CONCENTRATION OF CARS IN A ROAD
WITH NON-STATIONARY INFLOW AND OUTFLOW

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1. **Introduction.** The main objective of the present paper is to determine the concentration of cars at each point of a stretch of highway of finite length. The concentration will be given as a function of the position in the highway and the time that elapsed from the start of the process. Cars enter that stretch of the highway from one end of it, called the entrance, and can only leave it from the other end called the exit. We assume that the flow at the entrance is non-steady and given by a function of the time t. We also assume that the flow out of the exit is non-steady and given by another function of the time t. We assume that there exists a limiting concentration, called the jam concentration, where cars can no longer pass in a part of the road when the concentration in that part reaches the jam concentration. The system starts with the exit closed and the concentration at each point of the road at its limiting value. At time t=0 the flow starts out of the system through the exit and in the system from the entrance.

Considering the traffic flow from a macroscopic point of view, the only conservation law possible states that the rate of change of the concentration with respect to time is equal to the rate of change of flow with respect to distance at any point in the road. This can be explained in the following way. Let C(x,t) represent the concentration of cars at the point x at time t, i.e., the number of cars per unit distance at the point x and at time t, and let q(x,t) be the flow of cars, i.e., the number of cars crossing the position x per unit time at time t. Then we can see that

$$[q(x+\Delta x, t) - q(x, t)]\Delta t = [C(x, t+\Delta t) - C(x, t)]\Delta x,$$

thus

$$\frac{\partial q}{\partial x} + \frac{\partial C}{\partial t} = 0 .$$
For an explanation of this equation and its consequences see M. J. Lighthill and G. B. Whitham [5], and S. C. De [4]. If $D(x,t)$ denotes the diffusion coefficient at the point $x$ and time $t$, then we can see that
\begin{equation}
q(x,t) = -D(x,t) \frac{\partial C(x,t)}{\partial x}.
\end{equation}

Thus assuming $D$ to be independent of $x$ and $t$ equation (1.1) reduces to
\begin{equation}
\frac{\partial C(x,t)}{\partial t} = D \frac{\partial^2 C(x,t)}{\partial x^2},
\end{equation}
which is the equation of diffusion. Thus we can consider cars as molecules undergoing simple diffusion with the proper boundary and initial conditions.

2. The Concentration Function. Consider a stretch of uniform highway of length $2a$. Cars can only enter that stretch from one end of it at $a$, where the concentration is given by $K_1(t)$, and cars can only leave it from the other end, at $-a$, where the concentration is given by $K_2(t)$. Let $C(x,t)$ be the concentration function for $xt[-a,a]$ at time $t$, and let $C'$ be the jam concentration for that stretch of highway. Considering cars as molecules undergoing a simple diffusion process, we can show that the concentration $C(x,t)$ is the solution of the following boundary value problem:
\begin{equation}
\frac{\partial C(x,t)}{\partial t} = D \frac{\partial^2 C(x,t)}{\partial x^2},
\end{equation}

\begin{align*}
(2.2) & \quad C(-a,t) = K_1(t), \\
(2.3) & \quad C(a,t) = K_2(t), \\
(2.4) & \quad C(x,0) = C' \quad \text{for } x \in [-a,a].
\end{align*}

Let $\tilde{f}(s)$ denote the Laplace transform of the function $f(t)$, i.e.,
\[ \tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt. \]
Then, in view of the initial condition (2.4), the Laplace transform of (2.1) is

\begin{equation}
\frac{d^2 \overline{C}(x,s)}{dx^2} - s \overline{C}(x,s) + s^{-1} = 0 ,
\end{equation}

and the Laplace transforms for (2.2) and (2.5) are given by

\begin{equation}
\overline{C}(-a,s) = \overline{K}_1(s)
\end{equation}

and

\begin{equation}
\overline{C}(a,s) = \overline{K}_2(s) .
\end{equation}

Solving the ordinary differential equation (2.5) we get

\begin{equation}
\overline{C}(x,s) = G \exp\left(\frac{s}{D} x\right) + H \exp\left(-\frac{s}{D} x\right) + \frac{c'}{s}.
\end{equation}

Using (2.6) and (2.7) in (2.8) we can see that

\begin{equation}
\overline{K}_1(s) = G \exp\left(-\frac{s}{D} a\right) + H \exp\left(\frac{s}{D} a\right) + \frac{c'}{s},
\end{equation}

and

\begin{equation}
\overline{K}_2(s) = G \exp\left(\frac{s}{D} a\right) + H \exp\left(-\frac{s}{D} a\right) + \frac{c'}{s}.
\end{equation}

