SEQUENTIAL ONE-SAMPLE GROUPED RANK TESTS

by

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on

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This manuscript has been prepared for publication in the Bulletin of the International Statistical Institute in association with the presentation of this material at the 37th Session of the International Statistical Institute, London, September 3-11, 1969. This paper should be regarded as a preliminary report on this research and more detailed technical reports are in preparation.

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Consider the hypothesis that the distribution of a random
variable $Z$ is symmetric about the origin, $H_0: F(z) + F(-z) = 1$ for all $z$,
where $F(z)$ is an absolutely continuous c.d.f. with p.d.f. $f(z)$, the
functional form of $f(z)$ being unknown. For fixed $N$, let $Z_1, \ldots, Z_N$ be $N$
independent observations on $Z$. Wilcoxon (1945) proposed the signed
rank test for this situation with the intended alternative to $H_0$ being
a location change on the point of symmetry.

In this paper, we propose sequential extensions of the test with
observations taken in groups of size $N$ with the decision to take an
additional group following the rules of Wald (1947). To develop
sequential extensions, it is necessary to specify a class of alternatives
to $H_0$. We discuss this in terms of one group of size $N$.

Lehmann (1953) introduced a special class of alternatives for
rank tests. Let $Y_1, \ldots, Y_n$ be $n$ positive $Z$'s and let $X_1, \ldots, X_m$
be the absolute values of $m$ negative $Z$'s, $m + n = N$. The $Y$'s have the conditional
c.d.f. $F_+(y) = P(Z < y | Z > 0)$ and the $X$'s have the conditional c.d.f.

\[ F_-(x) = P(-Z < x | Z < 0); F(z) = F(0)[1-F_-(z)] + [1-F(0)]F_+(z), \quad -\infty < z < \infty. \]

In this problem, we rewrite $H_0: F_+(u) \equiv F_-(u)$, $F(0) = 1/2$ and consider
alternatives $H$: $F_+(u) = h[F_-(u)]$, $F(0) = \rho$, $0 < \rho < 1$, where $h$ is a
differentiable, nondecreasing function on $[0,1]$, $h(0) = 0$, $h(1) = 1$.

The alternative class $h_a: F_+(u) \equiv 1 - [1-F_-(u)]^k$, $k > 0$, is chosen and
discussed in detail by Weed (1968); here $h(u) = 1 - (1-u)^k$, $H_0$
results when $k = 1$, and $F(0) = k/(k+1) = \rho$, the latter resulting from a
requirement that $\lim_{u \rightarrow 0^+} h'[F_-(u)] = \rho/(1-\rho)$ to avoid a jump discontinuity
in $f(z)$ at the origin. The alternative $H_a$ provides a location change
when $k \neq 1$; it also introduces a change in shape of the distribution of
Z involving skewness. By specifying \( k < 1 \) or \( k > 1 \), we can determine
the location shift and its magnitude.

Let \( S_1 < \ldots < S_n \) be the ordered ranks of \( Y_1, \ldots, Y_n \) in the joint
ranking of \( |Z_1|, \ldots, |Z_N| \). Lehmann demonstrated that the probability
distribution of \( S_1, \ldots, S_n \), conditional on \( n \), depends only on \( h \). Under \( H_a \)
and observed rank configuration \( s_1, \ldots, s_n \), we find that

\[
P(s_1, \ldots, s_n; n|N,k) = \left[ k/(k+1) \right]^N \frac{\Gamma(s_{n+1} - s_{j+1}) \Gamma(s_{n+1} - s_{j-1})}{\prod_{j=1}^{n} \Gamma(s_{n+1} - s_{j+1}) \Gamma(s_{n+1} - s_{j-1})}
\]

(1)

where \( s_{n+1} = N+1 \), \( s_0 = 0 \), and \( a_j = (n-j+1)(k-1) \) for given \( N \) and \( k > 0 \).
The form (1) is valid for a specific observed occurrence of \( n \) positive
Z's and for a specific observed allocation of the ranks \( s_1, \ldots, s_n \) to them.

We propose two, one-sample, sequential, grouped rank tests, the
sequential rank configuration test and the sequential signed rank test,
both comparing \( H_0 \) with \( H_a \) for \( k = k_1 \) specified. Groups of \( N \)
independent observations on \( Z \) are obtained sequentially, \( N \geq 2 \). For
group \( \gamma \), we observe \( s_{1,\gamma}, \ldots, s_{n,\gamma} \) corresponding with \( s_1, \ldots, s_n \) above
and compute the probability ratio based on (1),

\[
k_\gamma(N,k_1,1) = 2^N P(s_{1,\gamma}, \ldots, s_{n,\gamma}; n|N,k_1),
\]

