SEQUENTIAL ONE-SAMPLE RANK TESTS

by

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Introduction and Summary

This paper summarizes more detailed reports by Weed and Bradley (1969a), Weed, Bradley and Govindarajulu (1969), and Weed and Bradley (1969b). Sequential, one-sample, rank tests have been developed with the referenced papers dealing with basic methodology, modified tests, and Monte Carlo studies, respectively. Emphases in this paper are placed on computational aspects of the work as deemed appropriate to the purposes of the conference.

The Wilcoxon (1945) signed rank method is reviewed. Sequential methods are developed for observations taken in groups of size N, the groups taken sequentially. Within-group, signed ranks are used and a model developed that permits calculation of group probability ratios for two procedures, one based on the within-group configurations of signed ranks and one based on the within-group rank sums. Modified tests with observations taken singly are also described when new sets of signed ranks are obtained as each new observation is taken. Wald's method of sequential analysis is used.

1Research supported in part by the Army, Navy and Air Force under an Office of Naval Research Contract and by the National Institute of General Medical Sciences under a training grant. The assistance of National Science Foundation grants to the Florida State University Computing Center is acknowledged. Reproduction in whole or in part is permitted for any purpose of the United States Government.
Monte Carlo studies of the sequential grouped procedures have been conducted. Calculations are described and examples of results given.

**Basic Methodology**

Two basic sequential one-sample ranking procedures are considered. The first involves observations taken in groups of size \( N \) with signed ranks obtained within groups and the groups taken sequentially. The second has observations taken sequentially one by one with signed ranks obtained at each step for the totality of observations available. The grouped procedure has greater practical utility while the ungrouped or modified procedure has much theoretical interest.

In review, Wilcoxon (1945) presented the signed ranks test for symmetry for the one-sample problem. Let \( Z_1, \ldots, Z_N \) be independent random variables with pdf \( f \) and cdf \( F \). The null hypothesis \( H_0 \) is that \( f \) is symmetric about the origin (without loss of generality). This hypothesis is particularly appropriate when \( Z_1, \ldots, Z_N \) are the differences of paired observations. The usual alternative to \( H_0 \) is that \( f \) is symmetric about some point other than the origin, a shift in location. We modify the alternative for convenience in sequential tests. In applications, Wilcoxon ranked \( |Z_1|, \ldots, |Z_N| \) and retained the signs of \( Z_1, \ldots, Z_N \) with the resulting absolute ranks. Let \( R_1, \ldots, R_m \) and \( S_1, \ldots, S_n \) be the ordered ranks of the absolute negative and positive \( Z \)'s, respectively. Note that \( m \) is the number of negative \( Z \)'s and \( n \), the number of positive \( Z \)'s respectively, \( m \) and \( n \) are random variables, \( m + n = N \). Wilcoxon used the smaller of the rank sums, \( V = \sum_{i=1}^{m} R_i \) and \( W = \sum_{j=1}^{n} S_j \), and presented a table of critical values of the statistic for tests of significance.
The hypothesis of symmetry about the origin may be expressed as

\[ H_0: \ F(z) + F(-z) = 1, \text{ or} \]
\[ f(z) = f(-z), \text{ for all } z. \]  

(1)

We choose a Lehmann alternative to (1). Let

\[ F_+(z) = P(Z \leq z | Z \geq 0), \]
\[ F_-(z) = P(-Z \leq z | Z < 0), \ z \geq 0. \]  

(2)

Lehmann (1953) took \( F_+ = h(F_-) \) where \( h \) is a continuous, differentiable, strictly increasing cdf on \([0,1]\). \( H_0 \) is achieved when \( h(u) = u \), \( \lambda = F(0) = 1/2 \), and other choices of \( h \) lead to departures from \( H_0 \). We choose

\[ h(u) = 1 - (1-u)^k, \quad \lambda = k/(k+1), \quad k > 0 \]  

