FUNDAMENTAL RELATIONSHIPS FOR TRAFFIC FLOW MODELS
WITH APPLICATIONS

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The opinions, findings and conclusions expressed in this
publication are those of the author and not necessarily those of
the Bureau of Public Roads.
This report provides fundamental relationships of use in the modeling of traffic flow by the theory of stochastic point processes. Certain conventional but unrealistic assumptions, such as that of independent gaps between vehicles or that of zero lengths of vehicles, are avoided as premises. Thus the results are widely applicable and cover situations other than traffic flow.

As described in the Summary, and elsewhere in the report, the author has benefitted greatly from the collaboration of Professor M. R. Leadbetter, Department of Statistics, University of North Carolina, who has provided gracious assistance and has allowed the author to use a pre-publication version of Leadbetter (1969). Of course, any flaws discovered in the present report are the responsibility of the author.
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0. **Summary.** The report provides fundamental relationships for traffic flow models which are stationary, orderly stochastic point processes.

Section 1 provides a careful description of how traffic flow may be considered as such, along with precise definitions of the random variables about which the various relationships are concerned. (There is some overlap here with the introduction to an earlier report, Serfling (1968), but the material in common is better expressed in the present case and Section 1 now constitutes a tidy introductory "package" that subsequent reports may rely upon simply by reference to the present report.)

Sections 2, 3, 4 and 5 present a variety of fundamental theoretical relationships for traffic flow models (and stationary, orderly point processes in general). Some of these results have appeared earlier in the literature but have been overlooked in traffic flow theory. Some have appeared previously but under restrictions much more severe or cumbersome than necessary. Finally, some of the results seem to be new. As may be seen from the presentation of the relationships, the foundations of the results are due to work of M. R. Leadbetter and, moreover, the advances achieved in the present report are due to the collaboration of M. R. Leadbetter. However, any defects in the present report must be attributed solely to its author.

Section 6 deals with a qualitative application of the relationships, namely an interpretation of the so-called renewal function in the context of processes which are not renewal processes.
Sections 7, 8 and 9 illustrate the use of the relationships in the cases of the Poisson process, the Erlang process and the Pólya process. Interesting new results are unearthed for the Erlang and Pólya processes.

Finally, Section 10 describes a major purpose in having such relationships available -- the analysis of new models that arise for consideration as potential traffic flow models.
1. Basic notation, definitions and description. In this section the view of traffic flow as a stochastic point process is set forth. The natural modes of observation and types of statistics are distinguished. The assumption, of which use shall be made, that traffic flow is a "stationary and orderly" stream of events is examined and characterized. Corresponding to the various types of data that might be collected, certain random variables of primary concern are defined precisely in the framework of the mathematical model. Finally, in the setting thus established, we discuss the main objectives of the present report, which concerns the relationships between the key random variables.

**View of traffic flow as a stochastic point process**

The discussion throughout this report utilizes the view of traffic flow as a stochastic point process. Each realization of such a process consists of a sequence of points \( \{P_i\}_{i=1}^{\infty} \), on the real line, which represent the points of occurrence of the events under study. In modeling traffic flow in this way, one may let the real line represent either a time axis or a space axis. A traffic flow thus generates a random sequence of points \( \{P_i\} \), where the points are the positions (on the axis) of the front bumpers of the successive vehicles. If the axis is taken in space, the sequence \( \{P_i\} \) represents a snapshot of the traffic flow taken at a fixed instant in time. If the axis is a time axis, the points \( \{P_i\} \) denote the time instants at which the vehicle front bumpers pass a fixed point in space (a fixed point on the roadway). Some of the relationships between vehicles in space and in time have been given by Wardrop (1952).
The viewing of traffic flow as a point process in no way introduces unrealistic features into the model. On the contrary, this is the simplest and most natural way to characterize traffic flow without restricting generality. Thus, for example, it is not necessary in this view to assume that the random gaps between vehicle front bumpers are statistically independent random variables. Again, it is not necessary in this view to assume that vehicles have zero lengths. Both of these assumptions are popular for mathematical convenience but are well-known to be unrealistic except for traffic flow of very light density. Accordingly, the use of such premises would seriously limit the scope of application of any conclusions obtained.

More detailed discussion of premises, both to be adopted and not to be adopted, will be given below in more precise language. Let us mention here, however, that one of the popular and conventional assumptions about traffic flow, namely stationarity, shall be utilized in this report. As expressed by Ashton (1966), p. 77, the notion implies that "equal intervals of time (or space) are equally likely to contain equal numbers of events (or vehicles)." While it is helpful to have at hand such an unmathematical verbalization of any assumption considered, the inherent vagueness of such expressions makes it difficult to obtain precise quantitative conclusions or to explore the possibilities for interesting conclusions. Thus it is important to have also at hand mathematical formulations of any notions to be utilized.

The treatment in this report shall pertain not only to stationary traffic flow models but actually to a large class of stationary stochastic point processes. The assumption of stationarity is essential only to
Modes of observation, types of statistic

Two methods for initiating observation (in time or space) of a stream of traffic will be considered. In the synchronous mode, the observation interval commences immediately after the point of occurrence of an event (vehicle front bumper). Asynchronous observation begins at a point "randomly located" with respect to the stream of events, that is, at an arbitrarily chosen point $t_0$ on the measurement axis. The terminology here is that of Haight (1963), p. 97.

For each mode of observation, and whether in time or space, the following types of statistic are of basic interest:

(i) the intervals between successive events;
(ii) the interval to the $n$-th event after observation begins;
(iii) the count of the number of events in an observation interval of length $t$.

In the above, $n$ may be any fixed integer $> 0$ and $t$ may be any fixed real number $> 0$. Types (i) and (ii) are called "interval statistics," while type (iii) is a "count" statistic.

Stationary orderly streams of events

The basic assumptions on the traffic flow model shall be formulated in terms of conditions on the family of random variables $\{N(s,t)\}$ all $s,t$, where $N(s,t)$ denotes the number of events occurring in the interval $(s,t)$. More precisely, $N(s,t)$ represents the function $N(s,t,\omega)$ defined on the elements $\omega = (\ldots, P_{-1}, P_0, P_1, \ldots)$ by

\[ N(s,t,\omega) = \text{the number of points of } \omega \text{ in the interval } (s,t). \]
Thus, for fixed \((s,t)\), \(N(s,t,\omega)\) is a random variable as \(\omega\) varies in the space \(\Omega\) of outcomes. Following the usual practice in probability literature, we shall suppress \(\omega\) in our notation and thus speak of \(N(s,t)\).

As mentioned earlier, we shall suppose that the traffic flow is a stationary point process. The notion of stationarity has been defined in various ways, not all equivalent. We shall adopt a conventional definition: the process is stationary if for every finite collection of observation intervals \((s_1,t_1], \ldots, (s_k,t_k]\), choice of integers \(r_1, \ldots, r_k\) and real number \(h > 0\),

\[
P[N(s_1+h,t_1+h) = r_1, 1 \leq i \leq k] = P[N(s_1,t_1) = r_1, 1 \leq i \leq k].
\]

That is, the joint distribution of the numbers of events in each of \(k\) observation intervals \((s_1,t_1], \ldots, (s_k,t_k]\) remains unchanged if the intervals are translated by the same amount \(h\). Therefore, in discussing stationary streams, what applies to \(N(0,t)\) also applies to \(N(s,s+t)\) for any \(s\).

As regards traffic flow, stationarity is compatible with the notion that the conjectural model on an infinite measurement axis should retain the properties of the actual traffic flow in a segment \([A,B]\) to which the theoretical model is to be applied via a data-fitting procedure.

Every stationary stream of events has associated with it a unique intensity \(\lambda\), namely a value \(0 \leq \lambda \leq \infty\) such that

\[
P[N(0,t) \geq 1] = \lambda t + o(t), \ t \to 0.
\]

The meaning of (1.2) is that the probability of at least one event occurring in a given observation interval is asymptotically proportional to the length of the interval, as the length of the interval is allowed to decrease to zero. The result is due to Khintchine (1955).
Also, it is given that the intensity is positive unless \( P[N(0,t) \geq 1] = 0 \). Thus for applications to traffic flow it is reasonable to assume that \( \lambda > 0 \). It is desirable to have further that \( \lambda < \infty \), and indeed this will be concluded below.

