ba}{n}}. \]  \hspace{1cm} (2.35)

Equation (2.34) requires \( n \geq 3 \) to satisfy \( f > 0 \). The approximation (2.33) is not good for \( n = 1 \), and neither is (2.35). However, for \( n = 1 \),
\[ E \frac{1}{K_1} = 1 \] and no approximation is needed. For \( m = 1 \) and \( 2 \leq n \leq 6 \), equations (2.17) through (2.21) give \( E \frac{1}{K_n} \) exactly. Auxiliary work may be needed for \( m > 1 \) and intermediate values of \( n \). For large \( n \), (2.34) and (2.35) should give acceptable results.

**Example 2.11**

(a) Let \( f = 10, b = 1, a = .5 \). From (2.34) \( n[\log(n)-1] = 10 \) yields \( n \approx 9 \) and \( m = \frac{1}{\log 9} \sqrt{\frac{.5}{9}} \approx 1 \).

Use \( n = 9, m = 1 \).

(b) Let \( f = 10, b = 400, a = .5 \). As in (a), \( n \approx 9 \), and

\[ m = \frac{1}{\log 9} \sqrt{\frac{200}{9}} \approx 2. \]

Use \( n = 9, m = 2 \).

(c) Let \( f = 0.1, a = .5, b = 8 \). Then (2.34) gives \( n \approx 3 \) and from (2.35), \( m \approx 1 \). With \( m = 1 \), if EA is computed using exact results for \( E \frac{1}{K_n} \), \( n = 2, 3, 4 \) as provided by (2.17) through (2.19) with \( m = 1 \), one obtains

- \( n = 2: \ EA = 7.08 \)
- \( n = 3: \ EA = 6.85 \)
- \( n = 4: \ EA = 7.20. \)

So far \( n = 3, m = 1 \) appears acceptable. However, suppose \( m = 3 \) and \( n = 1 \). Then \( E \frac{1}{\sum_{r=1}^{3} K_{1,r}} = \frac{1}{3} \) and from (2.31) \( EA = 5.97. \)

Thus for small \( n \), (2.34) and (2.35) do not always give the best design.
2.5 Extension to \( M \) types of components per trial

Suppose \( n(r) \) components of type \( r = 1, \ldots, M \) are exposed simultaneously to the same realization of \( X \in S_2 \). Suppose for the \( r \)th type of component

\[
P(T > t | X(\cdot)) = e^{-a_r X(t)}, \quad t \geq 0
\]

and the vector \((a_1, \ldots, a_M)\) of \( M \) positive constants is known. The problem is to estimate \( \nu \) from a vector of observed failure times

\[
t = (t_{1,1}, \ldots, t_{1,n(1)}; \ldots; t_{M,1}, \ldots, t_{M,n(M)}).
\]

Here \( t_{r,i} \) is the observation of \( T_{r,i} \), the random failure time of the \( i \)th component of type \( r \). Form the distinct ordered failure time vector

\[
D(t) = (t_{[1]}, \ldots, t_{[k]}), \quad 0 < t_{[1]} < \ldots < t_{[k]}
\]

and the \( M \times k \) matrix

\[
C(t) = \begin{bmatrix}
c_{1,1} & \cdots & c_{i,k} \\
\vdots & \ddots & \vdots \\
\vdots & & \ddots \\
c_{M,1} & \cdots & c_{M,k}
\end{bmatrix},
\]

where \( c_{r,j} \) represents the number of type \( r \) components failing at time \( t_{[j]} \).
\[ P(T > t) = Ee_{r=1}^{M} \sum_{i=1}^{n(r)} a_{r} X(t_{r,i}) \]
\[ - \sum_{j=1}^{k} \sum_{r=1}^{M} a_{r} c_{r,j} X(t_{[j]}) \]
\[ = Ee_{j=1}^{k} \left[ \frac{s^{*}_{j}}{X(t_{[j]}) - X(t_{[j-1]})} \right] \]

where

\[ s^{*}_{j} = \sum_{i=j}^{k} \left( \sum_{r=1}^{M} a_{r} c_{r,i} \right), j = 1, \ldots, k+1. \]  