Therefore

\begin{align*}
\overline{K}_1(s) + \overline{K}_2(s) &= 2G \cosh\left(\frac{s}{D} a\right) + 2H \cosh\left(\frac{s}{D} a\right) + 2\frac{c'}{s} \\
&= 2(G + H) \cosh\left(\frac{s}{D} a\right) + 2\frac{c'}{s},
\end{align*}

and

\begin{align*}
\overline{K}_1(s) - \overline{K}_2(s) &= -2G \sinh\left(\frac{s}{D} a\right) + 2H \sinh\left(\frac{s}{D} a\right) \\
&= 2(H - G) \sinh\left(\frac{s}{D} a\right) .
\end{align*}
Therefore, solving for $H$ and $G$, we get

$$H = \left[ \bar{K}_1(s) - \bar{K}_2(s) \right]/4 \sinh\left( \sqrt{\frac{s}{D}} \right) a$$

$$+ \left[ \bar{K}_1(s) + \bar{K}_2(s) \right]/4 \cosh\left( \sqrt{\frac{s}{D}} \right) a$$

$$- C\sqrt{2s} \cosh\left( \sqrt{\frac{s}{D}} \right) a,$$

and

$$G = \left[ \bar{K}_1(s) + \bar{K}_2(s) \right]/4 \cosh\left( \sqrt{\frac{s}{D}} \right) a - \left[ \bar{K}_1(s) - \bar{K}_2(s) \right]/4 \sinh\left( \sqrt{\frac{s}{D}} \right) a$$

$$- C\sqrt{2s} \cosh\left( \sqrt{\frac{s}{D}} \right) a.$$  

Therefore

$$(2.11) \quad \bar{C}(x,s) = \left[ \bar{K}_1(s) + \bar{K}_2(s) \right] \cosh\left( \sqrt{\frac{s}{D}} \right) x/2 \cosh\left( \sqrt{\frac{s}{D}} \right) a$$

$$- \left[ \bar{K}_1(s) - \bar{K}_2(s) \right] \sinh\left( \sqrt{\frac{s}{D}} \right) x/2 \sinh\left( \sqrt{\frac{s}{D}} \right) a$$

$$- C^\prime \cosh\left( \sqrt{\frac{s}{D}} \right) x/s \cosh\left( \sqrt{\frac{s}{D}} \right) a + C^\prime$$

$$= \frac{\bar{K}_1(s)}{2} \left[ \cosh\left( \sqrt{\frac{s}{D}} \right) x/ \cosh\left( \sqrt{\frac{s}{D}} \right) a \right]$$

$$- \sinh\left( \sqrt{\frac{s}{D}} \right) x/ \sinh\left( \sqrt{\frac{s}{D}} \right) a$$

$$+ \frac{\bar{K}_2(s)}{2} \left[ \cosh\left( \sqrt{\frac{s}{D}} \right) x/ \cosh\left( \sqrt{\frac{s}{D}} \right) a \right]$$

$$- \sinh\left( \sqrt{\frac{s}{D}} \right) x/ \sinh\left( \sqrt{\frac{s}{D}} \right) a$$

$$- C^\prime \cosh\left( \sqrt{\frac{s}{D}} \right) x/s \cosh\left( \sqrt{\frac{s}{D}} \right) a + C^\prime.$$
Since the Laplace transform of
\[
\frac{k}{2\sqrt{\pi}t^3} \exp\left(\frac{k^2}{4t}\right)
\]
is given by \(\exp - k\sqrt{s}\), then, using the convolution, we find that the
inverse Laplace transform for \(\bar{K}_1(s) \exp - k\sqrt{s}\) is given by
\[
\frac{1}{2\sqrt{\pi}} \int_0^t K_1(t-y) \frac{k}{y^3} \exp\left(-\frac{k^2}{4y^2}\right)dy
\]
\[
= \frac{2}{\sqrt{\pi}} \int_{k/2\sqrt{t}}^{\infty} K_1(t - \frac{k^2}{4y^2}) \exp(-y^2)dy
\].

Now since
\[
\sinh(y\sqrt{s})/\sinh(z\sqrt{s}) = \sum_{n=0}^{\infty} \{\exp - [z(2n+1) - y]\sqrt{s}
- \exp - [z(2n+1) + y]\sqrt{s}\}
\]
and
\[
\cosh(y\sqrt{s})/\cosh(z\sqrt{s}) = \sum_{n=0}^{\infty} (-1)^n \{\exp - [z(2n+1) - y]\sqrt{s}
+ \exp - [z(2n+1) + y]\sqrt{s}\}
\]
then the inverse Laplace transform for
\[
\bar{K}_1(s) \cosh\left(\frac{K}{\sqrt{\Delta}}\right)s) / \cosh\left(\frac{\alpha}{\sqrt{\Delta}}\right)s)
\]
is given by
\[
\frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \left\{ \int_{\ell/2\sqrt{t}}^{\infty} K_1(t - \frac{\ell^2}{4y^2}) \exp(-y^2)dy
+ \int_{m/2\sqrt{t}}^{\infty} K_1(t - \frac{m^2}{4y^2}) \exp(-y^2)dy \right\},
\]
where
\[ \ell = \frac{a}{\sqrt{D}}(2n+1) - \frac{x}{\sqrt{D}}, \]
and
\[ m = \frac{a}{\sqrt{D}}(2n+1) + \frac{x}{\sqrt{D}}. \]

