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AN IMMIGRATION AND FRAGMENTATION STOCHASTIC PROCESS

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SUMMARY

A population of different types of particles is considered where immigration of the various particle types into the population is allowed. The immigration causes "fragmentation" or branching of all the existing target particles in the population. An immigrating particle can also fragment into the different types. The joint moment generating function for the number of particles in the m-th generation, and the expected number of particles in the m-th generation are given. Asymptotic expected values, and limits of random variables for a k-type process in discrete random environment with and without immigration are obtained.

1. INTRODUCTION

A population of $n_{0,t} \geq 0$ target particles of type $teT = \{1,2,\ldots\}$ exist in the 0-th generation. Between the 0-th and 1-st generations, an immigrant particle of type $i \in \mathbb{T}$ joins the population with probability $p_i \geq 0$, $\sum_{i \in \mathbb{T}} p_i = 1$. This event causes "fragmentation" of all the existing target particles into various numbers of new particles of types $n \in \mathbb{T}$. We denote by

\begin{equation}
X_{i,t} = (X_{i,t}^1, X_{i,t}^2, \ldots)
\end{equation}

the random vector whose $r$-th component is the random variable equal to the number of new particles of type $r$ produced by a target particle of type $teT$ when fragmented by an immigrant particle of type $i \in \mathbb{T}$. We
denote the moment generating function (m.g.f.) of \(X_{i,t}\) by \(F_{i,t}(s)\), where \(0 \leq s_i < 1\) and \(i \in T\).

Immediately after fragmenting the target particles, the immigrant is itself allowed to fragment into different numbers of new particles. We denote by

\[(1.2) \quad Y_i = (Y_{i,1}^1, Y_{i,2}^2, \ldots)\]

the random vector whose \(r\)-th component is the random variable equal to the number of new particles of type \(r\) produced by an immigrant particle of type \(i \in T\) which is fragmented. We denote the m.g.f. of \(Y_i\) by \(G_i(s)\), where \(0 \leq s_i < 1\) and \(i \in T\).

The new particles produced between the 0-th and 1-st generations form the population of target particles for the immigrant that comes between the 1-st and 2-nd generations. Then the process repeats itself on the succeeding generations.

By an appropriate choice of the parameters \(T, \{p_i\}\), and m.g.f.'s \(\{F_{i,t}(s)\}, \{G_i(s)\}\), one can obtain the well known Harris k-type process and a generalization of this process to include immigration. Furthermore, one can obtain a generalization of the Smith and Wilkinson [2] model to a k-type process in discrete random environment where immigration is allowed. The latter model includes the extension of the Harris model to allow immigration as a special case (see Sections 3 and 4).

Throughout the remainder of this paper it will be assumed that at each stage of the process, the type of immigrant to join the population depends only on the probability distribution \(\{p_i\}\). We will also assume that for fixed \(i \in T\), the random vectors \(Y_i, X_{i,1}, X_{i,2}, \ldots\) are independent, and that a target particle in the population fragments into numbers of new particles of the various types independently of how a neighboring target particle may fragment.
2. MOMENT GENERATING FUNCTION FOR THE FRAGMENTATION PROCESS

We will obtain results concerning the m.g.f. of the random vector

\[(2.1) \quad N_m = (N(1,m), N(2,m), \ldots)\]

whose \(r\)-th component is a random variable equal to the number of
particles of type \(r\) in the \(m\)-th generation, where \(\mathcal{M} = \{0,1,2,\ldots\}\).

We let \(H_m(s)\) be the m.g.f. of \(N_m\), and we have the following

**Theorem.** For \(m \geq 1\) and \(s = (s_1, s_2, \ldots)\) where \(0 < s_k < 1\),

\[(2.2) \quad H_m(s) = \sum_{i \in \mathcal{T}} p_i \sum_{i \in \mathcal{T}} F_{i,1}(s) \cdots \sum_{i \in \mathcal{T}} F_{i,2}(s) \cdots .\]

**Proof.**

Let \(E(i)\) denote the event that an immigrant particle of type
\(i\) joins the population between the \(m\) and \((m+1)\)-st generations, \(\mathcal{M}\).