The Poisson process with parameter \( \lambda \) is a familiar example of a stationary stream of events having intensity \( \lambda \). The Poisson (\( \lambda \)) process is that for which the joint probability of observing \( r_1, \ldots, r_k \) events, respectively, in any \( k \) non-overlapping intervals \( (s_1, t_1], \ldots, (s_k, t_k] \) is given by

\[
P[N(s_i, t_i) = r_i, i=1, \ldots, k] = \prod_{i=1}^{k} \frac{(-\lambda(t_i-s_i))^{r_i}}{r_i!},
\]

for each choice of non-negative integers \( r_1, \ldots, r_k \). It follows that the Poisson (\( \lambda \)) process is stationary (in the sense of (1.1)), has independent increments (i.e., the numbers of events in non-overlapping intervals are mutually independent random variables), and that the distribution of the number of events in any interval of length \( t \) is given by

\[
P[N(0,t) = r] = e^{-\lambda t} \frac{(\lambda t)^r}{r!} \quad (r=0,1,\ldots).
\]

We see that \( P[N(0,t) \geq 1] = 1 - e^{-\lambda t} \), from which (1.2) follows easily. Thus the "intensity" of a Poisson (\( \lambda \)) process is indeed the parameter \( \lambda \).

Returning to stationary streams in general, we say that a stream is orderly if

\[
P[N(0,t) > 1] = o(t), \quad t \to 0.
\]

The meaning of (1.5) is, in view of (1.2), that the probability of more than one event in \( (0,t] \) is negligible relative to that of exactly one event, as \( t \to 0 \).
It is reasonable to assume orderliness for traffic flow because of the non-zero lengths of vehicles. If $t$ is chosen small enough, the interval $(0, t]$ has length shorter than the length of any vehicle, so that $P[N(0, t) > 1] = 0$ for all $t$ small enough. Hence (1.5) trivially holds. Therefore, we shall suppose that the traffic flow is an orderly point process.

In the case of a stationary orderly stream, it follows by a theorem of Korolyook (see Khintchine (1955), p. 42) that

$$E[N(0, t)] = \lambda t,$$

that is, the mean number of events in an interval equals the intensity of the process times the length of the interval. (Relation (1.6) is not necessarily true of an arbitrary stationary process.) For a traffic flow, therefore, we have that the intensity is equal to the density, where "density" represents cars per unit interval (of time or space). Moreover, it is obviously reasonable for a traffic flow process, again because of the non-zero lengths of vehicles, to assume that the density is finite. Then it may also be concluded, as anticipated earlier, that the intensity is finite.

The results presented in this report shall apply to stochastic point processes which are stationary and orderly and whose intensity $\lambda$ satisfies $0 < \lambda < \infty$. As demonstrated above, this is a reasonable class of models for traffic flow. We note that the class includes the Poisson processes, which are frequently considered in traffic flow modeling because of the mathematical tractability. However, the class is much broader, and this in a useful way, because within it one may avoid two properties of the Poisson model that are unrealistic for traffic flow.
These features are (a) that more than one event may possibly occur in an interval \((0,t]\) no matter how small \(t\) is chosen, and (b) the property of independent increments mentioned earlier.

_The random variables of primary concern_

We have discussed above the family of random variables \(\{N(s,t)\}_{s,t}\) and by stationarity it suffices to consider the family \(\{N(0,t)\}_{t}\). (It is assumed that \(s \leq t\) wherever \(N(s,t)\) is expressed.) We shall employ the simpler notation \(N(t)\) for \(N(0,t)\) and thus speak of the family \(\{N(t)\}_{t\geq0}\).

The random variable \(N(t)\) may take values \(k = 0,1,\ldots\) and its probability distribution shall be denoted by

\[
(1.7) \quad v_k(t) = P[N(t) = k] \quad (k = 0,1,\ldots).
\]

The distributions defined by (1.7) represent the marginals of the family \(\{N(t)\}_{t\geq0}\). While the joint distributions of the members of this family are also of interest, we shall be concerned primarily with the marginal distributions in order to retain mathematical simplicity and because information about the marginals implies information about the joint distributions (see Section 5).

The random variable \(N(t)\) which we have been discussing at some length is a "count statistic" corresponding to the asynchronous mode of observation. In the same mode we can observe the "interval statistics"

\[
(1.8) \quad T_n = \text{length of interval from } t = 0 \text{ to the } n\text{-th subsequent event, for each } n = 1,2,\ldots, \text{ with } T_0 = 0, \text{ and}
\]
\[(1.9) \quad Y_n = T_n - T_{n-1} \]

= length of interval between the \((n-1)\)-th and \(n\)-th
events subsequent to \(t = 0\),

for each \(n = 1, 2, \ldots\). The distribution function of \(T_n\) is determined
by the \(\{v_k(t)\}\) as follows. Denoting the d.f. by \(G_n\), we have

\[G_n(t) = P[T_n \leq t]\]

\[= P[N(t) \geq n]\]

\[(1.10) \quad = \sum_{k=n}^{\infty} v_k(t).\]

The problem of specifying random variables analogous to \(N(t)\),
\(T_n\), and \(Y_n\) for the synchronous mode of observation is not simple, for
it has not been made clear in the probability literature what is meant
by "arbitrarily selected event." The approach taken here shall circum-
vent this difficulty rather than resolve it.

Define, following Leadbetter (1966),

\[(1.11) \quad F_n(t) = \lim_{\delta \to 0} P[N(t) \geq n \mid N(-\delta, 0) \geq 1], \quad t \geq 0,\]

for each \(n = 1, 2, \ldots\). The term in the right-hand side is the conditional
probability that the \(n\)-th event after the point 0 occurs in the interval
\((0, t]\), given that an event has occurred in the interval \((-\delta, 0]\). The
limit \(F_n(t)\) exists and may be interpreted as the conditional probability
that the \(n\)-th event after the point 0 occurs in the interval \((0, t]\), given
that "an event has occurred at the point 0". As Leadbetter shows, \(F_n(\cdot)\) is
a distribution function for each fixed \( n = 1, 2, \ldots \) and hence may be interpreted as the distribution function for the length of interval between an arbitrary event and the \( n \)-th subsequent event. Yet \( F_n(\cdot) \) is defined as the limit of a conditional probability rather than as the distribution function of an actual random variable. Further discussion is available in Leadbetter (1966, 1969).

Let \( S_n \) be a random variable having \( F_n(\cdot) \) as its distribution function, for each \( n = 1, 2, \ldots \), and let \( S_0 = 0 \). Let \( X_n = S_n - S_{n-1} \) for \( n = 1, 2, \ldots \). Finally, for each \( t > 0 \), let \( M(t) \) be a "count" random variable taking values \( k = 0, 1, \ldots \), with \( \text{Prob}[M(t) = k] = u_k(t) \), where

\[
u_k(t) = F_k(t) - F_{k+1}(t) \quad (k = 0, 1, \ldots).
\]

Thus the random variable \( M(t) \) may be interpreted as the number of events in an interval of length \( t \) following an arbitrary event.

Thus \( \{X_n\}, \{S_n\} \) and \( \{M(t)\} \) are artificial random variables having interpretations in the synchronous mode of observation which are analogous to the roles in the synchronous mode of observation of the observable random variables \( \{Y_n\}, \{T_n\}, \{N(t)\} \). Likewise the probability distributions \( \{u_k(t)\}_{k=0}^{\infty} \) for each \( t > 0 \) and \( F_n(t) \) for each \( n = 1, 2, \ldots \) may be interpreted as the synchronous-mode analogues of the asynchronous-mode distributions \( \{v_k(t)\}_{k=0}^{\infty} \) and \( G_n(t) \).

**Relationships**

Several types of relationship are of interest.

As noted earlier, there is a correspondence between measurements on a time axis concerning events at a point in space and measurements on a space axis concerning events at a moment in time. The reader is
referred to Wardrop (1952) for these. It is assumed in the following that one or the other type of measurement axis has been selected.

