SOME WEAK LAWS FOR RANDOM ELEMENTS
IN NORMED LINEAR SPACES

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1. **Summary**

Some weak laws of large numbers for random elements in normed linear spaces are given in this paper. The main result (Theorem 3.5) is a weak law of large numbers for separable normed linear spaces.

**Theorem 3.5:**

Let \( \{V_n\} \) be a sequence of weakly uncorrelated, identically distributed random elements in a separable normed linear space. If \( E|V_1| < \infty \) and \( EV_1 \) exists, then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} V_k - EV_1 = 0 \text{ in probability.}
\]

An extension of Theorem 3.5 and results for product sequences of random variables and random elements are also given. Examples and counterexamples are provided throughout the paper to facilitate the reader.

The necessary background is outlined in Section 2.

2. **Introduction**

Unless otherwise stated \( X \) will be a real normed linear space with elements, \( x \)'s, and norm, \( || \cdot || \). The \( \sigma \)-field generated by the open subsets of \( X \) will be denoted by \( B \). Let \( (W,F,P) \) denote a probability space and let \( V \) be a function from \( W \) into \( X \). If \( V^{-1}(B) \in F \), then \( V \) is called a random element in \( X \).

If \( V \) is a random element in \( X \), then \( f(V) \) is a random variable for each \( f \in X^* \) (the topological dual of \( X \)). Nedoma [5] has shown that the sum of 2 measurable functions in \( X \) is not necessarily a measurable function in \( X \). Hence, the sum of 2 random elements may not be a random element. However,
Lemma 2.1 (an extension of a result in [1] and [3]) is sufficient for this paper since the results in Section 3 are for separable spaces. In particular, Lemma 2.1 states that the sum of two random elements $V$ and $Z$ in a separable normed space $X$ is a random element since $f(V + Z) = f(V) + f(Z)$ is a random variable for each $f \in X^*$. 

**Lemma 2.1:**

If $X$ is separable, then $V$ is a random element if and only if $f(V)$ is a random variable for each $f \in X^*$.

By using Lemma 2.1, random elements in $n$-dimensional real Euclidean space $(\mathbb{R}^n)$ and the space $\ell = \{x \in \mathbb{R}^\infty : \sum |x_n| < \infty\}$ are easily characterized. Specifically, $V$ is a random element in $\mathbb{R}^n$ if and only if $V = (V_1, \ldots, V_n)$ where each $V_i$, $1 \leq i \leq n$, is a random variable with respect to $(\mathbb{W}, F, P)$. Let $V = (V_1, V_2, \ldots)$ be a random element in $\ell$, then $\{V_n\}$ is an absolutely pointwise summable sequence of random variables. Moreover, any sequence of random variables which is absolutely summable with probability 1 defines a random element in $\ell$. Random elements in the spaces $c_0$, $c$, and $\ell_p(p > 1)$ can be characterized similarly (see [7] for the definitions of these spaces).

Before extending the definitions of independence and identically distributed to random elements, an additional measurability result is needed.

**Lemma 2.2:**

Let $V$ be a random element in $X$ and let $A$ be a random variable, then $AV$ is a random element in $X$. 
Definition 2.3:

A finite set of random elements \( \{V_1, \ldots, V_n\} \) is said to be independent if

\[
P[V_i \in B_i : 1 \leq i \leq n] = P[V_1 \in B_1] \cdots P[V_n \in B_n]
\]

for every \( B_1, \ldots, B_n \in \mathcal{B} \). A family of random elements is said to be independent if every finite subset is independent.

Definition 2.4:

The random elements \( V \) and \( Z \) are identically distributed if

\[
P[V \in D] = P[Z \in D]
\]

for each \( D \in \mathcal{B} \). A family of random elements is identically distributed if every pair of random elements is identically distributed.

The expected value of a random element in a normed linear space can be defined by a Pettis integral [6]. A random element \( V \) is said to have expected value \( EV \) if there exists an element \( EV \in X \) such that

\[
E[f(V)] = f(EV) \text{ for each } f \in X^*.
\]

If the expected value exists, then it is unique since \( X^* \) is separating over \( X \) [7]. When \( X \) is complete and \( E\|V\| < \infty \), then either separability or reflexivity is a sufficient condition for the existence of \( EV \). For \( X = R^D(n \geq 1) \) the expected value is the usual one. Additional properties of the expected value are given in Pettis [6] and Mourier ([3] and [4])

Example 2.5 illustrates the ideas of the previous paragraph.
Example 2.5:

Let $X = \mathbb{R}^{(\omega)}$, the space of finite, real sequences with $\| x \| = \sup_n |x_n|$. Let $V = d^N$ with probability $\lambda/n^2$ where $d^1 = (1,0,0,...)$, $d^2 = (0,1,0,...)$, ..., and $\lambda = 6/\pi^2$. Since $E\|V\| < \infty$, $E[f(V)]$ exists for each $f \in X^*$. If $f(m) = E[f(V)]$ for each $f \in X^*$, then $m = \lambda(1,1/4,..., 1/n^2,...)$. Thus, EV does not exist since $m \in R^{(\omega)}$. If $X$ were the space of null convergence sequences, $c_0$, then EV would exist and $EV = m$.