Given that \(E(i)\) occurs and that \(N_{m-1} = (n_1, n_2, \ldots)\) we note

that

\[(2.3) \quad N_m = Y_1 + \sum_{t \in \mathcal{T}} \sum_{j=1}^{n_t} X_{i,t,j}\]

where \(X_{i,t,j}\) is distributed as \(X_{i,t}\). Since the random vectors

\(Y_1, X_{i,1,1,1}, \ldots, X_{i,1,n_1}, X_{i,2,1}, \ldots, X_{i,2,n_2}, \ldots\)

are independent, it follows

that

\[(2.4) \quad \left( \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \ldots \right) \prod_{j \in \mathcal{T}} \sum_{i \in \mathcal{T}} p_{j}(N_m = (r_1, r_2, \ldots)|E(i)), N_{m-1} = (n_1, n_2, \ldots)\]

\[= G_1(s) \prod_{i \in \mathcal{T}} f_{i,t}(s)^{n_t}.\]
Assuming that the type of immigrant to join the population is independent of the size of the population at the time of immigration and the type of immigrants that preceded it, we have

\[ P(E(i), N_{m-1} = (n_1, n_2, \ldots)) = P(E(i)) P(N_{m-1} = (n_1, n_2, \ldots)) \]

\[ = p_i P(N_{m-1} = (n_1, n_2, \ldots)). \]

Multiplying both sides of (2.4) by \( P(E(i), N_{m-1} = (n_1, n_2, \ldots)) \) yields

\[ (\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \ldots) \prod_{j \in T} s^j P(N_m = (r_1, r_2, \ldots), E(i), N_{m-1} = (n_1, n_2, \ldots)) \]

\[ = p_i G_i(s) \prod_{t \in T} [F_i, t(s)] \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \ldots P(N_{m-1} = (n_1, n_2, \ldots)). \]

It follows that

\[ (\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \ldots) \prod_{j \in T} s^j P(N_m = (r_1, r_2, \ldots), E(i), N_{m-1} = (n_1, n_2, \ldots)) \]

\[ = \sum_{i \in T} (\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \ldots) p_i G_i(s) \prod_{t \in T} [F_i, t(s)] \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \ldots P(N_{m-1} = (n_1, n_2, \ldots)). \]

We recognize the right side of (2.6) as the m.g.f.

\[ \sum_{i \in T} p_i G_i(s) H_{m-1}(F_{i,1}(s), F_{i,2}(s), \ldots). \]

Since we can interchange summations in the left side of (2.6) and observe that
\begin{align*}
(2.8) \quad \sum_{i \in T} \left( \sum_{n_1 = 0}^{\infty} \sum_{n_2 = 0}^{\infty} \cdots \right) \left( \sum_{r_1 = 0}^{\infty} \sum_{r_2 = 0}^{\infty} \cdots \right) \Pi_{j \in T} r_j^{n_j} \\
&= P(N_m = (r_1, r_2, \ldots), E(i), m_{m-1} = (n_1, n_2, \ldots)) \\
&= \left( \sum_{r_1 = 0}^{\infty} \sum_{r_2 = 0}^{\infty} \cdots \right) \sum_{i \in T} \left( \sum_{n_1 = 0}^{\infty} \sum_{n_2 = 0}^{\infty} \cdots \right) \Pi_{j \in T} r_j^{n_j} \\
&= P(N_m = (r_1, r_2, \ldots), E(i), m_{m-1} = (n_1, n_2, \ldots)) \\
&= \left( \sum_{r_1 = 0}^{\infty} \sum_{r_2 = 0}^{\infty} \cdots \right) \Pi_{j \in T} r_j^{n_j} P(N_m = (r_1, r_2, \ldots)) \\
&= \Pi_m (s).
\end{align*}

Combining the equalities (2.6) and (2.8), we obtain the desired result, (2.2).

Before continuing, we point out that we have assumed

\begin{equation}
(2.9) \quad N_0 = (n_{0,1}, n_{0,2}, \ldots)
\end{equation}

with probability one, and hence that

\begin{equation}
(2.10) \quad H_0(s) = \Pi_{t \in T} s_{0,t}^{n_{0,t}}.
\end{equation}

Therefore, (2.2) can be solved recursively. This, and other related results, can be found in Section 4.
3. MOMENTS AND ASYMPTOTIC RESULTS

Let

\[ \mu_{i,t} = \frac{\partial}{\partial s_t} g_i(s) \bigg|_{s=1} = E[Y_i^t] \]  

be the expected number of particles of type \( t \in T \) produced by a fragmenting immigrant of type \( i \in T \). Define

\[ \lambda_{*,v} = \sum_{i \in T} p_i \lambda_{i,v,t} \]  

and

\[ U = (\mu_{*,1}, \mu_{*,2}, \ldots) \]  