(3)

and give details in Weed and Bradley (1969a) and Weed (1968). In particular, some choices of \( h \) lead to a jump discontinuity in \( f \) at the origin and the second equality of (3) prevents this. We use as a model \( F \) such that

\[ F(0) = \lambda = k/(k+1), \]
\[ F_+(z) = 1 - [1-F_-(z)]^k, \ z \geq 0, \]
\[ F(z) = \lambda[1-F_-(z)], \ z < 0, \]
\[ = \lambda + (1-\lambda)F_+(z), \ z \geq 0. \]  

(4)

Given (4), we may restate (1) as

\[ H_0: \ k = 1 \]  

(5)
and take the alternative as

$$H_a: \ k = k_1, \ k_1 > 1. \quad (6)$$

We consider only the one-sided test here but note that the two-sided test is given in the references. The nature of (6) is discussed in the references also and Weed and Bradley (1969a), Table 1, give the means $\mu(k)$ and standard deviations $\sigma(k)$ of $Z$ in a "normal" example\(^2\). These are reproduced here in Table 1. Note that $k > 1$ leads to a negative $\mu(k)$; the symmetrically opposite case occurs with parameter $k^* = 1/k$.

Calculations in Table 1 were effected by means of numerical integration. Through use of a computer, Table 1 provides an indication of the role of $k$; further insight results from the result that $P(X<Y) = 1/(k+1)$ where $X$ is the absolute value of a negative $Z$ and $Y$ the value of a positive $Z$, $X,Y$ independent.

**Table 1. Means and Standard Deviations for $F$ when $F_- = \Phi_-$**

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>.500</th>
<th>.556</th>
<th>.600</th>
<th>.667</th>
<th>.714</th>
<th>.750</th>
<th>.777</th>
<th>.800</th>
<th>.833</th>
<th>.857</th>
<th>.900</th>
</tr>
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<tr>
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<td>1.25</td>
<td>1.5</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
<td>3.5</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>$\mu(k)$</td>
<td>0</td>
<td>-.143</td>
<td>-.244</td>
<td>-.376</td>
<td>-.459</td>
<td>-.515</td>
<td>-.555</td>
<td>-.590</td>
<td>-.629</td>
<td>-.658</td>
<td>-.705</td>
</tr>
<tr>
<td>$\sigma(k)$</td>
<td>1</td>
<td>.927</td>
<td>.874</td>
<td>.804</td>
<td>.760</td>
<td>.730</td>
<td>.709</td>
<td>.690</td>
<td>.672</td>
<td>.658</td>
<td>.637</td>
</tr>
</tbody>
</table>

We consider sequential probability ratio tests. The model (4) permits calculation of the required probability ratios. For the grouped sequential tests, we need

\(^2\)When $k = 1$, take $F = \phi$, $\phi$ being the standard normal cdf. When $k > 1$, let $F_- = \Phi$ and $F_+ = 1 - (1-\phi)^k$, $F(0) = k/(k+1)$. $\phi_-$ for $\Phi$ corresponds with $F_-$ for $F$. 

\begin{equation}
P(n,s_1, \ldots, s_n | k, N) = \left( \frac{k}{k+1} \right)^N \prod_{j=1}^n \frac{\Gamma(s_{n+1} - s_{j+1} + \alpha_j)}{\Gamma(s_{n+1} - s_j - 1 + \alpha_j)} \frac{\Gamma(s_{n+1} - s_{j+1} + \alpha_j)}{\Gamma(s_{n+1} - s_j - 1 + \alpha_j)} \frac{\Gamma(s_{n+1} - s_{j+1} + \alpha_j)}{\Gamma(s_{n+1} - s_j - 1 + \alpha_j)} \end{equation}