(2.36)

By independent increments

\[ P(T > t) = e^{\frac{1}{\nu} \sum_{j=1}^{k} \Delta N^{* t}_{j}[j]} \]
\[ - \frac{1}{\nu} \sum_{j=1}^{k} N^{*}(s^{*}_{j}) \Delta t_{[j]} \]
\[ = e^{\frac{1}{\nu} \sum_{j=1}^{k} N^{*}(s^{*}_{j}) \Delta t_{[j]}}, \]  

(2.38)

where \( \Delta N^{*}_{j} = N^{*}(s^{*}_{j}) - N^{*}(s^{*}_{j+1}) \). This is an extension of (1.24) and (1.25) for \( m = 1 \), with \( s^{*}_{j} \) replacing \( s_{j} \). The theory of Section 2.1 applies immediately. In particular the MLE \( \hat{\nu} \) of \( \nu \) is

\[ \hat{\nu} = \sum_{j=1}^{k} \Delta N^{* t}_{j}[j] \]

and given \( k \), \( \hat{\nu} \sim \Gamma(k, k/\nu) \).
2.6 Relaxation of \( X(0) = 0 \) a.s.

The assumption \( X \in S_2 \) implies \( X(0) = 0 \) a.s. Suppose instead that \( X(0) \) is a nondegenerate r.v., but retain

(i) a.s. \( \inf X(0) = 0 \),

(ii) \( X(0) \) is infinitely divisible,

(iii) \( X \) has independent increments.

The physical significance of permitting \( P\{X(0) > 0\} > 0 \) is that failures at time zero can occur with positive probability when \( X \) is the underlying Gaver model wear process. However, the assumption of independent increments permits one to accommodate the zero failure times in a sample \( (t_1, \ldots, t_n) \) and to work with the positive \( t_i \) in estimating \( \nu \).

More specifically, from (1.12) for \( t > 0 \)

\[
\log \mathbb{E} e^{iu[X(t)-X(0)]} = \frac{1}{v} t \int_0^\infty (e^{iu \nu} - 1)dN(\nu)
\]

and now assume the existence of a nondecreasing function \( N_0(\nu), \nu > 0 \) such that \( N_0(\infty) = 0 \),

\[
\int_0^\infty \frac{s}{2\pi} \lim_{n \to \infty} \frac{CN_n(s)}{s} < \infty, \text{ and}
\]

\[
\log \mathbb{E} e^{iuX(0)} = \int_{0^+} \frac{\nu}{1 + \nu^2} dN_0(\nu)
\]

Suppose \( N(\cdot) \) is known, and for the time being \( N_0(\cdot) \) is not known, but is functionally independent of \( \nu \). For an observed failure time vector

\[
\tilde{t} = (t_1, \ldots, t_n) \in \tilde{T}^n,
\]

let

\[
\tilde{D}(0) = (0), \tilde{D}(\tilde{t}) = (0, t_{[1]}, \ldots, t_{[k]}), \tilde{t} \neq 0
\]

(2.39)

\[
\tilde{C}(0) = (n), \tilde{C}(\tilde{t}) = (c_0, c_1, \ldots, c_k), \tilde{t} \neq 0
\]

(2.40)

where \( c_0 > 0 \) is the number of zero failure times and \( c_j \) and \( t_{[j]} \),

\[ j = 1, \ldots, k, \text{ are as before.}\]