Similarly, let

\[ \lambda_{i,v,t} = \frac{\partial}{\partial s_t} F_i,v(s) \bigg|_{s=1} = E[x_{i,v}^t] \]  

be the expected number of particles of type \( t \in T \) produced by a target particle of type \( v \in T \) when fragmentated by an immigrant of type \( i \in T \). Define

\[ \lambda_{*,v,t} = \sum_{i \in T} p_i \lambda_{i,v,t} \]  

and
\[
\Lambda = \begin{bmatrix}
\lambda_{*,1,1} & \lambda_{*,1,2} & \cdots \\
\lambda_{*,2,1} & \lambda_{*,2,2} & \cdots \\
\vdots & \ddots & \ddots \\
\vdots & & \ddots \\
\vdots & & & \ddots
\end{bmatrix}
\]

Our first result concerns a relationship between \(N_{m+1}\) and \(N_m\).

**Lemma.** For \(m \in M\),

\[
E[N_{m+1} | N_m] = U + N_m \Lambda
\]

with probability one.

**Proof.**

From (2.3)

\[
E[N_{m+1} | E(i \ldots, N_m = (n_1, n_2, \ldots))] = E[Y_i] + \sum_{t \in T} \sum_{j=1}^{n_t} E[X_{i,t,j}]
\]

\[
= E[Y_i] + \sum_{t \in T} n_t E[X_{i,t,1}]
\]

\[
= (\mu_i, 1, \mu_i, 2, \ldots)
\]

\[
+ \sum_{t \in T} n_t (\lambda_{i,t,1}, \lambda_{i,t,2}, \ldots)
\]

Multiplying both sides by \(p_i\) and summing on \(i \in T\) we obtain

\[
E[N_{m+1} | N_m = (n_1, n_2, \ldots)] = U + \sum_{t \in T} n_t (\lambda_{*,t,1}, \lambda_{*,t,2}, \ldots)
\]

Hence, (3.7) follows. It is now possible to prove the more general
THEOREM. For \( n, m \in \mathbb{M}, n \geq 1, \)

\[
E[N_{m+n} | N_m] = U \sum_{j=0}^{n-1} \Lambda^j + \frac{N_m}{\lambda} \Lambda^n
\]

with probability one, where \( \Lambda^0 = I. \)

Proof.

Using \( E[X|Y] = E[E[X|Z]|Y], \) we have

\[
E[N_{m+n} | N_m] = E[E[N_{m+n} | N_{m+n-l}] | N_m]
\]

\[
= E[U + \frac{N_{m+n-l}}{\Lambda} | N_m]
\]

by the previous lemma. Hence,

\[
E[N_{m+n} | N_m] = U + E[N_{m+n-l} | N_m] \Lambda,
\]

and iteration yields the result, (3.8).

From (3.8) we obtain

COROLLARY. For \( m \geq 1, \)

\[
E[N_m] = U + E[N_{m-1}] \Lambda.
\]

We also have the following

THEOREM. For \( m \geq 1, \)

\[
E[N_m] = U \sum_{j=0}^{m-1} \Lambda^j + \frac{N_0}{\Lambda} \Lambda^m.
\]
We will next make assumptions concerning the matrix $\Lambda$ to obtain the limiting properties of $E[N_m]$.

Assumption A. Suppose every particle of type $\nu \in T$ when fragmented by any immigrant of type $\nu' \in T$ produces the same mean number of new particles of each type $\nu, \nu' \in T$. Then, $\lambda_{i,\nu,t} = \lambda_{i,\nu',t}$ for every $i, \nu, \nu', t \in T$, and hence,

$$\lambda_t = \lambda_{*,\nu,t} \quad \text{for every } \nu, t \in T$$

for every $v, t \in T$ and

$$\Lambda = \begin{bmatrix}
\lambda_1 & \lambda_2 & \cdots \\
\lambda_1 & \lambda_2 & \cdots \\
\vdots & \vdots & \ddots \\
\vdots & \vdots & \ddots \\
\end{bmatrix}$$

(3.11)

It can be shown that

$$\Lambda^{m+1} = c^m \Lambda$$

(3.12)

where

$$c = \sum_{t \in T} \lambda_t.$$

Combining these results with (3.9) we obtain

$$E[N_m] = (\sum_{j=0}^{m-1} c^j) U_A + \sum_{k=0}^{m-1} c^k N_0 A.$$

(3.13)

This completes the proof of the following
THEOREM. Under the assumption (3.10), that $\lambda_t = \lambda,_{v,t}$ for every $v, t \in T$,

$$\lim_{m \to \infty} E[N_{t,m}] = \frac{1}{1-c} U \Lambda$$

if $c < 1$

and for every $t \in T$, if $\lambda_t > 0$

$$\lim_{m \to \infty} E[N(t,m)] = \infty$$

if $c \geq 1$.