The relationships between interval statistics and count statistics may be considered within each of the two modes of observation, synchronous and asynchronous. Within each mode, we may relate the probability distribution of the interval length between \( t = 0 \) and the \( n \)-th subsequent event to the probability distribution of the count in an arbitrary interval of given length. This may be done by expressing the distribution function of the interval random variable in terms of the distribution functions of the count random variables or, alternatively, in terms of the moments of the count random variables. (Section 2)

Another type of relationship is that between the synchronous and asynchronous modes of observation. Such may be expressed for each type of random variable, interval and count. For example, it will be seen that the factorial moments of \( M(t) \) are related in a fairly simple way to those of \( N(t) \). (Section 3)

Further relationships which are mixtures of the above types may be deduced. Many relationships already in the literature are of the mixed kind. For example, Buckley (1967) considers a certain relation between the interval from an arbitrary event to the \( n \)-th subsequent event (synchronous mode) and the count in an arbitrary interval of given length (asynchronous mode). The more systematic treatment in the present report should somewhat unify the various results scattered in the literature, as well as provide some new results and insights. (Section 4)

Certain relationships between the joint distributions of several \( N(s_1, t_1) \) and the univariate distributions of the individual \( N(s_1, t_1) \) shall be given. In particular, the correlation between any \( N(s_1, t_1) \) and
and $N(s_2,t_2)$ may be deduced purely from knowledge of the marginal

distributions $\{v_k(t)\}_{k=0}^{\infty}$ for all $t > 0$. (Section 5)

Using all these relationships, one may conclude much about a
traffic flow model directly from knowledge of $\{v_k(t)\}_{k=0}^{\infty}$, $t > 0$. The
latter information may be converted into other forms by the relations.
Or one can use the relations to develop insight into aspects of the model
not readily apparent otherwise. For example, the role of the renewal
function (which in the case of a renewal process determines the process)
in the context of non-renewal processes may be assessed. Finally, it is
noted that the availability of these relationships makes it convenient to
define, construct or fit a model completely from the standpoint of the
distributions $\{v_k(t)\}$, rather than piece by piece from numerous,
uncoordinated standpoints.
2. **Relations between intervals and counts.** The relations between intervals and counts shall be expressed separately for each mode of observation, synchronous and asynchronous, with the corresponding relations of mixed type considered in Section 4. The random variables involved shall be related through their probability distributions. For example, synchronous intervals and counts shall be related via relations between the interval distribution functions $F_n(\cdot)$, $n = 1, 2, \ldots$, and the count distributions $\{u_k(t)\}_{k=0}^{\infty}$, $t > 0$. Likewise, for the asynchronous case, the functions $G_n(\cdot)$, $n = 1, 2, \ldots$, shall be related to the functions $\{v_k(t)\}_{k=0}^{\infty}$, $t > 0$.

Some useful and simple relations follow immediately from the definitions in Section 1. These are

**Relations (2.1).** For each $n = 1, 2, \ldots$,

\[
(2.1A) \quad F_n(t) = \sum_{k=n}^{\infty} u_k(t) \quad (t > 0)
\]

and

\[
(2.1B) \quad G_n(t) = \sum_{k=n}^{\infty} v_k(t) \quad (t > 0).
\]

**Relations (2.2).** For each $t > 0$,

\[
(2.2A) \quad u_k(t) = F_k(t) - F_{k+1}(t) \quad (k = 0, 1, \ldots)
\]

and

\[
(2.2B) \quad v_k(t) = G_k(t) - G_{k+1}(t) \quad (k = 0, 1, \ldots).
\]

The latter relations are the converses of the former.
For many purposes, the above relations are the most convenient and satisfactory and should not be overlooked. On the other hand, for considerations involving moments of the count distribution, it is useful to have relationships expressed in terms of these moments. In this regard, we shall provide alternative relations which link the factorial moments of the count variables to the cumulative distribution functions of the interval variables. (Relations among factorial moments and other kinds of moments are available in standard literature, e.g. Parzen (1960).)

By the $k$-th factorial moment of a "count" random variable $Z$ is meant the quantity

\[(2.3) \quad E[Z(Z-1)\cdots(Z-k+1)] = \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1) P[Z=k],\]

for each $k = 1, 2, \ldots$. The $k$-th factorial moments of $M(t)$ and $N(t)$ are thus, respectively,

\[(2.4) \quad \alpha_k(t) = k! \sum_{j=k}^{\infty} \binom{j}{k} u_j(t)\]

and

\[(2.5) \quad \beta_k(t) = k! \sum_{j=k}^{\infty} \binom{j}{k} v_j(t),\]

for each $k = 1, 2, \ldots$ and each $t > 0$.

The alternatives to relations (2.1) and (2.2) are as follows.

**Relations (2.6).** For each $n = 1, 2, \ldots$,

\[(2.6A) \quad F_n(t) = \sum_{k=n}^{\infty} (-1)^{k-n} \binom{k-1}{n-1} \frac{\alpha_k(t)}{k!} \quad (t > 0)\]

and

\[(2.6B) \quad G_n(t) = \sum_{k=n}^{\infty} (-1)^{k-n} \binom{k-1}{n-1} \frac{\beta_k(t)}{k!} \quad (t > 0),\]
where it is assumed that these series converge absolutely.

### Relations (2.7).

For each $t > 0$,

$$
(2.7A) \quad \alpha_k(t)/k! = \sum_{n=k}^{\infty} \frac{(n-1)}{k-1} \frac{F_n(t)}{F_n(t)} \quad (k = 0, 1, \ldots)
$$

and

$$
(2.7B) \quad \beta_k(t)/k! = \sum_{n=k}^{\infty} \frac{(n-1)}{k-1} \frac{G_n(t)}{G_n(t)} \quad (k = 0, 1, \ldots).
$$

Before proving relations (2.6) and (2.7), let us illustrate relations (2.1) and (2.6) for the case of a Poisson $(\lambda)$ process.

For such a process, we have

$$
(2.8) \quad u_k(t) = v_k(t) = e^{-\lambda t} (\lambda t)^k/k!
$$

and

$$
(2.9) \quad \alpha_k(t) = \beta_k(t) = (\lambda t)^k.
$$

Thus (2.1) and (2.6) state, respectively,

$$
(2.10) \quad F_n(t) = G_n(t) = \sum_{k=n}^{\infty} e^{-\lambda t} (\lambda t)^k/k! \quad (t > 0)
$$

and

$$
(2.11) \quad F_n(t) = G_n(t) = \sum_{k=n}^{\infty} (-1)^{k-n} \frac{(k-1)}{n-1} (\lambda t)^k/k! \quad (t > 0).
$$

In particular, (2.10) and (2.11) each yield the familiar result that $F_1(t) = G_1(t) = 1 - \exp(-\lambda t)$, $t > 0$, i.e., that the interval between consecutive events in a Poisson process has a negative exponential distribution.
Relations (2.6) and (2.7) are special cases of the following lemmas, which shall be proved for an arbitrary "count" distribution \((p_k)_{k=0}^{\infty}\), where \(p_k\) denotes the probability of a count equal to \(k\), \(\sum_{0}^{\infty} p_k = 1\). Let \(P_n = \sum_{k>n} p_k\), the probability of a count of at least \(k\), and let \(\gamma_k = k! \sum_{n\geq k} \binom{n}{k} p_n\), the \(k\)-th factorial moment of the count distribution.

**Lemma 2.1.** If \(\gamma_k\) is finite, then

\[(2.12) \quad \frac{\gamma_k}{k!} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} P_n.\]

**Proof:** It is well-known (e.g. Feller 1957, p. 62) that \(\binom{n}{k} = \sum_{j=k}^{n} \binom{j-1}{k-1}\).

Hence, from the definition of \(\gamma_k\), we obtain

\[(2.13) \quad \frac{\gamma_k}{k!} = \sum_{n=k}^{\infty} \sum_{j=k}^{n} \binom{j-1}{k-1} P_n.\]

The series in (2.13) converges absolutely, so that rearrangement of terms is valid, yielding (2.12) easily.

We obtain (2.7) from the preceding lemma and (2.6) from the next lemma.

**Lemma 2.2.** We have

\[(2.14) \quad P_n = \sum_{k=n}^{\infty} (-1)^{k-n} \binom{k-1}{n-1} \frac{\gamma_k}{k!},\]

provided that this series converges absolutely.

**Proof:** By Lemma 2.1, the series in (2.14) may be written as

\[(2.15) \quad \sum_{k=n}^{\infty} (-1)^{k-n} \sum_{j=k}^{\infty} \binom{j-1}{k-1} P_j,\]

which in turn may be written

\[(2.16) \quad \sum_{j=n}^{\infty} P_j \sum_{k=n}^{\infty} (-1)^{k-n} \binom{k-1}{n-1} \binom{j-1}{k-1}.\]
The coefficient of $P_j$ in (2.16) may be reduced to

\[(2.17) \quad \binom{j-1}{n-1} \sum_{k=0}^{j-n} (-1)^k \binom{j-n}{k}.
\]

By the formula (Feller (1957), p. 61)

\[(2.18) \quad \sum_{k=0}^{m} (-1)^k \binom{j-n}{k} = (-1)^m \binom{j-n-1}{m},
\]

for $m$ an integer $\geq 0$, the quantity in (2.17) equals 1 if $j = n$ and equals 0 if $j > n$. Hence the series in (2.16) is simply $P_n$. 
3. **Relations between synchronous and asynchronous variables.**

In this section are given relationships between synchronous and asynchronous counts and between synchronous and asynchronous intervals, with relations of mixed type appearing in Section 4. As in the previous section, the relationships are expressed in terms of the relevant probability distributions.