Then
\[ P(T > t) = E e^{-\sum_{j=0}^{k} c_j X(t[j])} \]
\[ = E e^{-c_0 X(0) - \sum_{j=1}^{k} s_j [X(t[j]) - X(t[j-1])] - \sum_{j=1}^{k} \Delta N^* t[j]} \]
\[ = e^{-N_0^*(c_0) - \frac{1}{\nu} \sum_{j=1}^{k} \Delta N^* t[j]} \]
\[ = e^{-\sum_{j=1}^{k} \Delta N^* t[j]} \cdot \frac{1}{\nu^k}, \quad (2.41) \]

since \( X(0) \) is independent of \( X(t[j]) - X(t[j-1]), j = 1, \ldots, k \). Then

\[ P(\tilde{D}(t) | \tilde{C}(t)) = R(t) e^{-\sum_{j=1}^{k} \Delta N^* t[j]} \frac{1}{\nu^k}, \quad (2.42) \]

where \( R(t) \) normalizes \( P(\tilde{D}(t) | \tilde{C}(t)) \) and does not involve \( \nu \). Let \( \tilde{k}_n \)
represent the random dimension of \( \tilde{C}(t) \). For \( \tilde{k}_n = 1 \), corresponding to \( t = 0 \), the situation is degenerate. No information regarding \( \nu \) is available. For \( \tilde{k}_n = k+1 > 1 \), the techniques of Section 2.1 applied to
\( (2.42) \) give as before

\[ \hat{\nu} = \frac{1}{\nu^k} \sum_{j=1}^{k} \Delta N^* t[j]. \quad (2.43) \]

Further, for \( k \geq 1 \)

\[ P(\tilde{k}_n = k) = \sum_{c_0=0}^{n} \binom{n}{c_0, n-c_0} P(T_j = 0, j = 1, \ldots, c_0; T_j > 0, j = c_0 + 1, \ldots, n) \times \sum_{c_{n-c_0, k-1}} P(C^{-1}(c_{n-c_0, k-1})) , \]
where \( c_{n-c_0,k-1} = (c_1, \ldots, c_{k-1}) \). Since

\[
P\{T_j = 0, j = 1, \ldots, c_0; T_j > 0, j = c_0 + 1, \ldots, n\}
= E[1 - e^{-X(0)}] e^{-c_0 X(0)} e^{-(n-c_0)X(0)},
\]

knowledge of \( N_0^\infty(u) = \int_{0^+} (1 - e^{-uv}) dN_0(v) \) is necessary in establishing the distribution of \( \tilde{K}_n \). However, even though \( N_0^\infty \) is unknown, in the manner of the consistency proof of Theorem 2.2, as \( n \) increases \( \hat{V} \) is consistent.

If \( N_0 \) is a function of \( \nu \), then the tie-configuration involving the number of failures at \( t = 0 \) is a function of \( \nu \). In the conditional density (2.42) \( R(t) \) now involves \( \nu \), so that in differentiating the log likelihood with respect to \( \nu \), the tie-configuration probability now enters and complicates the likelihood equation.

As an example, suppose \( X(t) = Y(t + t_0) \), where \( Y(t) \sim P(\frac{1}{\nu} t; a) \) and \( t_0 > 0 \). In effect, we look at \( Y(t) \) starting at time \( t_0 \), at which point some wear and failures may have already occurred. Loosely speaking, if \( \nu \) is small, we would expect more failures at time \( t = 0 \) than if \( \nu \) is large. If one is willing to ignore the number of zero-time failures, in effect throwing information away, than we can still use equation (2.43) and the previously developed theory.
III. INFERENCE, ANTELMAN-SAVAGE MODEL

3.1 Multiple trials, one component per trial

In this chapter the wear process is assumed to follow the Antelman-Savage model. In this section the MLE \( \hat{\nu} \) of the parameter \( \nu \) in the representation

\[
\log E e^{iuX(t)} = \frac{1}{\nu} \int_{0^+}^{\infty} (e^{iu\nu} - 1)dN(\nu)
\]

(cf. Equation (1.12), Definition 1.10) is derived. The distribution of \( \hat{\nu} \) is ascertained, and minimum variance, unbiasedness, consistency, and asymptotic normality are demonstrated.

In Section 3.2 when \( X \) is a Poisson process and the number of components per trial increases, the existence of a consistent estimate is shown.

