CONCENTRATION OF CARS IN A MACROSCOPIC
STUDY OF TRAFFIC FLOW, I. STEADY STATE.

I. N. Shimi

FSU Statistics Report M156

February, 1969
Department of Statistics
Florida State University
Tallahassee, Florida

Prepared under Contract No. FH-11-6890, Bureau of Public Roads,
Federal Highway Administration, U. S. Department of Transportation.
The opinions, findings and conclusions expressed in this
publication are those of the author and not necessarily those of
the Bureau of Public Roads.
1. **Introduction.** Recently an interesting macroscopic theory of traffic flow was introduced and studied in detail in a number of papers [1], [2], [6] - [11]. In these papers a statistical approach to traffic flow was developed, which is similar to the approach used in Statistical Mechanics to study the Boltzmann equation of the kinetic theory of gases. The main concern in these papers is to determine the speed distribution function $f(x,v,t)$, where $f(x,v,t)dx dv$ is the probability of a car being in $[x, x + dx]$ whose velocity is in the range $[v, v + dv]$ at time $t$. All important hydrodynamical properties depending on the position and velocity of a single car may be determined whenever the function $f(x,v,t)$ is known. In the above mentioned papers it was shown that the function $f(x,v,t)$ satisfies the Boltzmann-type integral differential equation

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0(f),
\]

where $0(f)$ is the sum of two operators

\[
0(f) = D(f) + I(f).
\]

The operator $D(f)$ is a linear functional in $f$ representing the relaxation process describing the desires of the molecules (drivers) to return to some desired speed distribution, and the nonlinear functional $I(f)$ corresponds to interaction processes between cars. An example of the operator $0(f)$ which was studied extensively in [1], [2], [6] - [11] is given by

\[
0(f) = -\frac{f - f^0}{T} + c(<v> - v)(1 - P)f,
\]

where $f^0$ is the one-car desired speed distribution function, assumed to exist,
T is the relaxation time, the concentration (cars per unit distance) \( c(x,t) \) satisfies

\[
c = \int_0^\infty f dv = \int_0^\infty f_0 dv,
\]

the average speed

\[
<v> = \frac{1}{c} \int_0^\infty v f dv,
\]

and

\[
P \text{ is the probability of passing.}
\]

The stationary homogeneous solution of equation (1.1) satisfies the integral equation

(1.4) \( D(f) + I(f) = 0. \)

If the solution of equation (1.4) is given by

(1.5) \( f = F(c,v) \)

then the local steady state speed distribution function for the inhomogeneous case is given by \( f = F(c(x,t),v) \), where \( c(x,t) \) is the local value of the concentration. Prigogine, I., Herman, R., and Anderson, R. [11] arrived at this result by an argument similar to the Chapman-Enskog method used in the kinetic theory of gases. In this case we can see that the dependence of \( f \) on the location \( x \) and the time \( t \) appears only through the local concentration \( c(x,t) \). Thus the local average velocity becomes a functional of \( c(x,t) \).
\[ (1.6) \quad \langle v(x,t) \rangle = \frac{1}{c(x,t)} \int_{0}^{\infty} vF(c(x,t),v) dv = \langle v[c(x,t)] \rangle. \]

A complete discussion of these ideas and their consequences for the homogeneous time-independent case is given in [1], [2], [6] - [11].

As a consequence of the above discussion, it is clear that our main efforts should be to determine the concentration \( c(x,t) \). The method used to determine the concentration \( c(x,t) \) is the same as the one used in the study of diffusion processes of gas molecules.

We are going to determine the concentrations for the following traffic flow problem. Consider a stretch of highway composed of two adjacent parts of length \( a \) and \( b \). These two parts of the road are assumed to have different road characteristics. Cars enter that stretch of the highway from one end of it, called the entrance, and can only leave it from the other end, called the exit. We assume that the cars arrive at the entrance with constant concentration at all times, that is, there is a steady flow of cars at the entrance. We also assume that there exist two limiting concentrations for the two parts of the road. Cars can no longer pass in that part of the road when the concentration in that part reaches its limiting value. The system starts with the exit closed and the concentrations in each part at its limiting value. At time \( t = 0 \) the flow starts out of the exit. Our objective is to determine the concentrations in the two parts of the road as functions of the time and the location in the road.