where \( \alpha_j = (n-j+1)(k-1) \), \( s_0 = 0 \), \( s_{k+1} = N+1 \), \( s_1, \ldots, s_k \) being a realization of \( S_1, \ldots, S_n \), the ranks of the positive observations and \( n \) the number of positive observations, \( n \) a random variable with binomial distribution with parameters \( N \) and \( 1 - \lambda = 1 - F(0) \). The probability in (7) is of the joint occurrence of \( n \) and \( s_1, \ldots, s_n \) for a sample size \( N \), model (4), and parameter \( k \). Derivation of (7) is given by Weed (1968) and Weed and Bradley (1969a); alternative forms are also established in these references. The form useful for computations has

\begin{equation}
P(n,s_1, \ldots, s_n | k, N) = N! \left( k/(k+1) \right)^N \prod_{i=1}^N \left[ \left( \sum_{j=N+1-i}^{N} \delta_j \right) k^+ \right] \end{equation}

where \( \delta_i = 0 \) if the \( i \)th ordered \( |Z| \) is for a \( Z > 0 \) and \( \delta_i = 1 \) otherwise. The probability ratio for testing \( H_0 \) in (5) versus \( H_a \) in (6) for the group of \( N \) independent observations is

\begin{equation}
l(N,k_i,1) = 2^N \left( \frac{k_i}{k_i+1} \right)^N \prod_{j=1}^n \frac{\Gamma(N+1-s_j + \alpha_j)}{\Gamma(N+1-s_j - 1 + \alpha_j)} \frac{\Gamma(N+1-s_j + \alpha_j)}{\Gamma(N+1-s_j - 1 + \alpha_j)} \frac{\Gamma(N+1-s_j + \alpha_j)}{\Gamma(N+1-s_j - 1 + \alpha_j)} \end{equation}

where the notation is as before except that \( \alpha_j = (n-j+1)(k_i-1) \). When independent groups of observations are taken sequentially, the form (8) applies to each group; for the \( \gamma \)th group, the probability ratio is \( l(N,k_i,1) \) and this is given by (8) with \( n, \alpha_j, s_j \) replaced by \( n, \alpha_j, s_j \). For the sequential test at the end of \( t \) groups, the complete probability ratio is

\[ P_{lt}/P_{0t} = \prod_{\gamma=1}^{t} l(N,k_i,1), \quad t = 1,2, \ldots \]
We designate this sequential test as the sgrc-test, the sequential, one-sample, grouped rank configuration test. A second test may be devised based on \( W_\gamma \), the rank sum \( W \) for the \( \gamma \)th group. Note that

\[
P(W_\gamma = w | N, k) = P(w | N, k) = \sum_S P(n_\gamma, s_{1\gamma}, \ldots, s_{n_\gamma} | N, k)
\]  

(9)

where the sum over \( S \) is over those values of \( n_\gamma, s_{1\gamma}, \ldots, s_{n_\gamma} \) such that

\[
\sum_{i=1}^{n_\gamma} s_{i\gamma} = w.
\]

The probability ratio for the \( \gamma \)th group is

\[
L_\gamma (N, k_1, l) = P(w_\gamma | N, k_1) / P(w_\gamma | N, l)
\]

and the complete probability ratio after \( t \) groups is

\[
P_{lt} / P_{0t} = \prod_{\gamma=1}^{t} L_\gamma (N, k_1, l).
\]  

(10)

We designate this second test as the sgsr-test, the sequential grouped signed ranks test. Definitions of the tests are complete when we note that we use the method of Wald (1947): Let \( A = (1-\beta)/\alpha \) and \( B = \beta/(1-\alpha) \) when \( \alpha, \beta \) are probabilities of Type I and Type II errors. Terminate the tests with acceptance of \( H_a \) when \( P_{lt} / P_{0t} > A \) or \( P_{lt} / P_{0t} > A \) and with acceptance of \( H_0 \) when \( P_{lt} / P_{0t} < B \) or \( P_{lt} / P_{0t} < B \). Take another group of observations if \( B < P_{lt} / P_{0t} < A \) or \( B < P_{lt} / P_{0t} < A \).