From (1.28) the joint distribution of failure times of \( n \) components per trial in the Antelman-Savage model with \( X \in S_2 \) is given for \( t \in T^n \) by

\[
\overline{F}(t) = e^{-\frac{1}{\nu} \sum_{m=1}^{n} \int_{(m-1)(n+1-m)}^{(n+1-m)} t^{(m)} - \sum_{j=m}^{n} t^{(j)} - (n+1-m)\tau} d\tau}
\]

where for \( u \geq 0 \) and \( N \) as in (3.1)
\[ N^*(u) = \int_{0+}^{\infty} (1-e^{-uv})dN(v). \]

For \( n = 1 \) and \( t > 0 \) (3.2) reduces to

\[ F(t) = e^{-\int_0^t N^*(t-\tau)d\tau} \]

\[ \overline{F}(t) = e^{-\int_0^t \int_0^1 (e^{-vt}-1)dv}dN(v) \]

\[ = e^{-\int_0^t \int_0^1 (e^{-vt}-1)dv}dN(v). \]  \hspace{1cm} (3.3)

The density of a single failure time is

\[ f(t) = \frac{1}{v} N^*(t) \overline{F}(t). \]  \hspace{1cm} (3.4)

The likelihood of the observed failure times \( t = (t_1, \ldots, t_m) \) from \( m \) independent trials is

\[ L(t) = \prod_{j=1}^{m} f(t_j) = \frac{1}{v^m} \prod_{j=1}^{m} N^*(t_j) \overline{F}(t_j), \]

whence

\[ \log L(t) = -m \log v + \sum_{j=1}^{m} \log N^*(t_j) - \frac{1}{v} \sum_{j=1}^{m} z(t_j), \]  \hspace{1cm} (3.5)

where

\[ z(t) = \int_{0+}^{\infty} \frac{1}{v} (e^{-vt}-1)dv. \]  \hspace{1cm} (3.6)

Since \( N^* \) does not involve \( v \), the solution to
\[
\frac{\partial}{\partial \nu} \log L(t) = -\frac{m}{\nu} + \frac{1}{\nu^2} \sum_{j=1}^{m} z(t_j) = 0
\] (3.7)

as a function of \( \nu \) is

\[
\hat{\nu} = \frac{1}{m} \sum_{j=1}^{m} z(t_j).
\] (3.8)

**Theorem 3.1**

The estimate \( \hat{\nu} \), as given by (3.8), for the Antelman-Savage model with \( X \in S_2 \), \( m \) independent trials, and one component per trial, has a gamma distribution with mean \( \nu \) and variance \( \nu^2/m \).

**Proof:**

From (3.6) \( z(t) \) has derivative

\[
z'(t) = \int_{0^+}^{\infty} (1-e^{-\nu t}) dN(\nu),
\]

which is positive for all \( t > 0 \). Thus \( z(t) \) is continuous and increasing for \( t > 0 \), and so is invertible. Let \( T \) be the random failure time of one component. From (3.3) and (3.6), for all \( x > 0 \)

\[
P(z(T) > x) = P(T > z^{-1}(x)) = e^{-\frac{1}{\nu}z(z^{-1}(x))} = e^{-\frac{1}{\nu}x}.
\]

Therefore \( z(T) \) is exponentially distributed with mean \( \nu \). It follows that

\[
\sum_{j=1}^{m} \frac{z(T_j)}{m} \sim \Gamma(m, \nu) \text{ in the notation (1.11)}.
\]

Q.E.D.
Remark: Unbiasedness and consistency as \( m \to \infty \) follow by inspection, and the asymptotic normality of the gamma distribution is well-known. From (3.7)

\[
E \left[ \frac{\partial}{\partial \nu} \log L(T) \right]^2 = E \left[ \frac{1}{\nu} \sum_{j=1}^{m} z(T_j) - \frac{m}{\nu} \right]^2
\]

\[
= \frac{m^2}{\nu^4} E \left[ \sum_{j=1}^{m} \frac{z(T_j)}{m} - \nu \right]^2
\]

\[
= \frac{m}{\nu^2}
\]

so that the Cramér-Rao lower bound on variances of unbiased estimates of \( \nu \) is \( \nu^2/m \). Since \( \text{Var} \hat{\nu} = \nu^2/m \), \( \hat{\nu} \) has minimum variance.