In this paper the solutions giving the concentrations will be particularly suitable for large \( t \), thus steady state solutions will be given.
Solutions whose functional forms are particularly suitable for small $t$ will be given in another paper [12], these will give the non-stationary solutions for the concentrations.

This work is important in cases of moderate and high concentrations, since under such conditions the usual assumption of independence of locations and of speeds of the cars breaks down.

2. The concentration functions. Let $c_1(x,t)$ and $c_2(x,t)$ represent the concentrations in the part of the road from $-a$ to 0 and from 0 to $b$ respectively. Let $c^-$ and $c^-'$ be the limiting concentrations in $[-a,0]$ and $[0,b]$ respectively. Considering cars as molecules in a diffusion process, we are led to the following system of partial differential equations and boundary conditions.

The diffusion equations in the two parts of the road are

\begin{equation}
\frac{\partial c_1}{\partial t} = D_1 \frac{\partial^2 c_1}{\partial x^2} \quad \text{for} \ -a < x < 0,
\end{equation}

and

\begin{equation}
\frac{\partial c_2}{\partial t} = D_2 \frac{\partial^2 c_2}{\partial x^2} \quad \text{for} \ 0 < x < b,
\end{equation}

where $D_1$ and $D_2$ are the diffusion coefficients in the respective parts of the road, $D_1$ and $D_2$ are constant parameters independent of concentrations, time and location within the respective parts of the road. At the boundary $x = 0$ the equality of flow in and out of an element of road on this boundary is expressed by
(2.3) \[ D_1 \frac{\partial c_1}{\partial x} = D_2 \frac{\partial c_2}{\partial x} \] at \( x = 0 \).

We also assume that the concentration at \( b \) is constant and equal to \( c \) for all \( t \). Thus the flow in the system through the entrance at \( b \) is stationary. With no loss of generality we may assume that the concentration at \( x = -a \) is zero (in general this is assumed to be a constant) thus

(2.4) \[ c_1(-a,t) = 0 \quad \text{for} \quad t > 0. \]

We shall also introduce two parameters \( k_0 \) and \( k_b \), called the partition coefficients. These constants give the relationship between the concentrations on both sides of the partitions in the road at \( x = 0 \) and \( x = b \). Thus

(2.5) \[ c_2(b,t) = k_b c, \]

(2.6) \[ c_2(0,t) = k_0 c_1(0,t), \]

and the initial conditions \( c_1(x,0) = c^- \) for \(-a \leq x < 0\), and \( c_2(x,0) = c^- \)

for \( 0 < x \leq b \) satisfy

(2.7) \[ c^- = k_0 c^- \]

and

(2.8) \[ c^- = k_b c. \]

Let \( \tilde{f}(s) \) denote the Laplace transform of the function \( f(t) \), i.e.,
\[ f(s) = \int_0^\infty e^{-st}f(t)dt . \]

Then in view of the initial conditions the Laplace transforms of (2.1), (2.2) and (2.3) are

\[ D_1 \frac{d^2 c_1(x,s)}{dx^2} - sc_1(x,s) + c^- = 0, \]  
\[ D_2 \frac{d^2 c_2(x,s)}{dx^2} - sc_2(x,s) + c'^- = 0, \]

and

\[ D_1 \left( \frac{dc_1(x,s)}{dx} \right)_{x=0} = D_2 \left( \frac{dc_2(x,s)}{dx} \right)_{x=0}. \]

Solving the ordinary differential equations (2.9) and (2.10) we get

\[ c_1(x,s) = A \exp\left(\sqrt{\frac{s}{D_1}} x \right) + B \exp\left(-\sqrt{\frac{s}{D_1}} x \right) + \frac{c^-}{s} \]

and

\[ c_2(x,s) = G \exp\left(\sqrt{\frac{s}{D_2}} x \right) + H \exp\left(-\sqrt{\frac{s}{D_2}} x \right) + \frac{c'^-}{s}. \]