A basic relation between synchronous and asynchronous counts follows immediately from relation (2.2A) of the previous section in conjunction with a formula of Leadbetter (1966) given by relation (4.1) of the next section. We obtain

**Relation (3.1).** For each $t > 0$,

$$u_n(t) = -\lambda^{-1} d^+ \{ \sum_{k=0}^{n} v_k(t) \} \quad (n = 0, 1, \ldots).$$

Here $\lambda$ denotes the intensity of the point process and $d^+$ denotes right-hand differentiation with respect to $t$.

A basic relation between synchronous and asynchronous intervals follows from relation (3.1) in conjunction with the relations (2.1).

From (2.1A) and (3.1) we have

$$1 - F_n(t) = \sum_{k=0}^{n-1} u_k(t)$$

$$= -\lambda^{-1} d^+ \{ \sum_{k=0}^{n-1} \sum_{j=0}^{k} v_j(t) \},$$

which by (2.1B) implies

$$1 - F_n(t) = -\lambda^{-1} d^+ \{ \sum_{k=0}^{n-1} [1 - G_{k+1}(t)] \}.$$ 

Therefore,
Relation (3.2). For each \( n = 1, 2, \ldots \),

\[
(3.2) \quad F_n(t) = 1 - \lambda^{-1} D^+ \{ \sum_{k=1}^{n} G_k(t) \} \quad (t > 0).
\]

An alternative to (3.1) is to relate the moments of the synchronous and asynchronous counts. In the case of factorial moments, a particularly elegant relationship holds, a result obtained in collaboration with M. R. Leadbetter.

Relation (3.3). For each \( t > 0 \),

\[
(3.3) \quad \alpha_k(t)/k! = \lambda^{-1} D^+ \{ \beta_{k+1}(t)/(k+1)! \} \quad (k = 1, 2, \ldots)
\]

That is, the \( k \)-th factorial moment of \( M(t) \) is \( 1/\lambda(k+1) \) times the right-hand derivative of the \((k+1)\)-th factorial moment of \( N(t) \).

The proof of (3.3) is as follows. From (3.1) we have

\[
u_n(t) = \lambda^{-1} D^+ \{ G_{n+1}(t) \},
\]

whence

\[
(3.4) \quad \sum_{n=k}^{\infty} \binom{n}{k} \int_0^t \frac{u_n(s)ds}{n!} = \lambda^{-1} \sum_{n=k}^{\infty} \binom{n}{k} G_{n+1}(t).
\]

By (2.7B), the right-hand side of (3.4) is \( \lambda^{-1} \beta_{k+1}(t)/(k+1)! \). Hence the left-hand side of (3.4) converges absolutely and may be written

\[
\int_0^t \left[ \sum_{n=k}^{\infty} \binom{n}{k} u_n(s) \right] ds,
\]

in which the integrand is, by definition, \( \alpha_k(t)/k! \). It follows that

\[
(3.5) \quad \lambda(k+1) \int_0^t \alpha_k(s) ds = \beta_{k+1}(t).
\]
Therefore,

\[ \beta_{k+1}(t) \text{ is absolutely continuous with density } \lambda(k+1)a_k(t). \]

Moreover, since \( F_n(t) \) is continuous from the right for each \( n \), so are \( u_n(t) \) and (by (2.7A) and dominated convergence) \( a_n(t) \). Therefore, being a right-continuous density of \( \beta_{k+1}(t) \), the quantity \( \lambda(k+1)a_k(t) \) is, by the fundamental theorem of calculus, the right-hand derivative of \( \beta_{k+1}(t) \). This proves (3.3).
4. Relations of mixed type. In this section synchronous intervals are related to asynchronous counts. Such relations not only have intrinsic interest but also are useful as tools in obtaining relations of direct type as considered in previous sections.

The first relationship is one shown by Leadbetter (1966).

Relation (4.1). For each \( n = 1, 2, \ldots \),

\[
F_n(t) = 1 + \lambda^{-1} D^+ \left\{ \sum_{k=0}^{n-1} (n-k) v_k(t) \right\} \quad (t > 0).
\]

(Here \( \lambda \) denotes the intensity of the point process and \( D^+ \) denotes right-hand differentiation with respect to \( t \).) Certain implications of (4.1) have been given and utilized earlier in the literature but with rather heuristic foundation. For example, in the case that \( v_k(t) \) is twice-differentiable, the density for the "length of interval between an arbitrary event and the \( n \)-th subsequent event" is thus given by

\[
F_n'(t) = \lambda^{-1} \sum_{k=0}^{n-1} (n-k) v_k''(t) \quad (t > 0),
\]

a result obtained by McFadden (1958) and utilized in the theory of zero-crossing problems (e.g., see Levenbach (1963)). The particular case of (4.2) for \( n = 1 \), i.e.,

\[
F_1'(t) = \lambda^{-1} v_0''(t),
\]

goes back to Kohlenberg (1953), as noted by Myers (1964). More recently, the McFadden result has been used by Buckley (1967).

An alternative to relation (4.1), expressing \( F_n(t) \) in terms of the factorial moments of \( N(t) \), follows immediately from the relations (2.6A) and (3.3). We have
Relation (4.2). For each \( n = 1, 2, \ldots \),

\[
F_n(t) = \frac{\lambda^{-1}}{\binom{k-2}{n-1}} \sum_{k=n+1}^{\infty} (-1)^{k-n-1} \binom{k-2}{n-1} \mathbb{D}^+ \frac{\beta_k(t)}{k!} \quad (t > 0),
\]

provided that the given series converge absolutely.

Variants of the above were produced by Rice (1945) and Longuet-Higgins (1962) for the case of events being axis crossings by a stationary normal process. Recently, Leadbetter (1969) has strengthened the mathematical foundation of these results and at the same time generalized to arbitrary stationary and orderly streams of events by proving a result of form (4.2) under the assumption that, for each \( t \), \( N(t) \) has a probability generating function \( P_t(z) = \sum_0^\infty v_k(t) z^k \) which is finite in the region \( |z| < \rho \), for some \( \rho > 2 \). This condition implies absolute convergence of the series in (4.2). On the other hand, absolute convergence of that series implies finiteness of \( P_t(z) \) in the region \( |z| \leq 2 \). Thus relation (4.2) is obtained in the present report under conditions very slightly weaker than those in Leadbetter (1969). (Also, it is noted, this improvement is due in part to Leadbetter because of the use of (3.3) in obtaining (4.2).)

The converse to (4.2) is

Relation (4.3). for each \( t > 0 \) and \( k = 1, 2, \ldots \),

\[
\frac{\beta_k(t)}{k!} = \frac{\lambda}{\binom{k-2}{n-1}} \sum_{n=k-1}^{\infty} \frac{n-1}{k-2} \int_0^t F_n(u) \, du.
\]

The result follows easily from (2.7) and (3.5), by the dominated convergence theorem.
5. **Relations between joint and marginal distributions.** In this section we shall treat correlation of various kinds.

First let us obtain the correlation between the asynchronous counts in two non-overlapping intervals of the measurement axis. Denote the variance of \( N(t) \) by \( \sigma^2(t) \), \( t > 0 \). Considered as a function of \( t \), the "variance function" \( \sigma^2(t) \) suffices to determine the correlations also. (This has been partially dealt with previously by Lewis and Govier (1964) and by Cox and Lewis (1966).) Denote by \( c(s,d,t) \) the covariance of \( N(0,s) \) and \( N(s+d,s+d+t) \). That is,

\[
(5.1) \quad c(s,d,t) = E[N(0,s)N(s+d,s+d+t)] - E[N(0,s)]E[N(s+d,s+d+t)].
\]

As noted earlier, in formula (1.6), we have \( E[N(0,x)] = \lambda x \), so that

\[
(5.2) \quad c(s,d,t) = E[N(0,s)N(s+d,s+d+t)] - \lambda^2 s t.
\]

By stationarity, \( c(t,d,t) \) is the covariance of the counts in any two intervals of length \( t \) which are separated by an interval of length \( d \).