Example 3.2

In the notation (1.10) for \( X(t) \sim P(1/\nu t; a) \) with \( a > 0 \) known, \( N(\nu) = -I_{(0,a)}(\nu) \). From (3.6)

\[
z(t) = \frac{e^{-at} - 1 + at}{a}
\]

and from (3.8)

\[
\hat{\nu} = \frac{m}{\nu} \sum_{j=1}^{m} e^{-j-1+at_j} \frac{1}{am}
\]

Standard procedures of inference concerning \( \nu \) based on \( \hat{\nu} \), including those discussed in Section 2.4, can be employed.
3.2 Multiple components per trial

Because each sample function \( h(t) \) of \( H(t) = \int_0^t X(\tau) d\tau, \ X \in S_2 \), is continuous, the probability of two or more tied failure times among \( n \) components is zero. However, in developing the MLE of \( \nu \), difficulties arise, which are discussed in Section 3.3. As for the method of moments, the r.v. \( \bar{T} = \frac{1}{n} \sum_{j=1}^{n} T_j / n \) has variance

\[
\text{Var} \bar{T} = \frac{1}{n} \text{Var} T_1 + \frac{n(n-1)}{2n} \text{Cov}(T_1, T_2).
\]

Lack of independence amongst the failure times prevents the covariance term from vanishing, and hence \( \text{Var} T \neq 0 \) as \( n \to \infty \). The method of moments does not provide a consistent estimate.

Suppose now that the wear process is \( H(t) = \int_0^t X(\tau) d\tau \) with \( X(t) \sim P(\lambda t; a) \). As mentioned in the remark to Definition 1.10, the parameter \( \lambda \), as opposed to \( \frac{1}{\nu} \), is chosen here for convenience. Assume \( a = 1 \). Theorem 3.3 demonstrates the existence of a consistent sequence of estimates of \( \lambda \) as \( n \to \infty \), where \( n \) is the number of failure times observed from a single trial.

If \( \{X_n\} \) is a sequence of r.v.'s and \( \{a_n\} \) is a sequence of real numbers, then write \( X_n = o_p(a_n) \) if for each positive \( \epsilon \) there exists a finite \( N_\epsilon \) such that for each \( n > N_\epsilon \), \( P[|X_n/a_n| < \epsilon] > 1 - \epsilon \).

**Theorem 3.3**

Let \( H(t) = \int_0^t X(\tau) d\tau, \ X(t) \sim P(\lambda t; 1) \), and suppose the distribution of a single failure time \( T \) is given by
\[
\bar{F}(t|X(\cdot)) = P(T > t|X(\cdot)) = e^{-H(t)}, \quad t \geq 0.
\]

For each integer \( n \) define the empirical d.f. \( F_n(t) \) by

\[
F_n(t) = \frac{\text{number of failures by time } t}{n}
\]

for \( t > 0 \) and let

\[
u_n = (\log \sqrt{n})^{1/3}.
\]

Then for almost all realizations of \( H \)

\[
\hat{\lambda}_n = \frac{-2 \log[1-F_n(u_n)]}{\frac{2}{u_n^2}}
\]

is a consistent sequence of estimates of \( \lambda \).

**Proof:**

For each \( n \) and given \( H(\cdot) \), \( n[1-F_n(u_n)] \) is binomially distributed with parameters \( n \) and \( e^{-H(u_n)} \). Hence

\[
E[1-F_n(u_n)|H(\cdot)] = e^{-H(u_n)}
\]

\[
\text{Var}[1-F_n(u_n)|H(\cdot)] \leq 1/4n
\]

and one can write, given \( H(\cdot) \),

\[
1 - F_n(u_n) = e^{-H(u_n) + K/\sqrt{n}}, \quad (3.12)
\]

where \( K \) is a r.v. with zero mean and finite variance bounded in \( n \) and
H(·). In what follows, the symbol K may refer to different r.v.'s from line to line. For clarity when two such r.v.'s occur in the same expression, they will be distinguished by K, K'. From (3.12) for given H(·) and providing the argument of the logarithm is positive,

\[
\log[1-F_n(u_n)] = -H(u_n) + \log \left(1 + \frac{H(u_n)}{\sqrt{n}}\right). \tag{3.13}
\]