From (2.11), (2.12) and (2.13) a relation between A, B, G and H is given by

\[ \sqrt{D_1}(A-B) = \sqrt{D_2}(G-H). \]
Taking the Laplace transform of (2.6) we have

\[(2.15) \quad k_0 \overline{c_1}(0,s) = \overline{c_2}(0,s).\]

Using (2.12), (2.13) and (2.15) we get

\[k_0(A + B + \frac{c^{'}}{s}) = G + H + \frac{c^{''}}{s},\]

and using (2.7) this reduces to

\[(2.16) \quad k_0(A+B) = G + H.\]

In view of (2.4) we have \(\overline{c_1}(-a,s) = 0\); therefore if we put \(\xi = -a\) in (2.12) we have

\[(2.17) \quad A \exp\left(-\sqrt{\frac{s}{D_1}} a\right) + B \exp\left(\sqrt{\frac{s}{D_1}} a\right) + \frac{c^{'}}{s} = 0.\]

Also from (2.5) we have

\[\overline{c_2}(b,s) = k_b \frac{c}{s}.\]

Therefore from (2.13) we get

\[G \exp\left(\sqrt{\frac{s}{D_2}} b\right) + H \exp\left(-\sqrt{\frac{s}{D_2}} b\right) + \frac{c^{''}}{s} = k_b \frac{c}{s},\]

and using (2.8) this reduces to
\begin{equation}
G \exp \left( \sqrt{\frac{S}{D_2}} b \right) + H \exp \left( -\sqrt{\frac{S}{D_2}} b \right) = 0.
\end{equation}

Using (2.14), (2.16), (2.17) and (2.18) we can solve for \(A, B, G\) and \(H\),
and substituting in (2.12) and (2.13) we get

(I) \[
\tilde{c}_1(x, s) = \frac{c^*}{s} \left[ 1 - \frac{m \sinh(\gamma \sqrt{s}) + \ell \sinh(-\delta \sqrt{s})}{\ell \sinh(\omega \sqrt{s}) - m \sinh(u \sqrt{s})} \right] \text{ for } -a \leq x \leq 0,
\]

and

(II) \[
\tilde{c}_2(x, s) = \frac{c^*}{s} \left[ k_0 + \frac{2 \sinh\left( \sqrt{\frac{S}{D_2}} (x-b) \right)}{\ell \sinh(\omega \sqrt{s}) - m \sinh(u \sqrt{s})} \right] \text{ for } 0 \leq x \leq b,
\]

where

\[\ell = \frac{1}{k_0} + \sqrt{\frac{D_2}{D_1}}, \quad \quad \quad m = \frac{1}{k_0} - \sqrt{\frac{D_2}{D_1}},\]

\[w = \frac{a}{\sqrt{D_1}} + \frac{b}{\sqrt{D_2}}, \quad \quad \quad u = \frac{a}{\sqrt{D_1}} - \frac{b}{\sqrt{D_2}}\]

\[\gamma = \frac{x}{\sqrt{D_1}} + \frac{b}{\sqrt{D_2}} \quad \text{and} \quad -\delta = \frac{b}{\sqrt{D_2}} - \frac{x}{\sqrt{D_1}}.\]

Solutions with no restrictions on the parameters involved in the model
will be discussed in another paper. In the next section solutions
resulting from restricting the relative lengths of the two parts of the
road will be given.
3. Special solutions. In this section the inverse Laplace transforms of $\bar{\Omega}_1(x,s)$ and $\bar{\Omega}_2(x,s)$ will be given assuming three different conditions on the parameters $a$, $b$, $D_1$ and $D_2$.

(i) Assuming $\frac{a}{\sqrt{D_1}} = \frac{b}{\sqrt{D_2}}$.