Ungrouped modified tests are possible. Consider observations \( Z_1, Z_2, \ldots \) taken sequentially. At each stage \( t \) signed ranks \( s_{1t}, \ldots, s_{tt} \) are obtained and the probability ratio (8) is computed:

\[
\xi_t(t, k_1, l) = 2^t \binom{k_1}{k_1 - t} \sum_{j=1}^{n_t} \Gamma(t+1-s_{jt}+\alpha_{jt}) \Gamma(t+1-s_{jt}-\alpha_{jt}) \Gamma(t+1-s_{jt}+\alpha_{jt}) \Gamma(t+1-s_{jt}-\alpha_{jt}),
\]  

(11)
\[ \bar{a}_{jt} = (n_t-j+1)(k_1-l). \] The boundaries A and B are used with the statistic \( l_t(N,k_1,l) \) with decision rules as before. We call this the smrc-test, the sequential, modified, rank configuration test. A sequential, modified, signed rank test, smsr-test, may be obtained in the same way. It is based on

\[ L_t(t,k_1,l) = P(w_t|t,k_1)/P(w_t|t,1) \]

with

\[ P(w_t|t,k) = \sum_{S_t} P(n_t,s_{1t},\ldots,s_{tt}|t,k), \]

the sum over \( S_t \) being over those values of \( n_t, s_{1t}, \ldots, s_{tt} \) such that

\[ \sum_{i=1}^{n_t} s_{ih} = w_t. \]

Tables of values of \( \ln L_y(N,k_1,l) \) in (10) are given by Weed and Bradley (1969a) for \( N = 2(1)10, k_1 = 1(1)5 \). They also give ASN-values for the sgar- and sgrc-tests with \( \alpha = \beta = 0.05 \); these are reproduced here in Table 2. The middle value of \( k, k' \) in Table 2, requires definition. We follow Wald (1947, p. 176): Let \( \tau = \ln l_y(N,k_1,l) \) or \( \ln L_y(N,k_1,l) \) depending on which test is considered. Then \( k' \) is that value of \( k \) such that \( E_k(\tau) = 0 \) where, for any \( k > 0 \),

\[ E_k(\tau) = \sum \tau P(n,s_1,\ldots,s_n|N,k), \]

the sum being over all possible values of \( n \) and \( s_1,\ldots,s_n \) for fixed \( N \). It was necessary to use numerical methods to obtain \( k' \) for different group sizes \( N \) and alternatives \( k_1 \). It was found that \( k' \) depends on \( k_1 \) but very little if at all on \( N \): values of \( k' \) for the two tests are nearly
Table 2. ASN-values for the sgsr-test and the sgrc-test with $\alpha = \beta = .05$.

<table>
<thead>
<tr>
<th>Test $k = 1$ against $k = k_1 &gt; 1$</th>
<th>$N$</th>
<th>$k=0$</th>
<th>k=1</th>
<th>$k=k'$</th>
<th>$k=k_1$</th>
<th>$k=\infty$</th>
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<tr>
<td></td>
<td></td>
<td>both</td>
<td>sgsr</td>
<td>sgss</td>
<td>sgsr</td>
<td>sgss</td>
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<tr>
<td>$k_1=2$ (k'=1.41)</td>
<td>2</td>
<td>7.26</td>
<td>36.00</td>
<td>36.00</td>
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<td></td>
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<td>30.77</td>
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<td>27.55</td>
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<td>49.47</td>
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<td>11.84</td>
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<td>5.82</td>
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<td>10.38</td>
</tr>
</tbody>
</table>
identical. Let $L(k)$ be the OC-function, the probability of accepting $H_0$ when the population is indexed by $k$. $L(k)$ has the approximate values

$$1, (1-\alpha), \frac{\ln((1-\beta)/\alpha)}{\ln((1-\beta)/\alpha) - \ln(\beta/(1-\alpha))}, \beta, 0$$

for $k = 0, 1, k', k_1, \infty$.