Let \( J_n \) represent the number of jumps experienced by \( X(t) \) in the interval \([0,u_n]\). That is, \( J_n = X(u_n) \). For \( J_n = j \) and \( j = 0,1,\ldots \)

\[
H(u_n) = \int_0^{u_n} (u_n - \tau) dX(\tau) = \sum_{i=1}^{j} (u_n - \tau_i) = ju_n - \sum_{i=1}^{j} \tau_i,
\]

where \( \tau_i \) is the time of the \( i^{th} \) jump in \( X \). For \( J_n = j \) the vector \((\tau_1, \ldots, \tau_j)\) is distributed as the vector of order statistics from independent r.v.'s \( Y_i \), \( i = 1, \ldots, j \), each uniformly distributed on \([0,u_n]\). Thus

\[
E[H(u_n)|J_n=j] = \frac{ju_n}{2} \tag{3.14}
\]

\[
\text{Var}[H(u_n)|J_n=j] = \frac{ju_n^2}{12}. \tag{3.15}
\]

Then given \( J_n = j \)

\[
H(u_n) = \frac{ju_n}{2} + K\sqrt{\frac{ju_n^2}{12}} = u_n\left(j + K\sqrt{\frac{1}{3}}\right). \tag{3.16}
\]

Since \( \sqrt{n} = e_n^3 \) from (3.10), one has, with \( J_n = j \)
\[
\frac{H(u_n)}{\sqrt{n}} = H(u_n/2 + K^{1/2}u_n^2/12 - u_n^3}.
\]

Now notice \( J_n = o_p(u_n^{3/2}) \), so that
\[
\log \left( 1 + \frac{H(u_n)}{\sqrt{n}} \right) = o_p(1).
\]

Equation (3.13) becomes, given \( H(\cdot) \)
\[
\log[1 - F_n(u_n)] = -H(u_n) + o_p(1) = \frac{-u_n}{2} \left( J_n + K^{1/3} \right) + o_p(1).
\]

Then since \( J_n = o_p(u_n^{3/2}) \), from (3.11)
\[
\hat{\lambda}_n = \frac{J_n}{u_n} + \frac{K^{1/3}u_n}{3} + o_p(1) = \left( \frac{\lambda u_n}{u_n} + \frac{K^{1/3}u_n}{3} \right) + o_p(1)
\]
\[
= \lambda + o_p(1). \tag{3.17}
\]

Thus \( \hat{\lambda}_n \) is consistent as \( n \to \infty \).

Q.E.D.

Remark: Theorem 3.3 shows the existence of a consistent sequence of estimates \( \hat{\lambda}_n \). The properties of this sequence remain an open topic for research. With only minor modifications, the proof holds for \( a \neq 1 \), that is, set \( J_n = \frac{1}{a} X(u_n) \). Then when \( J_n = j \),
\[
H(u_n) = a(ju_n - \sum_{i=1}^{j} \tau_i). \] Equations (3.14) and (3.15) become
\[ E[H(u_n) | J_n] = au_n J_n / 2 \]

\[ \text{Var}[H(u_n) | J_n] = a^2 u_n^2 J_n / 12 \]

so that the analog of (3.16) is

\[ H(u_n) = \frac{au_n}{2} \left( j + k \sqrt{\frac{j}{3}} \right) \]

and (3.17) holds as for \( a = 1 \).