Moreover, under the symmetry assumption that

\[
(5.3) \quad c(s,d,t) = c(t,d,s) \quad (\text{all } s,t),
\]

we have that \( c(s,d,t) \) is the covariance of the counts in two intervals of lengths \( s \) and \( t \) separated by an interval of length \( d \). Although \( c(s,d,t) \) depends upon the joint probability distribution of \( N(0,s) \) and \( N(s+d,s+d+t) \), it shall now be seen that in fact knowledge of the marginal distributions \( \{ v_k(t) \}_{k=0}^{\infty} \) for all \( t \) suffices to determine the \( c(s,d,t) \)'s. From (5.2), it follows easily, writing \( N(s+d,s+d+t) = N(s,s+d+t) - N(s,s+d) \), that

\[
(5.4) \quad c(s,d,t) = c(s,0,d+t) - c(s,0,d).
\]
Also,

\[(5.5) \quad \sigma^2(s+t) = \sigma^2(s) + \sigma^2(t) + c(s,0,t) + c(t,0,s),\]

so that, under \((5.3)\),

\[(5.6) \quad c(s,d,t) = \frac{1}{2} [\sigma^2(s+d+t) - \sigma^2(s+d) - \sigma^2(d+t) + \sigma^2(s)].\]

Letting \(r(s,d,t)\) denote the correlation between the counts in any two intervals \((a,s+a)\) and \((a+s+d,a+s+d+t)\), we have \(r(s,d,t) = c(s,d,t)/\sigma(s)\sigma(t)\)

and therefore

Relation \((5.7)\). If \(r(s,d,t) = r(t,d,s)\), all \(s\) and \(t > 0\), then

\[(5.7) \quad r(s,d,t) = [\sigma^2(s+d+t) - \sigma^2(s+d) - \sigma^2(d+t) + \sigma^2(d)]/2\sigma(s)\sigma(t),\]

all \(s\) and \(t > 0\).

Of particular interest is the correlation \(r_n(t)\) between the counts within two intervals of length \(t\) separated by \(n = 1\) similar intervals. For such a choice of intervals, the symmetry assumption considered above holds automatically and thus

Relation \((5.8)\). For each \(t > 0\), and \(n \geq 1\),

\[(5.8) \quad r_n(t) = [\sigma^2((n+1)t) - 2\sigma^2(nt) + \sigma^2((n-1)t)]/2\sigma^2(t).\]

It follows that for point processes such that \(\sigma^2(t)\) is asymptotically proportional to \(t\) as \(t \to \infty\), say \(\sigma^2(t) \sim \Lambda t\) as \(t \to \infty\), we have \(r_n(t) \to 0\) as \(n \to \infty\), for each fixed \(t\), and conversely. The constant \(\Lambda\) need not equal the intensity \(\lambda\). Also, if \(\sigma^2(t)\) is exactly proportional to \(t\), then \(r_n(t) \equiv 0\), all \(n\) and \(t\), and conversely. In this case, however, the constant of proportionality must be \(\lambda\), as discussion in Cox and Lewis (1966), pp. 72-75, shows.
Let us now examine the correlations among the intervals between events. This has been studied by McFadden (1962) and Cox and Lewis (1966), but under restrictions somewhat cumbersome or not explicitly formulated and concerning "intervals" somewhat loosely defined. The correlations to be determined presently shall correspond to the context of a strictly stationary sequence of random variables \( \{ X_i \}_{-\infty}^{\infty} \). (By stationarity is meant that the joint probability distribution of any collection \( \{ X_{i_1}, X_{i_2}, \ldots, X_{i_m} \} \) is the same as that of \( \{ X_{i_1+k}, X_{i_2+k}, \ldots, X_{i_m+k} \} \), for any integer \( k \); that is, the joint distributions remain unaltered under simultaneous shifting of the subscripts of the \( X_i \)'s involved.) The question of whether an observed sequence of intervals between events may be regarded as stationary has not been satisfactorily resolved in the literature. (See Beutler and Leneman (1966) for discussion of this point.) Rather than pretend to have an observable stationary sequence \( \{ X_i \} \) of intervals between events, we shall consider a certain associated stationary sequence \( \{ X_i \} \) which may be interpreted as having the features of intervals between events. Precisely, we hypothesize a strictly stationary sequence \( \{ X_i \} \) for which the cum. distribution function of any sum \( \sum_{i=a+1}^{a+n} X_i \) of \( n \) consecutive \( X_i \)'s is the function \( F_n(\cdot) \) defined by (1.11), namely the function which we have seen may be interpreted as the distribution of the length of interval between an arbitrary event and the \( n \)-th subsequent event. In particular, the common marginal distribution of the individual \( X_i \)'s is \( F_1(\cdot) \). Assuming existence of the hypothetical sequence \( \{ X_i \} \), let \( \rho_n \) denote the correlation between the "intervals" \( X_0 \) and \( X_n, n \geq 1 \). By stationarity, \( \rho_n \) is thus the correlation between any two intervals which are separated
by exactly \( n - 1 \) intervals. As in the treatment for counts, these correlations may be determined by an appropriate "variance function."

Letting \( \sigma_n^2 \) denote the variance of \( \sum_1^n X_i \), we have easily that, for \( n \geq 2 \),

\[
(5.9) \quad \sigma_n^2 = n \sigma_1^2 + 2 \sum_{i=1}^{n-1} (n - i) \rho_i \sigma_1^2.
\]

Defining

\[
(5.10) \quad w_k = \left[ \sigma_{k+1}^2 - (k+1)\sigma_1^2 \right]/2\sigma_1^2 \quad (k = 1, 2, \ldots),
\]

we thus have

\[
(5.11) \quad w_k = \sum_{i=1}^{k} (k+1-i) \rho_i \quad (k = 1, 2, \ldots).
\]

It follows without difficulty that \( w_1 = \rho_1 \), \( w_2 - 2w_1 = \rho_2 \) and, for \( k \geq 3 \), \( w_k - 2w_{k-1} + w_{k-2} \) reduces to \( \rho_k \). Therefore,

**Relations (5.12).**

\[
(5.12A) \quad \rho_1 = (\sigma_2^2 - 2\sigma_1^2)/2\sigma_1^2
\]

and, for \( n \geq 2 \),

\[
(5.12B) \quad \rho_n = (\sigma_{n+1}^2 - 2\sigma_n^2 + \sigma_{n-1}^2)/2\sigma_1^2.
\]

The variance function \( \{\sigma_n^2\} \) and hence also the correlations \( \{\rho_n\} \) may be expressed directly in terms of the family of count distributions \( \{v_k(t)\}_{k=0}^{\infty}, t > 0 \). This shall be done under the assumption that

\[
(5.13) \quad v_0(\infty) = \lim_{t \to \infty} v_0(t) = 0,
\]

a condition clearly satisfied in any realistic traffic flow model.
Then, from Leadbetter (1966), for \( n \geq 1 \), the first two moments of the distribution \( F_n(\cdot) \) are

\[
(5.14) \quad \int_0^\infty t \, dF_n(t) = n\lambda^{-1}
\]

and

\[
(5.15) \quad \int_0^\infty t^2 \, dF_n(t) = 2\lambda^{-1} \sum_{k=0}^{n-1} (n - k) \int_0^\infty v_k(t) \, dt.
\]

Hence the variances \( \{\sigma_n^2\} \) are given by

\[
(5.16) \quad \sigma_n^2 = 2\lambda^{-1} \sum_{k=0}^{n-1} (n - k) \int_0^\infty v_k(t) \, dt - n^2 \lambda^{-2} \quad (n \geq 1).
\]

The use of (5.16) in relations (5.12) yields, after some routine reduction,

\textbf{Relation (5.17).} If \( v_0(\infty) = 0 \), then

\[
(5.17) \quad \rho_n = \frac{\lambda \int_0^\infty v_n(t) \, dt - 1}{2\lambda \int_0^\infty v_0(t) \, dt - 1} \quad (n \geq 1).
\]

Under restrictions not entailed in the present treatment, the above expression appears (in slightly different form) in Lewis (1961) and in McFadden (1962).
The implications of (5.17) are interesting. For example, as noted by McFadden (1962), if the point process is such that \( \rho_n \to 0 \) as \( n \to \infty \), then

\[
\lim_{n \to \infty} \int_0^\infty v_n(t) \, dt = \lambda^{-1}.
\]

(5.18)

In the same vein, if we have \( \rho_n \equiv 0 \) (\( n \geq 1 \)), then

\[
\int_0^\infty v_n(t) \, dt = \lambda^{-1} \quad (n \geq 1),
\]

(5.19)

and conversely. It is easily checked, e.g., that (5.19) holds in the case of a Poisson (\( \lambda \)) process.
6. The renewal function in non-renewal contexts. When traffic flow is being modeled by a renewal process, the entire theoretical analysis rests upon an important function known as the "renewal function." When traffic flow is being modeled by stochastic processes more suitable than the renewal kind, it is natural to inquire what the role and relevance of this key function might be in the more general setting. The object of the present section is to show that, roughly speaking, for a stationary orderly point process the associated renewal function is equivalent to knowledge of the second-order properties of the joint distributions of counts. This shall be made precise below.