Let $k$ be a finite positive integer. To consider a $k$-type process with immigration, I., and discrete random environment, R.E., we collapse the above model as follows:

Bona fide particle types are numbered from $1$ to $k$ and "particles" of type $k+1, k+2, \ldots$, referred to earlier, are the environmental states which affect the branching distributions of the bona fide target particles and hence cannot be produced nor can they have progeny. Hence, from the fragmentation process model, we have a $k$-type I.R.E. process when

$$n_{0,t} = 0 \quad \text{for } t > k,$$

$$N_0 = (n_{0,1}, \ldots, n_{0,k}, 0, 0, \ldots),$$

and

$$\Upsilon_i = 0 \quad \text{for } i > k,$$

$$\Theta = (\mu_{,1}, \ldots, \mu_{,k}, 0, 0, \ldots),$$

and

$$\chi^t_{1,v} = 0 \quad \text{for } t > k \text{ or } v > k,$$
\[ \Lambda = \begin{bmatrix} \lambda_{*,1,l} & \cdots & \lambda_{*,1,k} \\ \vdots & & \vdots \\ \lambda_{*,k,l} & \cdots & \lambda_{*,k,k} \\ 0 & & 0 \end{bmatrix} \]

and

\[ N_m = (N(1,m), \ldots, N(k,m), 0, 0, \ldots). \]

Thus, we also have

\[ F_{i,t}(s) = F_{i,t}(s_1, \ldots, s_k) \quad \text{for } 1 \leq t \leq k \text{ and } i \in T, \]

\[ H_m(s) = H_m(s_1, \ldots, s_k). \]

Clearly, we can claim all of our previous results in their present form by now letting

\[ N_0 = (n_0,1, \ldots, n_0,k), \]

\[ N_m = (N(1,m), \ldots, N(k,m)), \]

\[ U = (u_{*,1}, \ldots, u_{*,k}), \]

and
\[
\Lambda = \begin{bmatrix}
\lambda_1, & \lambda_2, & \ldots, & \lambda_k, \\
\lambda_1, & \lambda_2, & \ldots, & \lambda_k, \\
\vdots, & \vdots, & \ddots, & \vdots, \\
\lambda_1, & \lambda_2, & \ldots, & \lambda_k
\end{bmatrix}
\]

Considering a k-type I.R.E. process, we make the following

Assumption B: The matrix \( \Lambda \) is \( k \times k \), \( \Lambda > 0 \) and \( \Lambda^d >> 0 \) for some integer \( d > 0 \).

Then by Frobenius' theorem (see Section 2 of appendix to Karlin [1]) there exists a unique real eigenvalue \( \rho > 0 \) of maximum modulus with left and right eigenvectors \( u >> 0 \) and \( v >> 0 \) respectively, where

\[
u \Lambda = \rho u,
\]
(3.14)  \[\Lambda v = \rho v \]
(3.15)  \[u \cdot v = 1\]
(3.16)

Multiplying both sides of (3.9) by \( v \) gives

\[
E[N_m]v = U \sum_{j=0}^{m-1} \lambda_j^j v + N_0 \Lambda^m v = (\sum_{j=0}^{m-1} \rho^j)U \cdot v + \rho^m N_0 \cdot v
\]
(3.17)

by (3.15). We note that equality (3.17) gives a relationship between two scalars, and only limited knowledge about \( E[N_m] \) as \( m \to \infty \).
THEOREM. Assuming the components of $U$ are finite,

(i) if $\rho < 1$ then for every $t \in T$

$$\lim_{m \to \infty} E[N(t,m)] < \infty,$$

(ii) if $\rho > 1$ and $U > 0$ or $N_0 > 0$, or if $\rho = 1$ and $U > 0$,

then for some $t$, $1 \leq t \leq k$,

$$\lim_{m \to \infty} E[N(t,m)] = \infty.$$

4. LIMITING RANDOM VARIABLES FOR A $k$-TYPE PROCESS

We can also consider a $k$-type random environment (R.E.) process where none of the $k$-types of particles is allowed to immigrate into the population. In terms of the parameters of the fragmentation model, we need only further collapse the $k$-type I.R.E. process:

Let

(4.1) $p_i = 0$ for $1 \leq i \leq k$,

(4.2) $\sum_{i=k+1}^{\infty} p_i = 1$,

(4.3) $G_i(s) = 1$ for $1 \leq i \leq k$,

(4.4) $U = 0$,

and

$$F_{i,t}(s) = 1 \text{ if } t > k \text{ or } i \leq k.$$ 

Note that "particles" of types $k+1, k+2, \ldots$ are the environmental states and hence cannot be produced nor can they reproduce particles. In
general, particles for which (4.3) holds could be considered sterile.

Combining the above remarks with (2.2), the m.g.f. for a k-type R.E. process is

\begin{equation}
H_m(s) = \sum_{i=k+1}^{\infty} p_i H_{m-1}(F_{i,1}(s), \ldots, F_{i,k}(s)).
\end{equation}

If we define

\begin{equation}
f_i(s) = (F_{i,1}(s), \ldots, F_{i,k}(s))
\end{equation}

and the composition

\begin{equation}
f_i \circ f_j(s) = f_i(F_{j,1}(s), \ldots, F_{j,k}(s)),
\end{equation}

we can use the remarks concerning (2.9) and (2.10) with (4.5) to obtain

**THEOREM.** For \(m \geq 1\), the m.g.f. for a k-type R.E. process is

\begin{equation}
H_m(s) = \left( \prod_{i=k+1}^{\infty} \prod_{i=1}^{\infty} p_i \prod_{j=1}^{k} F_{i,j}(s) \right) \left( \sum_{l=1}^{m} f_{i_2} \circ f_{i_3} \circ \ldots \circ f_{i_m}(s) \right)^n_{0,t}.
\end{equation}

**Proof.**

The result follows from iteration of (4.5) and definitions (4.6) and (4.7).
We make special note of the fact that the moment generating function of $N_m$ in discrete random environment without immigration involves the composition of countably many different vectors $F_{\alpha}(s)$. For this reason, the m.g.f. techniques such as in [3] have only limited application. That is, one would have to work with expansions involving the matrices

$$ (4.9) \quad A^{(i)} = \left( \frac{\partial}{\partial s} F_{\alpha,\beta}(s) \right)_{s=1}, \quad 1 \leq \alpha, \beta \leq k. $$

Instead, we will consider the matrix

$$ A = \sum_{i \in T} p_i A^{(i)} $$

which was introduced in (3.6). The next theorem will enable us to find $\lim_{m \to \infty} H_m(s)$ where $H_m(s)$ is as given in (4.8).

**THEOREM.** For a k-type R.E. process if Assumption B holds with $\rho < 1$ then

$$ (4.10) \quad \lim_{m \to \infty} P(N_m = 0) = 1. $$

**Proof.**

Since $U = 0$, it follows from (3.9) that

$$ (4.11) \quad EN_m = N_0 A^m $$

where $N_0 = (n_0,1, \ldots, n_0,k)$. Hence for $v$ as in (3.15), we have

$$ (4.12) \quad EN_m v = N_0 v \rho^m \to 0 \quad \text{as } m \to \infty. $$

Since $v >> 0$ we have $EN_m \to 0$. The result follows.
The following corollary is an immediate consequence of (4.10).

**COROLLARY.** For a $k$-type R.E. process, if assumption B holds with $\rho < 1$ then

$$
\lim_{m \to \infty} H_{m}(s) = 1,
$$

where $H_{m}(s)$ is the m.g.f. given in (4.8).

Our next result concerns the case of $\rho = 1$.

**THEOREM.** For a $k$-type R.E. process, if Assumption B holds with $\rho = 1$, then

$$
\frac{N_{m} \nu}{W} \to W
$$

with probability 1 where $\nu$ is defined as in (3.15), $W \geq 0$ and $EW < \infty$.

Proof.

From (4.4) we know $U = 0$ and thus (3.8) reduces to

$$
E[N_{m} | N_{m-1}] = N_{m-1} \Lambda.
$$

It follows that

$$
E[N_{m} \nu | N_{m-1} \nu] = E[E[N_{m} \nu | N_{m-1}] | N_{m-1} \nu]
$$

$$
= E[N_{m-1} \Lambda \nu | N_{m-1} \nu]
$$

$$
= \frac{N_{m-1} \nu}{N_{m-1} \nu}.
$$

Next, we note that
\[ E[N_m v] = N_0 v \]

and so

\[ \sup_m E[N_m v] < \infty. \] (4.15)

Since \( N_m v \) is a martingale and (4.15) holds, the result follows.