3.3 Problems for further research

(1) Let \( X(t) \sim P(\frac{1}{\nu} t; a) \) be the wear process for a Gaver model. In general the wear process is unobservable, and both \( \nu \) and \( a \) may be unknown. The problem then is to obtain joint estimates of \( \nu \) and \( a \) from a sample \( t = (t_1, \ldots, t_n) \) of observed failure times of \( n \) identical components from a single trial. With \( q = e^{-a} \) and \( N^*(u) = 1 - q^u \), the likelihood is (cf. (1.51))

\[ L(t) = \exp \left( -\frac{1}{\nu} \sum_{j=1}^{k} (1-q^s j) \Delta t[j] \right) \frac{1}{\nu^k \prod_{j=1}^{k} (1-q^s j)} \]

and

\[ \frac{\partial}{\partial \nu} \log L = \frac{1}{\nu^2} \sum_{j=1}^{k} (1-q^s j) \Delta t[j] - \frac{k}{\nu} = 0 \]  \hspace{1cm} (3.18)

\[ \frac{\partial}{\partial q} \log L = \frac{1}{\nu} \sum_{j=1}^{k} s_j q^{s_j} \Delta t[j] \sum_{j=1}^{k} s_j q^{s_j-1} - \sum_{j=1}^{k} \frac{s_j q^{s_j-1}}{1-q^s j} = 0 \]  \hspace{1cm} (3.19)
A solution for \( \hat{\nu} \) in terms of \( \hat{q} \) from (3.18) is

\[
\hat{\nu} = \sum_{j=1}^{k} \frac{(1-\hat{q}^j) \Delta t[j]}{k}. \tag{3.20}
\]

The difficulty is in obtaining a solution for \( \hat{q} \). Substituting \( \hat{\nu} \) in (3.19) gives

\[
\sum_{j=1}^{k} s_j \hat{q}^{s_j-1} \Delta t[j] - \left[ \sum_{j=1}^{k} \frac{(1-\hat{q}^j) \Delta t[j]}{k} \right] \sum_{j=1}^{k} s_j \hat{q}^{s_j-1} \frac{s_j}{1-\hat{q}^s_j} = 0. \tag{3.21}
\]

When \( k = 1 \), then \( s_1 = n \) and (3.21) becomes the identity

\[
q^n-1 t[j] - q^n-1 t[1] = 0.
\]

Such a degeneracy may be expected when one tries to estimate two parameters from one failure time. For \( k > 1 \), solutions to (3.21) must be obtained by approximations or numerical methods. Once \( \hat{\nu} \) and \( \hat{q} \) are obtained, the properties of these joint estimates need to be established, for they are not based on independent observations.

(2) Problem (1) can be extended to \( m \) trials. A second extension is to different types of components within the same trial, as in Section 2.5, with the \( i \)th type having a parameter \( a_i \) requiring estimation.

(3) In the remarks following Definition 1.1, a deterministic component of \( X \in S \) is specifically excluded. However, some models use such a component to incorporate effects such as aging or corrosion. Suppose \( W(t) \) is the accumulated wear process and \( W(t) = X(t) + pt \),
\( t > 0 \), with \( X(t) \sim P(\lambda t; a) \). Suppose \( a > 0 \) and \( \rho > 0 \) are known. The problem is to estimate \( \lambda \) in the presence of the "nuisance" due to \( \rho \). This formulation applies to the Gaver model. A similar modification to the Antelman-Savage model can be made. Further, \( a \) and \( \rho \) may be unknown and require estimation.

(4) How should estimation be performed on data from censored or truncated samples? Suppose testing ceases after a preset number of failures has been observed, or after a certain time is reached. What procedure of estimation should be used on the available data?

(5) For the Antelman-Savage model, suppose \( H(t) = \int_0^t X(\tau) d\tau \), with \( X(t) \sim P(\lambda t; a) \) and both \( \lambda \) and \( a \) are unknown. As in (1), the problem is to obtain joint MLE's of \( \lambda \) and \( a \), when \( m \) trials are run with a single component per trial. Equation (3.5) becomes

\[
\log L(t) = -m \log v + \sum_{j=1}^{m} \log(1 - e^{-\lambda t_j}) - \frac{1}{v} \sum_{j=1}^{m} \frac{e^{-\lambda t_j}}{a}
\]

so that

\[
\frac{\partial}{\partial v} \log L(t) = -m + \frac{1}{v} \sum_{j=1}^{m} \frac{e^{-\lambda t_j}}{a} = 0 \quad (3.22)
\]

\[
\frac{\partial}{\partial a} \log L(t) = \sum_{j=1}^{m} \frac{t_j}{ae_{j-1}} - \frac{1}{v} \sum_{j=1}^{m} \frac{1-e^{-\lambda t_j(1+a)}}{a^2} = 0. \quad (3.23)
\]