We begin with a brief digression to define what is meant by a renewal process, renewal function, etc. Previously in this report, a point process has been considered to be a series of points distributed on the interval \(-\infty\) to \(+\infty\). Now we draw attention to processes corresponding to a series of points distributed on the interval \(0\) to \(+\infty\), and we call such a process a renewal process if the interval lengths between successive points are independent and identically distributed random variables. Obviously this is the natural way to model the situation of using an item (a light bulb, a machine, etc.) until failure and then replacing it by a new item of similar type. The lifetimes of the successive items put into service are assumed to be independent random variables having a common probability distribution. (See Smith (1958) and Parzen (1962) for further description.) We may represent such a process in terms of the synchronous counting process \(\{Z(t), t \geq 0\}\) generated by the successive points, where \(Z(t)\) denotes the number of points falling in the interval \((0,t]\). The counting is synchronous in the sense that
the process "starts off" at the point $t = 0$, just as if an event had occurred at $t = 0$. The renewal function is

$\text{(6.1)} \quad H(t) = E[Z(t)] \quad , t > 0,$

i.e., $H(t)$ is the mean number of "renewals" in the interval $(0, t]$.

The importance of $H(t)$ lies in the fact that, in the case of a renewal process, it completely determines the probability law of the process, as may be seen in Parzen (1962), p. 177.

Turning to the context of traffic flow, we note that a renewal model is appropriate only to the extent that the assumption of independent gaps between vehicles is valid, although this concern has not prevented the wide use of renewal models for traffic flow. Familiar examples are the Poisson process, in which the interval lengths are independent negative exponential random variables, and the Erlang model, in which the intervals are independent gamma variables. Efforts to obtain more suitable models include the random queues approach of Borel-Tanner-Miller (see Ashton (1966) for details and references), the Markov-renewal model approach (see Jewell (1965) for details and references), and the general point process approach central to the present report as well as to Serfling (1968). Any such efforts are in vain, of course, even if greater validity is achieved, unless sufficient mathematical tractability is achieved to overcome the appeal of the Poisson model from this standpoint.

Suppose now that an arbitrary stationary orderly point process is under consideration and, as heretofore, is represented by the family of random variables $\{N(t), t \geq 0\}$ discussed in Section 1. An associated family of random variables $\{M(t), t \geq 0\}$ was defined in Section 1 having analogous interpretations in the sense of synchronous counting. We shall
associate with our point process the "renewal function" $H(t)$ defined by

$$H(t) = E[N(t)] , \quad t > 0.$$  

In terms of the distribution functions $F_n(\cdot)$ by which $\{M(t), t \geq 0\}$ was defined, we have

$$H(t) = \sum_{n=1}^{\infty} F_n(t) , \quad t > 0.$$  

It should be noted that if $\{X_i, i \geq 1\}$ represents a stationary sequence of "intervals" initiated at $t = 0$ and such that the cumulative distribution of $\sum_{n=1}^{a+n} X_1$ is $F_n(\cdot)$ for any $a \geq 0$, then the associated counting process $\{Z(t), t > 0\}$, in which $Z(t)$ denotes the number of points in $(0,t]$, has as its "renewal function" $E[Z(t)]$ the same function given by (6.3). And if $\{X_i, i \geq 1\}$ were a renewal process in addition to the above requirements, the function given by (6.3) would be precisely the renewal function of the process. Thus $H(t)$ as defined by (6.2) is the appropriate "renewal function" to be associated with a non-renewal process $\{N(t), t \geq 0\}$.

Now that the $H(t)$ of (6.2) has been established as the "renewal function" in non-renewal contexts, we may proceed to interpret its role and meaning in the more general setting. First we note that $H(t)$ is actually the first factorial moment, $\alpha_1(t)$, of the random variable $M(t)$ and therefore relation (3.3) tells us that the second factorial moment of $N(t)$ is

$$\beta_2(t) = 2\lambda t^t H(z)dz \quad (t > 0).$$

Since $E[N(t)] = \lambda t$, it follows that the variance function $\sigma^2(t)$ of the asynchronous count process is
(6.5) \[ \sigma^2(t) = 2\lambda \int_0^t H(z) dz + \lambda t(1 - \lambda t) \quad (t > 0). \]

Conversely, the renewal function may be determined from the variance function by differentiating in (6.5), i.e.,

(6.6) \[ H(t) = [D^+(\sigma^2(t)) + 2\lambda^2 t - \lambda] / 2\lambda. \]

Therefore, knowledge of \( H(t) \) is equivalent to knowledge of \( \sigma^2(t) \).

Further, it was seen in Section 5 that the correlations between counts in distinct intervals are determined by \( \sigma^2(t) \). For a stationary orderly point process, therefore, \( H(t) \) corresponds precisely to knowledge of the second-order properties of the counts \( \{N(t)\} \) (and nothing more, quite in contrast to its all-encompassing role in the case of a renewal process).
7. The Poisson process. In this section the relationships that have been discussed in the previous sections shall be briefly exemplified for the familiar case of the Poisson process. The purpose here is merely illustration -- no novel results about the Poisson process shall be given (however, in subsequent sections the relationships shall disclose possibly new results for some well-known models.)

In a Poisson ($\lambda$) process, the probability of $k$ events in any interval of length $t$ is

$$(7.1) \quad v_k(t) = e^{-\lambda t} (\lambda t)^k / k! \quad (k = 0, 1, \ldots).$$

As mentioned in Section 2 without proof, we have

$$(7.2) \quad u_k(t) = v_k(t) \quad (k = 0, 1, \ldots)$$

and

$$(7.3) \quad \alpha_k(t) = \beta_k(t) = (\lambda t)^k \quad (k = 1, 2, \ldots).$$

It may be seen that (7.2) follows from relation (3.1). Likewise the first equality in (7.3) follows from relation (3.3), while the second equality may be verified directly. We see that the synchronous- and asynchronous-mode features are identical for a Poisson process. In each mode, the distribution of the interval length from $t = 0$ to the next event is negative exponential, $F_1(t) = G_1(t) = 1 - e^{-\lambda t}, t > 0$, as may be seen from relations (2.1) or (2.6).

The variance function for counts is found to be $\sigma^2(t) = \lambda t$, from which we find by relation (5.7) that $r(s, d, t) = 0$, all $s, d, t \geq 0$, i.e., the correlation between the counts in disjoint intervals is zero. Likewise, relation (5.17) implies that the correlation between two intervals between successive events is zero.
8. The Erlang process. Following Haight (1963), p. 60, a process with independent gamma intervals between events is called an Erlang process. In particular, if the gamma density adopted is

$$f(t) = (\theta m)^m t^{m-1} e^{-\theta t} / \Gamma(m) \quad (t > 0),$$

for a positive integer m, then the process is Erlang of order m.

The case m = 1 denotes, of course, a Poisson (\theta) process. In general, the Erlang process of order m may be generated by taking every m-th point of a Poisson process. The corresponding synchronous and asynchronous counting distributions are given in Haight (1963), § 4.4, and are quite complicated in form. In particular, the probability of a specified number of events in an interval of length t is given by

$$v_0(t) = \sum_{j=0}^{m-1} \frac{(1 - \frac{j}{m}) e^{-\theta t} (\theta t)^j}{j!}$$

and, for k > 1,

$$v_k(t) = \sum_{j=-m+1}^{m-1} \frac{(1 - \frac{|j|}{m}) e^{-\theta t} (\theta t)^{km+j}}{(km+j)!}.$$ 

Therefore, we have

$$\lim_{t \to 0} \frac{v_1(t)}{t} = \frac{\theta}{m}$$

and

$$\lim_{t \to 0} \frac{\sum_{k=2}^{\infty} v_k(t)}{t} = 0.$$ 

Thus the Erlang process is orderly and has intensity \(\lambda = \theta / m\), which is also, of course, the mean of the asynchronous count per unit interval.