When $V$ is a random element, $\|V\|$ is a non-negative random variable. Hence, the following form of the Markov inequality is valid:

$$P(\|V\| > \alpha) \leq E\|V\|^t/\alpha^t$$

where $t > 0$ and $\alpha > 0$.

Several types of convergence (norm convergence, weak convergence, compact convergence, etc.) are applicable for random elements in normed spaces. However, only types of norm convergence will be defined since the results in Section 3 are for convergence in the norm topology.

Definition 2.6:

A sequence $\{V_n\}$ of random elements is said to converge to the random element $V$:

(a) with probability 1 if

$$P(\lim_n \|V_n - V\| = 0) = 1,$$

(b) in probability if

$$\lim_n P(\|V_n - V\| > \epsilon) = 0$$

for each $\epsilon > 0$, or
(c) in mean with index $\alpha$ ($\alpha > 0$) if
\[
\lim_{n} E\|V_n - V\|^\alpha = 0.
\]

The same relationships among the different types of convergence which hold for random variables are still valid for random elements.

3. Weak Laws for Random Elements

Results for convergence in probability are given in this section. In particular, a weak law of large numbers (Theorem 3.5) is proved for weakly uncorrelated, identically distributed random elements. Results for product sequences of random elements and random variables are included, and an extension of Theorem 3.5 is provided.

The first result to be presented is an extension of a weak law of large numbers for uncorrelated random variables. The result is first proved for Banach spaces which have Schauder bases. The final version for separable normed spaces is achieved by observing that $c[0,1]$ has a Schauder basis and that each separable normed space can be embedded isometrically in $C[0,1]$.

**Definition 3.1:**

A sequence $\{b_n\} \subset X$ is a Schauder basis for $X$ if for each $x \in X$ there exists a unique sequence of scalars $\{t_n\}$ such that
\[
x = \lim_{n} \sum_{k=1}^{n} t_k b_k.
\]
A Schauder basis is a monotone basis if for each $x$ of the form $\lim_{n} \sum_{k=1}^{n} t_k b_k \in X$ the sequence $\{\sum_{k=1}^{n} t_k b_k\}$ is monotone increasing.
When \( X \) has a Schauder basis \( \{b_n\} \), a sequence of linear functionals \( \{f_n\} \) can be defined by \( f_n(x) = \langle x, b_n \rangle \) where \( x \in X \) and
\[
x = \lim_{n \to \infty} \frac{1}{t_n} \sum_{k=1}^{n} t_k b_k.
\]
These linear functionals are called the coordinate functionals (for the basis \( \{b_n\} \)). The coordinate functionals depend on the basis \( \{b_n\} \) and need not be continuous. However, as a consequence of the open-mapping theorem, the coordinate functionals for a Banach space are continuous (see Lemma 3.2). Finally, a sequence of linear functions \( \{U_n\} \) on \( X \) can be defined by
\[
U_n(x) = \sum_{k=1}^{n} f_k(x) b_k.
\]
The sequence \( \{U_n\} \) is called the sequence of partial sum operators (for the basis \( \{b_n\} \)).

**Lemma 3.2:**

(a) If \( X \) has a monotone basis, then each \( U_n \) is continuous and \( \|U_n\| \leq 1 \) for each \( n \).

(b) If \( X \) is a Banach space which has a Schauder basis, then each \( U_n \) is continuous and there exists an \( m > 0 \) such that \( \|U_n\| \leq m \) for all \( n \).

**Remark:**

The conditions in Lemma 3.2 are also sufficient for the continuity of each coordinate functional since \( \|U_n\| \leq m \) for each \( n \) implies that \( \|f_n\| \leq 2m \) for each \( n \).

The composition \( h(V) \) is a random element in the normed linear space \( Y \) if \( h \) is a continuous function from \( X \) to \( Y \) and \( V \) is a random element in \( X \).

This follows easily from Section 2 since the \( \sigma \)-field on \( X \) and \( Y \) are generated
by the open sets. Hence, \( JV = J(V) \) is a random element in \( X \) whenever \( V \) is a random element in \( X \) and \( J \) is a continuous partial sum operator on \( X \).