We can see that under this condition $u = 0$ and $w = \frac{2a}{\sqrt{D_1}}$, thus (1) reduces to

$$\bar{\Omega}_1(x,s) = \frac{c_0}{s} \left[ 1 - \frac{m \sinh(\gamma \sqrt{s})}{s \sinh(w \sqrt{s})} - \frac{\sinh(\delta \sqrt{s})}{\sinh(w \sqrt{s})} \right].$$

The inverse Laplace transform of $\frac{\sinh(\gamma \sqrt{s})}{\sinh(w \sqrt{s})}$ is given by formula (31), page 258 of [3] and is equal to

$$\frac{1}{w} \frac{\partial}{\partial \gamma} \theta_4 \left( \frac{\gamma}{2w}, \frac{\pi t}{w} \right),$$

where $\theta_4(x|y)$ is Jacobi's theta function given by

$$\theta_4(x|y) = 1 + 2 \sum_{n=0}^{\infty} (-1)^n (e^{i\pi y})^n \cos(2n\pi x).$$

Therefore

$$\frac{1}{w} \frac{\partial}{\partial \gamma} \theta_4 \left( \frac{\gamma}{2w}, \frac{\pi t}{w} \right) = \frac{2\pi}{w} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-\frac{\pi^2 n^2 t}{w^2}}}{w^2} \sin\left( \frac{n\pi y}{w} \right).$$

Thus the inverse Laplace transform of $\frac{\sinh(\gamma \sqrt{s})}{s \sinh(w \sqrt{s})}$ is given by
\[
\frac{2\pi}{w} \int_0^\infty (-1)^{n+1} e^{\frac{-\pi^2 n^2 t}{w^2}} \sin\left(\frac{n\pi y}{w}\right) dt = \frac{1}{2} \sum_{n=1}^\infty (-1)^{n+1} \frac{1}{n} \left[ 1 - \frac{\pi^2 n^2 t}{w^2} \right] \sin\left(\frac{n\pi y}{w}\right).
\]

In the last equality above we used the fact that

\[
\sum_{n=1}^\infty (-1)^{n+1} \frac{1}{n} \sin(n\theta) = \frac{\theta}{2}.
\]

The inverse Laplace transform of \(\frac{\sinh(-\sqrt{s})}{s \sinh(w\sqrt{s})}\) is given by formula (39), page 259 of [3] and is equal to

\[
\frac{-\delta}{w} + \frac{2}{\pi} \sum_{n=1}^\infty (-1)^n \frac{\sin\left(\frac{n\pi (-\delta)}{w}\right)}{n} e^{\frac{-\pi^2 n^2 t}{w^2}}.
\]

Therefore the inverse Laplace transform of \(c_1(x,s)\) is

\[(3.1) \quad c_1(x,t) = c \left[ 1 - \frac{m}{\lambda} \left( \frac{\gamma + 2}{\pi} \sum_{n=1}^\infty (-1)^n \frac{1}{n} e^{\frac{-\pi^2 n^2 t}{w^2}} \sin\left(\frac{n\pi y}{w}\right) \right) - \frac{2 n^2 t}{w^2} \sin\left(\frac{n\pi y}{w}\right) \right].
\]

This gives the concentration for \(x \in [-a,0] \) and \(t > 0\).
Under the same condition, that is \( \frac{a}{\sqrt{D_1}} = \frac{b}{\sqrt{D_2}} \), (II) reduces to

\[
\overline{c}_2(x,s) = c' \left[ k_0 + \frac{2 \sinh\left( \frac{s}{\sqrt{D_2}} (x-b) \right)}{\sqrt{s} \sinh\left( \frac{2a}{\sqrt{D_1}} \sqrt{s} \right)} \right], \quad \text{for } 0 \leq x \leq b.
\]