First let us examine the case m = 2, in which case

$$f(t) = 2 \theta^2 t e^{-\theta t} \quad (t > 0).$$
The correlations between the intervals between events are zero, of course, by the definition of an Erlang process. However, the correlations between the counts in two observation intervals are not zero and knowledge of their values can be useful in various ways. As seen in Section 5, it suffices to know the asynchronous variance function \( \sigma^2(t) \). For the Erlang process of order 2, a simple closed form for \( \sigma^2(t) \) is known (Cox and Lewis (1966), p. 79),

\[
(8.6) \quad \sigma^2(t) = \frac{1}{4} \Theta t + \frac{1}{8}(1-e^{-2\Theta t}).
\]

It follows from relation (5.7) that the correlation of the (asynchronous) counts in two intervals of lengths \( s \) and \( t \) separated by an interval of length \( d \) is

\[
(8.7) \quad r(s,d,t) = \frac{-\frac{1}{8} [e^{-2\Theta (s+d+t)} - e^{-2\Theta (s+d)} - e^{-2\Theta (d+t)} + e^{-2\Theta d}]}{2 \left[ \frac{1}{4} \Theta s + \frac{1}{8}(1-e^{-2\Theta s}) \right]^2 \left[ \frac{1}{4} \Theta t + \frac{1}{8}(1-e^{-2\Theta t}) \right]^2}.
\]

For \( d = 0 \), this becomes the correlation between the counts in two adjacent intervals. Denoting this latter correlation by \( K_2(s,t) \), we see that, for all \( d \geq 0 \),

\[
(8.8) \quad r(s,d,t) = K_2(s,d,t) e^{-2\Theta d}.
\]

This shows, in particular, that \( r(s,d,t) \to 0 \) at an exponential rate as the distance \( d \) increases between the two intervals.

In turning to the case of an order \( m \) greater than 2, we are immediately handicapped by there not being known any convenient formula such as (8.6) for the appropriate variance function. Numerical approximations have been determined by Whittlesey and Haight (1961), using as a starting point the exact formula
\[ (8.9) \quad \sigma^2(t) = \frac{\theta t}{m} - \left(\frac{\theta t}{m}\right)^2 + \frac{e^{-\theta t}}{m} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{(\theta t)^{jm+k}}{(jm+k)!} [j(j-1)m+2jk]. \]

It should be noted that the leading term of (8.9) is not the asymptotically correct linear approximation to \( \sigma^2(t) \) as \( t \to \infty \) (e.g., compare (8.9) for \( m = 2 \) with (8.6)), so that the series in (8.9) should not be construed as an "error of approximation" in any sense.

Below we shall obtain a new exact formula for the variance \( \sigma^2(t) \) and also provide useful upper and lower bounds for \( \sigma^2(t) \).

The second factorial moment of \( N(t) \) is, by definition and using (8.2),

\[ (8.10) \quad \beta_2(t) = \sum_{n=2}^{\infty} n(n-1) \sum_{u=-m+1}^{m-1} \frac{(1-n)}{m} e^{-\theta t} (\theta t)^{nm+u} / (nm+u)! . \]

Now, if \( jm < k \leq (j+1)m \), the term in \( (\theta t)^k \) appears in the above expansion not only for the combination of indices \( n = j, u = k - jm \) but also for the combination \( n = j + 1, u = (j+1)m - k \). It follows that

\[ (8.11) \quad \beta_2(t) = \sum_{j=1}^{\infty} \sum_{k=1}^{(j+1)m} \{j(j-1)[1-\frac{k-im}{m}] + j(j+1)[1-\frac{(j+1)m-k}{m}]\} \frac{e^{-\theta t} (\theta t)^k}{k!} . \]

The quantity in braces reduces to \( 2kj/m - j^2 - j \), yielding

\[ (8.12) \quad \beta_2(t) = \sum_{j=1}^{\infty} \sum_{k=1}^{(j+1)m} \frac{e^{-\theta t} (\theta t)^k}{k!} [2(\frac{k}{m})j - j^2 - j]. \]

Now we note that

\[ 2(\frac{k}{m})j - j^2 - j = (\frac{k}{m})^2 - \frac{k}{m} + (\frac{k}{m} - j)(\frac{k}{m} - j)^2 \]

\[ (8.13) \quad = (\frac{k}{m})^2 - \frac{k}{m} + (\frac{k}{m} - j)[1 - (\frac{k}{m} - j)]. \]

Also, for \( jm + 1 \leq k \leq (j+1)m \), it is seen easily that

\[ (8.14) \quad \frac{1}{m} \leq (\frac{k}{m} - j) \leq 1. \]
Hence, since for \(0 \leq p \leq 1\) we have \(0 \leq p(1-p) \leq \frac{1}{4}\), we have

\[(8.15) \quad 0 \leq \left(\frac{k}{m} - j\right)[1 - \left(\frac{k}{m} - j\right)] \leq \frac{1}{4}\]

whenever \(jm + 1 \leq k \leq (j+1)m\). Thus we may obtain upper and lower bounds for \(\beta_2(t)\) by applying, via (8.13), the bounds in (8.15) to the terms in (8.12). Denote the lower and upper bounds on \(\beta_2(t)\) by \(\underline{\beta}_2(t)\) and \(\overline{\beta}_2(t)\), respectively. Then

\[
\underline{\beta}_2(t) = \sum_{k=0}^{\infty} \frac{e^{-\theta t}(\theta t)^k}{k!} \left[\frac{k}{m} - \left(\frac{k}{m}\right)^2\right]
\]

\[
= \sum_{k=1}^{\infty} \frac{e^{-\theta t}(\theta t)^k}{k!} \left[\frac{k}{m} - \left(\frac{k}{m}\right)^2\right]
\]

\[
= (\frac{\theta t}{m})^2 \sum_{m=1}^{\infty} \frac{e^{-\theta t}(\theta t)^k}{k!} - (1 - \frac{1}{m})^2 \left\{ \sum_{m=1}^{\infty} \frac{e^{-\theta t}(\theta t)^k}{k!} \right\}
\]

\[(8.16) \quad = - (1 - \frac{1}{m})^2 \left\{ \frac{\theta t}{m}\right\} + (\frac{\theta t}{m})^2 + \frac{1}{m^2} \sum_{k=1}^{m-1} \frac{e^{-\theta t}(\theta t)^k}{k!} [k(m - k)].
\]

Likewise we obtain

\[(8.17) \quad \overline{\beta}_2(t) = \underline{\beta}_2(t) + \frac{1}{4}(1 - e^{-\theta t}) - \frac{1}{4} \sum_{k=1}^{m} \frac{e^{-\theta t}(\theta t)^k}{k!}
\]

We may convert to results on \(\sigma^2(t)\) by use of the relation

\[\sigma^2(t) = \beta_2(t) + \left(\frac{\theta t}{m}\right)^2 - (\frac{\theta t}{m})^2\]

Denote lower and upper bounds by \(\underline{\sigma}^2(t)\) and \(\overline{\sigma}^2(t)\), respectively. Then

\[(8.18) \quad \sigma^2(t) = \frac{\theta t}{m^2} + \sum_{k=1}^{m-1} \frac{e^{-\theta t}(\theta t)^k}{k!} \left[\frac{k}{m} \left(1 - \frac{k}{m}\right)\right] + \sum_{j=1}^{(j+1)m} \frac{e^{-\theta t}(\theta t)^k}{k!} \left[\frac{k}{m} \left(1 - \frac{k}{m}\right)\right]
\]

follows from (8.12), (8.13) and (8.16) and represents an exact formula for \(\sigma^2(t)\) in which the leading term is asymptotically equivalent to \(\sigma^2(t)\) as \(t \to \infty\). The lower bound \(\underline{\sigma}^2(t)\) corresponding to \(\underline{\beta}_2(t)\) is
\[
(8.19) \quad \sigma^2(t) = \frac{\theta t}{m^2} + \frac{40}{m^2} \sum_{k=1}^{m-1} \frac{e^{-\theta t}(\theta t)^k}{k!} \left[ \frac{k}{m}(1 - \frac{k}{m}) \right]
\]

Likewise, an upper bound is easily seen to be
\[
(8.20) \quad \sigma^2(t) = \frac{\theta t}{m^2} + \frac{m-1}{m^2} \sum_{k=1}^{m-1} \frac{e^{-\theta t}(\theta t)^k}{k!} \left[ \frac{k}{m}(1 - \frac{k}{m}) \right] + \frac{1}{4}(1 - e^{-\theta t}) - \frac{1}{4} \sum_{k=1}^{m-1} \frac{e^{-\theta t}(\theta t)^k}{k!},
\]

which in turn is exceeded by the simpler bound
\[
(8.21) \quad \sigma^2(t) = \frac{\theta t}{m^2} + \frac{1}{4}(1 - e^{-\theta t})
\]

obtained by using the relation \( \frac{k}{m}(1 - \frac{k}{m}) \leq \frac{1}{4} \) for \( 1 \leq k \leq m \) in (8.20).