**Theorem 3.3:**

Let \( X \) have a monotone basis or let \( X \) be a Banach space with a Schauder basis. Let \( \{ V_n \} \) be a sequence of identically distributed random elements in \( X \) such that \( E\|V_1\| < \infty \) and \( EV_1 \) exists. If \( \{ f(V_n) \} \) is a sequence of uncorrelated random variables for each coordinate functional \( f \in X^* \), then

\[
\frac{1}{n} \sum_{k=1}^{n} V_k \to EV_1 \text{ in probability.}
\]

**Proof:**

It can be assumed that \( EV_1 = 0 \) [otherwise, consider \( Z_n = V_n - EV_n \)] and that \( \|b_i\| = 1 \) for each \( i \) where \( \{b_n\} \) denotes the Schauder basis. Let \( e > 0 \) and \( d > 0 \) be given. In order that

\[
\frac{1}{n} \sum_{k=1}^{n} V_k \to 0 \text{ in probability}
\]

there must exist an \( N(e, d) \) such that

\[
P[\left\| \frac{1}{n} \sum_{k=1}^{n} V_k \right\| > e] < d \text{ for all } n \geq N(e, d).
\]

Let \( \lambda = \min(e/4, e/2m) \) where \( m \) is the positive constant such that \( \|U_n\| \leq m \) for all \( n \) (see Lemma 3.2). Let \( D = \{ x_1, x_2, \ldots \} \) be the countable dense subset of \( X \) formed by

\[
D = \left\{ \sum_{k} r_k b_k : \text{where } \{r_k\} \text{ is a finite rational sequence} \right\}.
\]

A sequence of identically distributed random elements \( \{TV_n\} \) can be defined such that the range of \( TV_1 \) is a subset of \( D \) and
\[ \| TV_n - V_n \| \leq \lambda \text{ for each } w \in W \text{ and each } n. \]

Since \( E \| V_1 \| < \infty \)

\[ E \| TV_1 \| = \sum_{i=1}^{\infty} \| x_i \| P[TV_1 = x_i] < \infty, \]

thus \( J \) can be chosen so that

\[ \sum_{i=j+1}^{\infty} \| x_i \| P[TV_1 = x_i] < \frac{3e^d}{16(m+1)}. \]

There exists a continuous partial sum operator \( J \) such that \( \{ x_1, \ldots, x_j \} \subseteq J(X) \)
and \( \| J \| \leq m. \)

Let \( Q = I - J \), then \( \| Q \| \leq \| I \| + \| J \| \leq 1 + m \) and \( x = J(x) + Q(x) \) for each \( x \in X. \) For any \( n, \)

(a) \[ \| 1/n \sum_{k=1}^{n} TV_k \| - \| 1/n \sum_{k=1}^{n} V_k \| \leq 1/n \sum_{k=1}^{n} \| TV_k - V_k \| \leq \lambda \leq e/4. \]

(b) \[ \| 1/n \sum_{k=1}^{n} JTV_k \| - \| 1/n \sum_{k=1}^{n} TV_k \| \leq 1/n \sum_{k=1}^{n} \| J \| \| TV_k - V_k \| \leq m\lambda \leq e/4. \]

(c) \[ P[\| 1/n \sum_{k=1}^{n} QTV_k \| > 3e/8] \leq 8/3e E \| QTV_1 \| \]

\[ \leq 8/3e \sum_{i=j+1}^{\infty} \| Q \| \| x_i \| P[TV_1 = x_i] < d/2. \]

From (a)

\[ P[\| 1/n \sum_{k=1}^{n} V_k \| > e] \leq P[\| 1/n \sum_{k=1}^{n} TV_k \| > 3e/4] \]

\[ \leq P[\| 1/n \sum_{k=1}^{n} JTV_k \| > 3e/8] + P[\| 1/n \sum_{k=1}^{n} QTV_k \| > 3e/8] \]
and from (b) and (c)

\[ P[\|1/n \sum_{k=1}^{n} JV_k\| > e/8] + d/2. \]

Let \( \{b_1, \ldots, b_t\} \) be the basis for \( J(X) \). By construction

\[ y = \sum_{i=1}^{t} f_i(y)b_i \]

for each \( y \in J(X) \) where \( f_1, \ldots, f_t \) are the coordinate functionals for the basis elements \( b_1, \ldots, b_t \). Hence,

\[ P[\|1/n \sum_{k=1}^{n} JV_k\| > e/8] = P[\| \sum_{i=1}^{t} f_i(1/n \sum_{k=1}^{n} JV_k)b_i \| > e/8] \]

\[ \leq P[ \sum_{i=1}^{t} |f_i(1/n \sum_{k=1}^{n} JV_k)| > e/8] \leq \sum_{i=1}^{t} P[|1/n \sum_{k=1}^{n} f_i(V_k)| > e/8t]. \]