**Asymptotic Properties**

The sgrc- and sgsr-tests terminate with probability one since they are sequential probability ratio tests with groups of observations and hence sequences of statistics $L_\gamma(N,k_1,l)$ and $L_\gamma(N,k_1,l)$ that are independent. The basic results of Wald on termination apply.

Weed, Bradley and Govindarajulu (1969) have considered termination of the smrc-test. Their results are summarized in the following theorem taken from that reference with necessary adjustments in notation.

**Theorem 1.** Suppose $S_\lambda(k_1,F_-,F_+)$ $\neq 0$ where

$$S_\lambda(k_1,F_-,F_+) = \ln2 - 1 + \ln(k_1/(k_1+1)) - B_\lambda(k_1,F_-,F_+)$$

and

$$B_\lambda(k_1,F_-,F_+) = \int_0^\infty \ln[\lambda(1-F_-(z)) + k_1(1-\lambda)(1-F_+(z))]d(\lambda F_-(z)+(1-\lambda)F_+(z)).$$

Let $T$ denote the number of stages before termination of the smrc-test.

Then

(i) $P(T>\tau) < \rho^\tau$ for sufficiently large $\tau$ and some $0 \leq \rho < 1$,

(ii) $P(T=\infty) = 1$,

(iii) $E(e^{\theta t}) < \infty$ for $\theta$ in some interval $(-\infty, \delta)$, $\delta > 0$. 
With the establishment of Theorem 1, validity of the bounds A and B for the smrc-test is established together with corresponding error probabilities $\alpha$ and $\beta$.

Work is in progress on termination of the smsr-test. This test is considerably more difficult to deal with and termination has not yet been established.

The authors are indebted to Savage and Sethuraman (1966) for results leading to the proof of Theorem 1 follow closely their work for two-sample procedures.

**Monte Carlo Studies**

Examination of properties of sequential tests may often be effected through Monte Carlo studies and use of a computer. Examples are the work of Bradley, Martin and Wilcoxon (1965) and of Bradley, Merchant and Wilcoxon (1966). The sgsr- and sgrc-tests have been so investigated and the results reported by Weed and Bradley (1969b). In this paper we give only an illustrative example.

Consider the sgsr-test. We consider two situations, data generated in accordance with the model (4), the L-model, and data generated from normal populations with standardized means chosen to match selected values of $k$, the $\Phi$-model.

We examine procedures for the L-model. We choose $\alpha = \beta = .05$, $N = 4$, $k_{1}^* = 1/4$ (to obtain positive location) and for simplified tabling we take $k_{1}^* = 1/k_{1} = 4$, $k^* = 1/k$. Studies are done for populations with $k^* = 1, 1.25, 1.5, 2, 2.5, 3, 3.5, 4, 5, 6$. Uniform random numbers $\omega$ on [0,1] are generated in the computer using subroutine RN1 of the **CDC Statistical Subroutines Reference Manual** (1966). The
uniform random numbers were converted to random observations from a
CDF \( F(z,k^*) \) such that \( F_+(z,k^*) = 1 - [1 - F_-(z,k^*)]^k \), \( F(0,k^*) = k/(k + 1) \)
or \( F_-(z,k^*) = 1 - [1 - F_+(z,k^*)]^k \), \( F(0,k^*) = 1/(k + 1) \), \( k^* > 1 \). Any
\( F(z,k^*) \) with the required properties that is convenient may be used;
we took \( F(z,k^*) = (1+z)^k/(1+k) \), \( z \leq 0 \), and \( F(z,k^*) = (1+k z)/(1+k) \),
\( z > 0, z \in [-1,1] \), \( F(0,k^*) = 1/(1+k) \). Then \( F_+(z,k^*) = z \), \( F_-(z,k^*) = 1 - (1-z)^k \),
z \( \in [0,1] \). Take two uniform random numbers \( u_1, u_2 \) and generate a positive
or negative observation \( z \) as \( u_1 \) exceeds or does not exceed \( F(0,k^*) \). If
\( u_1 > F(0,k^*) \), set \( z = u_2 \); if \( u_1 \leq F(0,k^*) \), set \( u_2 = F_-(|z|,k^*) = 1 - (1-|z|)^k \),
\( |z| = 1 - (1-u_2)^{1/k^*} \), \( z = -[1 - (1-u_2)^{1/k^*}] \). Observations are thus
generated to form a group of observations of size \( N \); successive groups
of observations are generated in the same way as needed. For each group
of observations, the signed rank statistic \( w \) is computed and the
corresponding value of \( \ln L(N,k_1^*,1) \) is obtained by table search in the
computer. After each group, \( P_{1t}/P_{0t} \) is computed from (10) and the
sequential test conducted through comparison of this probability ratio
with \( \ln A \) and \( \ln B \). Thus the sequential experiment is simulated. Five
hundred simulated experiments were conducted for each \( k^* \), \( N \) and \( k_1^* \) with
\( \alpha = \beta = .05 \). Results were tabulated as in Table 3.