Also, since \( \frac{k}{m}(1 - \frac{k}{m}) \geq \frac{1}{m}(1 - \frac{1}{m}) \) for \( 1 \leq k \leq m - 1 \), \( \sigma^2(t) \) exceeds the simpler lower bound
\[
(8.22) \quad \sigma^2_{\text{m}}(t) = \frac{\theta t}{m^2} + \frac{1}{m}(1 - \frac{1}{m}) \sum_{k=1}^{m-1} \frac{e^{-\theta t}(\theta t)^k}{k!}.
\]

In summary, we have, for all \( t \),
\[
(8.23) \quad \sigma^2_{\text{m}}(t) \leq \sigma^2(t) \leq \sigma^2(t) \leq \sigma^2(t) \leq \sigma^2(t).
\]

These bounds facilitate several purposes. For example, the rate of convergence of the asymptotic approximation of \( \sigma^2(t) \) by \( \theta t/m^2 \) can be studied theoretically. Secondly, close numerical approximation should be more easily obtainable. Further, one can use the bounds on \( \sigma^2(t) \) to place useful bounds on other quantities. For instance, parallel to the exact treatment of the correlations \( r(s,d,t) \) which was given above for \( m = 2 \), we can develop approximate results for the cases of \( m > 2 \).
9. The Pólya process. Following Haight (1967), p. 39, the process obtained by mixing the parameter of a Poisson process by a gamma distribution is called a Pólya process. The resulting distribution is known as Pólya-Eggenberger or negative binomial. Precisely, if we suppose that the parameter \( \mu \) of a Poisson (\( \mu \)) process has itself a gamma (\( m, \Delta \)) distribution, \( f(\mu) = \mu^{m-1} e^{-\mu/\Delta} \Delta^{-m} \Gamma(m)^{-1} \), then the resulting asynchronous counting distribution is

\[
(9.1) \quad v_k(t) = \int_0^\infty \frac{e^{-\mu t} (\mu t)^k}{k!} \cdot \frac{\mu^{m-1} e^{-\mu/\Delta}}{\Gamma(m) \Delta^m} \, d\mu,
\]

for each \( k = 0,1,\ldots \). We shall assume that \( m \) is an integer, in which case

\[
(9.2) \quad v_k(t) = \binom{m-1+k}{k} \left( \frac{1}{1+\Delta t} \right)^m \left( \frac{\Delta t}{1+\Delta t} \right)^k,
\]

for each \( k = 0,1,\ldots \). This is the negative binomial distribution with mean \( m \Delta t \). General background and further references on the Pólya process may be found in Haight (1967). The negative binomial distributions are among those having the curious property of being both a "mixed Poisson", as described above, and a "compound Poisson," obtained by letting a Poisson distribution be the mixing distribution for the parameter of some other distribution. A "compound Poisson" process thus corresponds to occurrences of multiple events at instants which form a Poisson process. However, the process consisting of \( \{v_k(t)\}_{k=0}^\infty, t > 0 \), as given by (9.2) does not fall into the latter category, as shall be seen below.

Concerning applications in traffic theory, see Ashton (1966), Ch. 10, for consideration of the negative binomial as a model for accidents and see Buckley (1967) for consideration of the negative binomial as a model for traffic flow.
In the present section various properties of the Polya process shall be obtained using the relationships of previous sections. The properties are evidently new, for the most part, and they shed considerable light on the relevance of the Polya process as a model for traffic flow.

It is easily verified that

\[
\lim_{t \to 0} \frac{v_1(t)}{t} = m\Delta
\]

and that

\[
\lim_{t \to 0} \frac{\sum_{k=2}^{\infty} v_k(t)}{t} = 0.
\]

Hence the Polya process given by (9.2) is orderly and has intensity \( \lambda = m\Delta \). Let us note also, for later reference, that

\[
\lim_{t \to \infty} v_0(t) = 0.
\]

The \( k \)-th factorial moment of the (asynchronous) count distribution is found by routine manipulations to be

\[
\beta_k(t) = k! (m-\frac{1+k}{R}) (\Delta t)^k
\]

for each \( t > 0 \) and \( k = 1, 2, \ldots \). As noted in Haight (1967), p. 40, the distinction between synchronous and asynchronous counting has been overlooked in some of the literature on the Polya process, especially concerning its application to accident causation. Here we may use relation (3.3) to obtain immediately the factorial moments of the synchronous count distribution, i.e.,
\( \alpha_k(t) = k! \binom{m+k}{k} (\Delta t)^k. \)

Thus the synchronous count distribution is of the form (9.2) with \( \Delta \) unchanged but \( m \) replaced by \( m+1 \). In particular, the renewal function is seen to be

\( H(t) = (m+1)\Delta t \) \hspace{1cm} \( t > 0 \).

The asynchronous mean function is \( \lambda t = m\Delta t \). The asynchronous variance function is seen to be

\( \sigma^2(t) = m\Delta t(1 + \Delta t). \)

Therefore, the index of dispersion, \( \sigma^2(t)/\lambda t \), is

\( I(t) = 1 + \Delta t = 1 + \frac{\lambda t}{m}, \)

which, given a fixed intensity \( \lambda \), depends also upon the parameter \( m \).

Thus the distribution is "overdispersed," relative to the Poisson distribution, whose index of dispersion is unity.

The features discussed thus far make the Pólya process seem plausible as a model for traffic flow. To this, we add the fact (to be shown below) of dependence among the intervals between events and also among the counts within separate intervals. On the other hand, the particular nature of this dependence shall seem rather to dispel this plausibility.

From the variance function (9.9) we obtain, via relation (5.7), that the correlation between the counts in two intervals of lengths \( s \) and \( t \) separated by a distance \( d \) is

\( r(s,d,t) = \left( \frac{\Delta s}{1 + \Delta s} \right)^{\frac{1}{2}} \left( \frac{\Delta t}{1 + \Delta t} \right)^{\frac{1}{2}}. \)

This result has been given previously by Lundberg (1940). In particular, the correlation \( r_n(t) \) between the counts in two intervals of length \( t \) separated by \( n - 1 \) similar intervals is
(9.12) \[ r_n(t) = \frac{\Delta t}{1 + \Delta t}. \]

The striking aspect of (9.11) and (9.12) is that the correlations do not depend upon \(d\) or \(n\), respectively, and thus do not \(\rightarrow 0\) as \(d \to \infty\) or \(n \to \infty\). We note also that the correlations depend upon the parameter \(\Delta\) but not upon the parameter \(m\).

The situation for the correlations of intervals between events is similar in character. Since \(V_0(\infty) = 0\), by (9.5), we may apply relation (5.17). For this purpose, we note that

\[
\int_0^{\infty} v_n(t) \, dt = \binom{m-1+n}{n} \int_0^{\infty} \left( \frac{1}{1+\Delta t} \right)^m \left( \frac{\Delta t}{1+\Delta t} \right)^n \, dt
\]

\[
= \binom{m-1+n}{n} \Delta^{-1} \int_0^1 u^n \frac{u^m-2}{1-u} \, du
\]

(9.13) \[
\frac{1}{\Delta(m-1)}.
\]

Therefore, applying (5.17), the correlation between two interval lengths which are separated by \(n-1\) intervals is given by

(9.14) \[
\rho_n = \frac{1}{m+1}.
\]

Again we obtain a value that does not depend upon \(n\) and thus does not \(\rightarrow 0\) as \(n \to \infty\). Note also that this value depends upon the parameter \(m\) but not upon the parameter \(\Delta\).

These results suggest that application of the Pólya process to traffic flow modeling requires careful consideration of the circumstances involved. While the Poisson process entails the severe assumptions of independence of counts in separate intervals and independence of intervals between events, the Pólya process seems too extreme in the other direction by not allowing the dependence to "die off" as the intervening "distance"
is increased. In any potential application, therefore, it must be considered whether this aspect of the dependence structure of the Pólya process is crucial or not.
10. **General applications.** It has been seen in previous sections how the relationships presented in this report may be used to shed new light on familiar models or to obtain previously known results under broader assumptions than heretofore allowed. We have also seen that the relationships serve to provide qualitative insights, such as the role of the renewal function in non-renewal contexts. In addition to these areas of application, we note that a primary application of these relationships is to the definition and theoretical analysis of new models that arise for consideration. Given an arbitrary model \( \{v_k(t)\}_{k=0}^{\infty}, \ t > 0, \) for consideration as a possible traffic flow model, one can build the picture by the use of these fundamental relationships. For instance, one can analyze whether the correlation structure of the potential model has reasonable features or not. Thus the results in this report support the efforts of researchers (e.g., Serfling 1968) to obtain suitable alternatives to the Poisson process. The examples considered, the Poisson, Erlang and Pólya processes, illustrate briefly the manner in which these fundamental relationships may be applied to analyze any given process.
REFERENCES