(2)

where \( k_1 > 0 \), \( k_1 \neq 1 \). After \( t \) groups of size \( N \) have been observed, the
complete probability ratio, \( p_{1t}/p_{0t} = \prod_{\gamma=1}^{t} k_\gamma(N,k_1,1) \), is formed. For
given \( \alpha \) and \( \beta \), specified probabilities of Type I and Type II errors
respectively, Wald's decision procedure for sequential probability
ratio tests is applied to \( p_{1t}/p_{0t} \) at each stage \( t = 1,2, \ldots \) until termi-
tion occurs. This is the sequential rank configuration test. The sequential signed rank test also follows from (1). The test is based on the rank sum \( W = \sum_{i=1}^{n} S_i \) where \( W \) is equivalent to Wilcoxon's (1945) signed rank statistic. For each group \( \gamma \), we observe \( w_\gamma \), a value of \( W \), and form the probability ratio,

\[
L_\gamma(N, k_\perp, 1) = P(w_\gamma | N, k_\perp) / P(w_\gamma | N, 1).
\]  

(3)

\( P(w_\gamma | N, k) \) is obtained from the appropriate sum of probabilities (1). The complete probability ratio after \( t \) groups is \( P_{1t} / P_{0t} = \prod_{\gamma=1}^{t} L_\gamma(N, k_\perp, 1) \) and this is again used in Wald's procedure.

Characteristics of these sequential, grouped rank tests have been studied extensively by Weed (1968). We can only regard this paper as a preliminary report and note that additional details will be given in subsequent papers now in preparation. An algorithm has been developed for the computation of (1) similar to that of Wilcoxon, Rhodes and Bradley (1963). Tables have been prepared giving the logarithm of \( L_\gamma \) in (3) for \( N = 2(1)10 \) and \( k_\perp = 2(1)5 \). The tables can be used also for \( k_\perp = 1/2, 1/3, 1/4, 1/5 \) through a symmetry property and for a proposed two-sided, sequential, signed rank test. Examples have been developed. The forms of special functions \( f(z) \) based on normal, logistic, and double exponential c.d.f's have been studied for various values of \( k \) and means and standard deviations computed. The OC- and ASN-functions of the two tests have been examined as obtained from approximations developed by Wald. The properties of the two tests were similar with the ASN-values of the signed rank test being slightly higher than
corresponding values of the rank configuration test. Extensive Monte Carlo studies of the sequential signed rank test were conducted under two data generation sources, models L and φ. Under the L-model, observations were generated with c.d.f. of the class $H_a$ indexed by $k$. Under the φ-model, observations were generated from a normal population with positive mean and unit variance. The mean of the normal was associated with a value of $k$ in an appropriate way. An examination of the Monte Carlo results shows that group size $N$ had a slight effect on the power of the test. For $k$ near $k_1$, the power values under the larger group sizes were greater than the corresponding values when $N$ was small. For $k$ near unity, the power values were smaller for the larger values of $N$.

It was concluded that, for large $N$, the complete probability ratio often significantly exceeded the boundaries of the decision procedure upon termination, thus providing more protection against an incorrect decision than was specified. A slight loss in power occurred under the φ-model relative to the L-model when $k$ was near $k_1$. In general, the power curves under the two models showed close agreement indicating robustness of the procedure for use with normal location-change alternatives of classical statistics. There was also close agreement between the ASN-values of the L- and φ-models. The Monte Carlo studies suggested that the optimum group size is between $N = 4$ and $N = 7$. A truncated sequential signed rank test was studied as were modified tests involving reranking of all observations as each new group of observations was obtained.

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Sommaire

Des observations independentes sont prises en groupes de taille $N$ progressivement. Une hypothèse de symétrie sur l'origine pour la

distribution, $H_0 : F(z) + F(-z) = 1$ pour tous les $z$, est comparée avec

une hypothèse alternative, $H_a : F_+(u) = 1 - [1 - F_-(u)]^k_1$, $k_1 > 0$, $k_1 \neq 1$,

où $F(z)$ est la distribution cumulative de population et $F_+(u) = P(Z\leq u | Z>0)$,

$F_-(u) = P(-Z\leq u | Z<0)$, $F(z) = F(0)[1-F_-(z)] + [1-F(0)]F_+(z)$, $-\infty < z < \infty$,

$Z$ étant le variable stochastique engendrent les observations. La méthode

progressive de Wald est développée pour deux tests dependant sur les rangs
des observations positives, $s_{1,\gamma}, \ldots, s_{n,\gamma}$, à groupe $\gamma$, $\gamma = 1,2,\ldots$.

Nous employons les statistiques du rapport des probabilités, une dependante

en $w_\gamma = \sum_{j=1}^n s_{j,\gamma}$ seulement. Le test dependant en $w_\gamma$ est une extension
du test de Wilcoxon de rangs signés (1945).

References


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13. Abstract
Independent observations are taken in groups of size \( N \) sequentially. An hypothesis of symmetry about the origin for the distribution is compared with an alternative of the Lehmann class. Wald's sequential analysis is used to develop two sequential rank tests, one of which is a sequential extension of Wilcoxon's signed rank test.

14. Key Words

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Signed Rank Test
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