Under the \( \Phi \)-model, observations were generated from the CDF
\( \Phi(z-u_z) \) where \( \Phi(z) \) is the standard normal CDF and \( u_z > 0 \). We wished to
match \( u_z \) to \( k^* \) so that a value of \( u_z \) corresponding to a location change
based on \( k^* \) could be determined. This was done with the help of Table 1
with \( u_z = -\mu(k^*)/\sigma(k^*) = \mu(k)/\sigma(k) \). The method of Muller (1959) was
used to generate groups of normal observations. The sequential experiment
was again simulated in the computer with 500 simulated experiments
conducted for each \( \mu(z) \), \( N \) and \( k^*_1 \) with \( \alpha = \beta = .05 \). Results for the two models are compared below in Table 3.

Table 3. Monte Carlo Values of the ASN and Power Functions for the sgrs-test.

<table>
<thead>
<tr>
<th>Design</th>
<th>( k^* )</th>
<th>( \mu_k )</th>
<th>Power</th>
<th>ASN</th>
<th>s.e. ASN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
<td>( \phi )</td>
<td>L</td>
<td>( \phi )</td>
<td>L</td>
</tr>
<tr>
<td>N=4</td>
<td>1</td>
<td>0</td>
<td>.022</td>
<td>-</td>
<td>12.46</td>
</tr>
<tr>
<td>( k^*_1 = 4 )</td>
<td>1.25</td>
<td>.1546</td>
<td>.086</td>
<td>.072</td>
<td>15.98</td>
</tr>
<tr>
<td>( \alpha = \beta = .05 )</td>
<td>1.5</td>
<td>.2792</td>
<td>.196</td>
<td>.244</td>
<td>20.40</td>
</tr>
<tr>
<td>2</td>
<td>.4769</td>
<td>.588</td>
<td>.608</td>
<td>23.38</td>
<td>23.44</td>
</tr>
<tr>
<td>2.5</td>
<td>.6034</td>
<td>.810</td>
<td>.838</td>
<td>21.54</td>
<td>22.54</td>
</tr>
<tr>
<td>3</td>
<td>.7048</td>
<td>.894</td>
<td>.922</td>
<td>18.76</td>
<td>19.29</td>
</tr>
<tr>
<td>3.5</td>
<td>.7830</td>
<td>.962</td>
<td>-</td>
<td>16.36</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>.8550</td>
<td>.984</td>
<td>.966</td>
<td>14.91</td>
<td>14.40</td>
</tr>
<tr>
<td>5</td>
<td>.9359</td>
<td>.996</td>
<td>.992</td>
<td>11.89</td>
<td>12.78</td>
</tr>
<tr>
<td>6</td>
<td>.9992</td>
<td>.998</td>
<td>.978</td>
<td>11.18</td>
<td>12.31</td>
</tr>
</tbody>
</table>