Also, the inverse Laplace transform for
\[ \bar{K}_1(s) \sinh\left(\frac{x}{\sqrt{D}}\sqrt{s}\right) / \sinh\left(\frac{a}{\sqrt{D}}\sqrt{s}\right) \]
is given by
\[ \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left\{ \int_{\ell/2\sqrt{t}}^{\infty} \bar{K}_1(t) \frac{e^{-\frac{t^2}{4y^2}}}{4y^2} \exp(-y^2)dy - \int_{m/2\sqrt{t}}^{\infty} \bar{K}_1(t) \frac{e^{-\frac{t^2}{4y^2}}}{4y^2} \exp(-y^2)dy \right\}, \]
where \( \ell \) and \( m \) are as given above.

Since the Laplace transform for \( \text{Erfc}(\frac{y}{2\sqrt{t}}) \) is \( \frac{1}{s} \exp(-\sqrt{s}) \),
where
\[ \text{Erfc}(x) = 1 - \text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-y^2}dy, \]
then the inverse Laplace transform for
\[ \cosh\left(\frac{x}{\sqrt{D}}\sqrt{s}\right) / s \cosh\left(\frac{a}{\sqrt{D}}\sqrt{s}\right) \]
is given by
\[ \sum_{n=0}^{\infty} (-1)^n \left\{ \text{Erfc}\left(\frac{2n+1-a-x}{2\sqrt{Dt}}\right) + \text{Erfc}\left(\frac{2n+1+x}{2\sqrt{Dt}}\right) \right\}, \]
\[ = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \left\{ \int_{\ell/2\sqrt{t}}^{\infty} \exp(-y^2)dy + \int_{m/2\sqrt{t}}^{\infty} \exp(-y^2)dy \right\}, \]
where \( \ell \) and \( m \) are as given above.
Thus, using the above results, we can see that the inverse Laplace transform for $\tilde{C}(x,s)$, given by (2.11), is

$$C(x,t) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \left\{ \int_{\ell/2\sqrt{t}}^{\infty} [K_1(t-\frac{\ell^2}{4y^2}) + K_2(t-\frac{\ell^2}{4y^2})] \exp(-y^2) dy \right\}$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left[ \int_{m/2\sqrt{t}}^{\infty} [K_1(t-\frac{m^2}{4y^2}) + K_2(t-\frac{m^2}{4y^2})] \exp(-y^2) dy \right]$$

$$- \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left[ \int_{\ell/2\sqrt{t}}^{\infty} [K_1(t-\frac{\ell^2}{4y^2}) + K_2(t-\frac{\ell^2}{4y^2})] \exp(-y^2) dy \right]$$

$$- \frac{2C}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \left\{ \int_{\ell/2\sqrt{t}}^{\infty} \exp(-y^2) dy + \int_{m/2\sqrt{t}}^{\infty} \exp(-y^2) dy \right\} + C' .$$

Therefore

$$C(x,t) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^{2n+1} \left\{ \int_{\ell/2\sqrt{t}}^{\infty} [K_1(t-\frac{\ell^2}{4y^2}) + K_2(t-\frac{\ell^2}{4y^2})] \exp(-y^2) dy \right\}$$

$$+ \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[ \int_{j/2\sqrt{t}}^{\infty} [K_1(t-\frac{j^2}{4y^2}) + K_2(t-\frac{j^2}{4y^2})] \exp(-y^2) dy \right]$$

$$- \frac{2C}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \left\{ \int_{\ell/2\sqrt{t}}^{\infty} \exp(-y^2) dy + \int_{m/2\sqrt{t}}^{\infty} \exp(-y^2) dy \right\} + C' ,$$

where

$$i = \frac{a}{\sqrt{D}} (4n+3) - \frac{x}{\sqrt{D}} ,$$

$$j = \frac{a}{\sqrt{D}} (4n+1) + \frac{x}{\sqrt{D}} ,$$

$$\ell = \frac{a}{\sqrt{D}} (2n+1) - \frac{x}{\sqrt{D}}$$

and

$$m = \frac{a}{\sqrt{D}} (2n+1) + \frac{x}{\sqrt{D}} .$$
3. **Remarks.** The results presented in this paper give, for the macroscopic models, the analogous conditions to the non-stationary rate of arrivals, and departures, in the microscopic models of traffic flow. Also, depending on the functional forms of the functions $K_1$ and $K_2$, different modes of arrivals can be considered. From equation (1.1) we can determine the flow function $q(x,t)$, and, at least in principle, can write the functional relation between $q$ and $C$, $q(x,t) = f(C(x,t))$ say, which was assumed in [4] and [5]. The results given in this paper, besides being of independent interest, shall be incorporated with the ideas developed in [1], [2], [6]–[12] to find the speed distribution function of cars at each point in the highway considered and at each time $t$. 
REFERENCES