Therefore the inverse Laplace transform for \( \overline{c}_2(x,s) \) is given by

\[
(3.2) \quad c_2(x,t) = c' \left[ k_0 - \frac{2}{l} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{\pi^2 n^2 t}{w^2}} \sin\left( \frac{n\pi (b-x)}{2a/\sqrt{D_2}} \right) dt \right]
\]

\[
= c' \left[ k_0 - \frac{(b-x)/\sqrt{D_1}}{2a/\sqrt{D_2}} - \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-\frac{\pi^2 n^2 t}{w^2}} \sin\left( \frac{n\pi (b-x)/\sqrt{D_2}}{2a/\sqrt{D_2}} \right) \right].
\]

The expressions giving \( c_1(x,t) \) and \( c_2(x,t) \) above are suitable for large \( t \) since the summation terms will be negligible. Thus if \( c_1(x) \)

\[
(3.3) \quad c_1(x) = c' \left[ 1 - \frac{b - \frac{2\delta}{\lambda w}}{2} \right], \quad x \in [-a,0]
\]

and

\[
(3.4) \quad c_2(x) = c' \left[ k_0 - \frac{(b-x)/\sqrt{D_1}}{2a/\sqrt{D_2}} \right], \quad x \in [0,b].
\]
ii) Assuming that \( \frac{a}{\sqrt{D_1}} \gg \frac{b}{\sqrt{D_2}} \).

If we assume that \( \frac{a}{\sqrt{D_1}} \) is very much larger than \( \frac{b}{\sqrt{D_2}} \) then \( u = w = \frac{a}{\sqrt{D_1}} \).

Thus (I) reduces to

\[
\overline{c}_1(x,s) = \frac{c}{s} \left[ 1 - \frac{m \sinh(\sqrt{s})}{(\xi-m)\sinh(a\sqrt{s}/\sqrt{D_1})} - \frac{\ell \sinh(-\sqrt{s})}{(\xi-m)\sinh(a\sqrt{s}/\sqrt{D_1})} \right].
\]

Thus the inverse Laplace transform of \( \overline{c}_1(x,s) \) is

\[
(3.5) \quad c_1(x,t) = \frac{c}{\pi} \left[ 1 - \frac{m}{\xi-m} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \left( \frac{\pi \sqrt{s}}{a} \right) \right] - \frac{\pi^2 n D_1 t}{\xi-m} \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \left( \frac{\pi \sqrt{s}}{a^2} \right) \]

for \(-a \leq x \leq 0\) and \( t \geq 0 \).

Under the same condition, that is \( \frac{a}{\sqrt{D_1}} \gg \frac{b}{\sqrt{D_2}} \), (II) reduces to

\[
\overline{c}_2(x,s) = \frac{c}{s} \left[ k_0 - \frac{2 \sinh((b-x)\sqrt{s}/\sqrt{D_2})}{(\xi-m)\sinh(a\sqrt{s}/\sqrt{D_1})} \right].
\]

Therefore the inverse Laplace transform of \( \overline{c}_2(x,s) \) is

\[
(3.6) \quad c_2(x,t) = \frac{c}{\pi} \left[ k_0 - \frac{2}{a(\xi-m)} \frac{\sqrt{D_1}}{D_2} - \frac{4}{\pi(\xi-m)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \left( \frac{n\pi (b-x)}{a} \frac{\sqrt{D_1}}{D_2} \right) \right], \text{ for } 0 \leq x \leq b \text{ and } t \geq 0.
\]
The steady state concentrations in this case are given by

\[(3.7) \quad c_1(x) = c' \left[ 1 - \frac{(m\gamma - \delta)\sqrt{D_1}}{a(\lambda-m)} \right], \quad \text{for} \quad -a \leq x \leq 0,\]

and

\[(3.8) \quad c_2(x) = c' \left[ k_0 - 2 \frac{b-x}{a(\lambda-m)} \sqrt{\frac{D_1}{D_2}} \right], \quad \text{for} \quad 0 \leq x \leq b.\]

(iii) Assuming that \(\frac{a}{\sqrt{D_1}} \ll \frac{b}{\sqrt{D_2}}\).