Table 3 constitutes an example of the results of the Monte Carlo studies. Entries under Power are simply proportions of simulated experiments (each entry based on 500 experiments) terminating with acceptance of \( H_a \). ASN-numbers are simply averages of termination numbers \( N \) given in terms of total numbers of observations and the final columns are intended to indicate the precisions of the ASN-numbers. Figure 1 gives plots of powers, \( \bar{\alpha}(k^*, \mu_z) \). Note that the horizontal scale is linear in \( \mu_z \) but not in \( k^* \); probability paper is used to establish the scale of the ordinate. It was observed that a probit
Figure 1. Power curves for the sgsr test under the design $N=h$, $k^*_1=4$ and $\alpha = \beta = .05$. 
model fitted this data well and the probit lines are drawn. A
standard probit analysis program was used to fit the relationship,

\[ \beta(k^*, \mu_z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a+by_z - 5}{c}} e^{-\frac{t^2}{2}} dt. \] (13)

\( \mu_z \) and \( k^* \) are related as discussed above. Estimates of \( a \) and \( b \) are
2.848 and 4.975 and 3.013 and 4.673 respectively for \( L \)- and \( \phi \)-models.
The unadjusted power curve in Figure 1 is for the \( L \)-model when
associated with \( \mu(k^*) \) instead of \( \mu_z \); the use of \( \mu_z \) is clearly necessary
for an approximately linear plot.

A comparison of corresponding ASN-values in Table 3 with those
of Table 2 shows results in Table 2 appreciably larger. Similarly,
from Figure 1, \( \alpha \) and \( \beta \) are smaller than the nominal values of \( .05 \). This
is probably due to an effect of grouping, the yielding of probability
ratios with excesses over the boundaries A and B at terminations of
tests.

Other information was available from the Monte Carlo studies and
is reported by Weed and Bradley (1969b). The shapes of the termination
number distributions may be studied as may the effects of various
truncation rules.

Discussion

Let us highlight the role of the computer in this research.
Computational work was involved at several stages:

1. Computations in Table 1 were developed through use of
numerical integration.
2. Tables of values of $P(w|N,k)$ in (9) and $L_x(N,k_l,l)$ were prepared and are given in Weed and Bradley (1969a).

3. Computer interpolations for $k'$ in Table 2 followed calculations of $E_k(r)$ in (12).

4. Monte Carlo studies exemplified in Table 3 were a major computer effort involving
   
   (a) Random number generation,
   (b) Random normal observation generation,
   (c) Simulation of sequential experimentations,
   (d) Summarization of results,
   (e) Probit analysis.

This paper illustrates the role of the computer in a sustained research effort in statistics. The work depends both on model building and theoretical investigation and on numerical techniques. It provides a clear demonstration of the necessity of computer availability in statistical research. Computing facilities used were quite comparable to those now available at the Scientific Computation Center. The Center will provide support for research in the Institute of Statistical Studies and Research as well as for other research in the United Arab Republic.
References


This is a summary paper based on several more detailed reports. Sequential, one-sample, rank tests have been developed with the referenced papers dealing with basic methodology, modified tests and Monte Carlo studies, respectively.

The Wilcoxon signed rank method is reviewed. Sequential methods are developed for observations taken in groups of size N, the groups taken sequentially. Within-group, signed ranks are used and a model developed that permits calculation of group probability ratios for two procedures, one based on the within-group configurations of signed ranks and one based on the within-group rank sums. Modified tests with observations taken singly are also described when new sets of signed ranks are obtained as each new observation is taken. Wald's method of sequential analysis is used.

Monte Carlo studies of the sequential grouped procedures have been conducted. Calculations are described and examples of results given.