A solution \( \hat{v} \) in terms of \( \hat{a} \) from (3.22) is

\[
\hat{v} = \frac{1}{m} \sum_{j=1}^{m} \frac{e^{-\lambda t_j}}{\hat{a}} \quad (3.24)
\]
The problem of obtaining \( \hat{a} \), using \( \hat{v} \) in (3.23) remains open.

(6) Maximum likelihood procedures were abandoned in Section 3.2 because of their complexity. For \( \mathbf{t} \in \overline{T}^m \) and \( X \in S_2 \), (3.2) is of the form

\[
\overline{F}(\mathbf{t}) = e^{-\frac{1}{v} g(\mathbf{t})},
\]

where

\[
g(\mathbf{t}) = \sum_{m=1}^{n} \int_{t_{(m-1)}}^{t_{(m)}} \mathbb{M}_k(\sum_{j=m}^{n} t_{(j)} - (n+1-m)\tau) d\tau.
\]

To get the density \( f(\mathbf{t}) \) requires the \( n \)th mixed partial derivative of \( \overline{F}(\mathbf{t}) \) with respect to \( t_1, \ldots, t_n \), which is of the form

\[
(-1)^n \frac{\partial^n \overline{F}(\mathbf{t})}{\partial t_1 \cdots \partial t_n} = e^{-\frac{1}{v} g(\mathbf{t})} \sum_{k=1}^{n} \frac{1}{v^k} g_k^*(\mathbf{t}),
\]

where each \( g_k^*(\mathbf{t}) \) is a sum of products of mixed partial derivatives of \( g(\mathbf{t}) \). Then

\[
\frac{\partial}{\partial v} \log \, L(\mathbf{t}) = \frac{1}{2} g(\mathbf{t}) - \frac{\sum_{k=1}^{n} \frac{k}{v^k} \frac{(k+1)g_k^*(\mathbf{t})}{v}}{\sum_{k=1}^{n} \frac{k}{v^k} g_k^*(\mathbf{t})},
\]

which is intractable even for moderate \( n \), causing maximum likelihood procedures to be difficult to use. (Instead, the sequence \( \{\hat{\lambda}_n\} \) of Theorem 3.3 was developed.) The properties of these MLE's and of \( \{\hat{\lambda}_n\} \) require exploration.
(7) What method of estimation should be used when the wear process is a linear combination of Antelman-Savage and G-aver models?

(8) If one uses an estimating procedure for the Gaver model when the true underlying model is an Antelman-Savage model, what will be the estimator's properties, and vice-versa?
REFERENCES


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Gaver [1963] and Antelman and Savage [1965] have proposed models for the distribution of the time to failure of a simple device exposed to a randomly varying environment. Each model represents cumulative wear as a specified function of a nonnegative stochastic process \( X \) with independent increments, and assumes the reliability of the device is conditioned upon realizations of this process. Let \( T \) be the random time to failure of the device. For \( t \geq 0 \) the Gaver model is 

\[
P(T > t | X(\cdot)) = e^{-X(t)} - \int_{0}^{t} e^{-X(\tau)} d\tau
\]

and the Antelman-Savage model is 

\[
P(T > t | X(\cdot)) = e^{X(t)} \int_{0}^{t} e^{-X(\tau)} d\tau
\]

From these models are derived the corresponding unconditional joint distributions for the random failure time vector \( T = (T_1, \ldots, T_n) \) of \( n \) independent, identical devices exposed simultaneously to the same realization of the wear process.

Conditions are given under which both models can give rise to identical one-dimensional failure time distributions. Joint failure time distributions are obtained in explicit form and the probabilities of ties and tie configurations are derived. Maximum likelihood estimates are obtained for certain relevant parameters for each of the models when \( X(t) \) is a process with stationary nonnegative increments.