We now note that if assumption (4.1) is modified so that

\[ P^{k+1} = 1 \] (4.16)

we obtain the ordinary \( k \)-type branching process discussed in [3]. This too can be thought of as a \( k \)-type R.E. process where the only environmental state to possibly occur is a "steady" state. It was proven in [3] that \( N_m \overset{p}{\rightarrow} 0 \). Combining this fact with the previous theorem yields the following

**THEOREM.** For the ordinary \( k \)-type branching process, if Assumption B holds with \( \rho = 1 \) and \( r_{k+1} (x) \neq \Lambda_x \), then

\[ N_m \overset{p}{\rightarrow} 0 \] (4.17)

with probability 1 as \( m \rightarrow \infty \).

Proof.

Since we have from [3] that \( N_m \overset{p}{\rightarrow} 0 \) we know that

\[ \frac{N_m v}{N_m} \overset{p}{\rightarrow} 0. \]

It follows from the previous theorem that

\[ \frac{N_m v}{N_m} \rightarrow 0. \]
with probability 1 as \( m \to \infty \). Hence, since \( y > 0 \) we have the desired result.

Next, we will again consider \( k \)-type I.R.E. processes. Using submartingale results similar to those above, we have the following

**THEOREM.** For a \( k \)-type I.R.E. process, if Assumption 3 holds with \( \rho \leq 1 \), then

\[
(m)^{-2+c}\frac{N_m}{V_m} \to 0
\]

with probability 1 where \( c \) is an arbitrary positive constant.

Proof.

Let

\[
V_n = \sum_{j=1}^{n} j^{-(2+c)}\frac{N_j}{N_j} Y.
\]

We note that

\[
E[V_n | V_{n-1}] = V_{n-1} + E[(n)^{-2+c}\frac{N_n}{N_n} Y | V_{n-1}].
\]

Since \( \frac{N_n}{n} Y \) is a positive random variable

\[
E[V_n | V_{n-1}] \geq V_{n-1}.
\]

Finally we note from (3.9) that

\[
E[V_n] = \sum_{j=1}^{n} (j)^{-2+c}E[N_j]Y
\]

\[
= \sum_{j=1}^{n} (j)^{-2+c}\left( \sum_{r=0}^{j-1} \lambda^r + N_0 \lambda^j \right) Y.
\]
Using (3.15) we have

\[ E[V_n] \leq \sum_{j=1}^{n} (j)^{(2+c)} [U \mathbf{v} \mathbf{J} + N_0 \mathbf{v}] \]

\[ = U \mathbf{v} \sum_{j=1}^{n} (j)^{(1+c)} + N_0 \mathbf{v} \sum_{j=1}^{n} (j)^{(2+c)}. \]

Hence

\[ \sup_{n} E[V_n] < \infty \]

and the result follows from the fact that \( \mathbf{v} >> 0 \).

5. AN OBSERVATION

We note that for a \( k \)-type I.R.E. process, the immigrant particles of types \( 1 \) to \( k \) can affect the branching of the target particles but are not affected by any of the environmental "particles" of type \( k+1, k+2, \ldots \). This can be altered as follows:

Allow the immigrant of type \( 1 \leq i \leq k \) to join the population of target particles. Then let all of them fragment or branch according to the environmental state \( j \) that then occurs. Replace the event \( E(i) \) by \( E(i, j) \) described above and \( p_i = P\{E(i)\} \) by \( p_{i,j} = P\{E(i, j)\} \) with \( \sum_{i=1}^{k} \sum_{j=k+1}^{\infty} p_{i,j} = 1 \). All the previous results will follow for this more general model, with the obvious notational changes.

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1. Originating Activity
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13. Abstract
    A population of different types of particles is considered where immigration of the various particles types into the population is allowed. The immigration causes "fragmentation" or branching of all the existing target particles in the population. An immigrating particle can also fragment into the different types. The joint moment generating function for the number of particles in the m-th generation, and the expected number of particles in the m-th generation are given. Asymptotic expected values, and limits of random variables for a k-type process in discrete random environment with and without immigration are obtained.

14. Key Words

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