In this case \(w = -u = \frac{b}{\sqrt{D_2}}\). Thus (I) reduces to

\[c_1(x,s) = \frac{c'}{s} \left[ 1 - \frac{m \sinh(\gamma\sqrt{s})}{(\lambda+m)\sinh(b\sqrt{s}/\sqrt{D_2})} - \frac{\ell \sinh(-\delta\sqrt{s})}{(\lambda+m)\sinh(b\sqrt{s}/\sqrt{D_2})} \right].\]

Thus the inverse Laplace transform \(c_1(x,s)\) is

\[(3.9) \quad c_1(x,t) = c' \left[ 1 - \frac{(m\gamma - \ell\delta)\sqrt{D_2}}{(\lambda+m)b} \right]
- \frac{2m}{\pi(\lambda+m)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\frac{b^2}{n^2D_2t}} \sin \left( \frac{n\pi\sqrt{D_2}}{b} \right)
- \frac{2\ell}{\pi(\lambda+m)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\frac{b^2}{n^2D_2t}} \sin \left( \frac{n\pi(-\delta)\sqrt{D_2}}{b} \right),\]

for \(-a \leq x \leq 0\) and \(t > 0\).
Under the same condition, that is \( \frac{a}{\sqrt{D_1}} \ll \frac{b}{\sqrt{D_2}} \), (II) reduces to

\[
\bar{c}_2(x,s) = \frac{c^*}{s} \left[ k_0 + \frac{2 \sinh((x-b)\sqrt{s}/D_2)}{(l+m) \sinh(b\sqrt{s}/D_2)} \right].
\]

Thus the inverse Laplace transform of \( \bar{c}_2(x,s) \) is

\[
(3.10) \quad c_2(x,t) = c^* \left[ k_0 - \frac{2(b-x)}{b(l+m)} \right]
\]

\[
- \frac{4}{\pi (l+m)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\frac{2D_2 t}{b^2}} \sin \left( \frac{n\pi (b-x)}{b} \right),
\]

for \( 0 \leq x \leq b \) and \( t > 0 \).

The steady state concentrations in this case are given by

\[
(3.11) \quad c_1(x) = c^* \left[ 1 - \frac{(m - \xi \delta)\sqrt{D_2}}{(l+m)b} \right], \quad \text{for} \ -a \leq x \leq 0,
\]

and

\[
(3.12) \quad c_2(x) = c^* \left[ k_0 - \frac{2(b-x)}{b(l+m)} \right], \quad \text{for} \ 0 \leq x \leq b.
\]

4. Remarks. For a specific model in which the operators \( D(f) \) and \( I(f) \), in (1.2), are given explicitly, the validity of the steady state theory can be checked, see [11]. If we can apply the steady state theory and if \( f = F(c,v) \) is the homogeneous steady state solution for (1.4), then in the inhomogeneous case a first approximation to the distribution function is
given by \( f(x,t,v) = F(c(x,t),v) \), where \( c(x,t) \) is the local value of the concentration given by (3.1), (3.5) or (3.9) for \(-a \leq x \leq 0\), and by (3.2), (3.6) or (3.10) for \( 0 \leq x \leq b\).

Another use of the concentration functions is in determining the probability of passing, \( P(x,t) \), at a given point \( x \) on the road at a specific time \( t \). We can assume that the probability of passing is a function only of the concentration, for example,

\[
P(x,t) = 1 - \frac{c(x,t)}{c_p} \quad \text{if} \quad c(x,t) < c_p,
\]

and

\[
P(x,t) = 0 \quad \text{if} \quad c(x,t) \geq c_p,
\]

where \( c_p \) is the limiting concentration in that part of the road. See [1] for discussion of this assumption.

It is not claimed, however, that the theory presented in this paper gives all the facts related to the statistical theory of traffic flow. But it can at least be used in conjunction with the work in [1], [2], [6] - [11] to generalize their results to the inhomogeneous non-stationary case.

A generalization of the analysis given in this paper is underway where we consider the case of non-stationary flow in the system at the entrance \( b \). This is equivalent to considering different modes of arrival in the microscopic case, that is when considering the individual behavior of the different cars.
REFERENCES