But,

\[ P[|1/n \sum_{k=1}^{n} f_i(V_k)| > e/8t] \rightarrow 0 \text{ as } n \rightarrow \infty \]

for each \( i = 1, \ldots, t \) since \( \{f_i(V_k)\} \) is an uncorrelated sequence of identically distributed random variables [in particular, \( E[f_i(V_k)]^2 < \infty \)]. Hence, there exists an \( N(e,d) \) such that

\[ \sum_{i=1}^{t} P[|1/n \sum_{k=1}^{n} f_i(V_k)| > e/8t] < d/2 \text{ for all } n \geq N(e,d). \]

Thus, for each \( n \geq N(e,d) \)

\[ P[\|1/n \sum_{k=1}^{n} V_k\| > e] \leq P[\|1/n \sum_{k=1}^{n} JV_k\| > e/8] + d/2 < d. \]
Lemma 3.2 shows that Theorem 3.3 is applicable to the spaces $c$, $c_0$, $R^{(\infty)}$, $\ell^p(p>1)$, $C[0,1]$, and $L^p(p>1)$. Theorem 3.5 will extend this result to all separable normed spaces. However, in Theorem 3.5 it is assumed that the random elements are weakly uncorrelated.

**Definition 3.4:**

Let $V$ and $Z$ be two random elements in $X$ such that $E[f(V)]^2 < \infty$ and $E[f(Z)]^2 < \infty$ for each $f \in X^*$. If $f(V)$ and $f(Z)$ are uncorrelated random variables for each $f \in X^*$, then $V$ and $Z$ are said to be weakly uncorrelated random elements.

A sequence of random elements is said to be weakly uncorrelated if every pair is.

**Theorem 3.5:**

Let $X$ be a separable normed linear space. Let $\{V_n\}$ be a sequence of weakly uncorrelated, identically distributed random elements in $X$. If $E\|V_1\| < \infty$ and $E(V_1)$ exists, then

$$
\frac{1}{n} \sum_{k=1}^{n} V_k \to EV_1 \text{ in probability.}
$$

**Proof:**

By Horvath ([4], p. 25) $X$ can be regarded as a dense linear subspace of a separable Banach space $\hat{X}$. Hence, $\{V_n\}$ can be regarded as a sequence of weakly uncorrelated, identically distributed random elements in $\hat{X}$. The separable Banach space $\hat{X}$ is isometric to a closed linear subspace $Y$ of $C[0,1]$, that is, there exists a $1-1$, linear function $h$ from $\hat{X}$ onto $Y$ and $\|h(x)\| = \|x\|$ for each $x \in X$. 
It is not hard to verify that \{h(V_n)\} is a sequence of identically distributed random elements in C[0,1], with \(E\|hV_1\| = E\|V_1\| < \infty\). Moreover, for any \(g \in C[0,1]'\) and for any \(m\) and \(n\) \((m \neq n)\)

\[
E[g(hV_n)g(hV_m)] = E[f(V_n)f(V_m)]
\]

\[
= E[f(V_n)]E[f(V_m)] = E[g(hV_n)]E[g(hV_m)]
\]

where \(h^* = f\) and \(h^*\) is the adjoint mapping of \(C[0,1]'\) into \((X)^*\). Hence, the random elements \(\{hV_n\}\) are also weakly uncorrelated.

By Theorem 3.3

\[
h(l/n \sum_{k=1}^{n} V_k) = l/n \sum_{k=1}^{n} hV_k + hEV_1 = EhV_1 \text{ in probability.}
\]

Hence,

\[
l/n \sum_{k=1}^{n} V_k + EV_1 \text{ in probability.} ///
\]

In Theorem 3.3 and Theorem 3.5 it was assumed that the random elements \(\{V_n\}\) were identically distributed. Example 3.6 shows that this condition cannot be relaxed by just imposing bounds on the moments of \(\|V_n\|\).

Moreover, Example 3.6 also shows that Theorems 3.7, 3.8, 3.9, and the strong law given by Mourier ([3] and [4]) for independent, identically distributed random elements \(\{V_n\}\) can not be similarly weakened by imposing bounds on the moments of \(\|V_n\|\).

Example 3.6:

Let \(X = \ell\) and let \(V_n \equiv d^n\). Notice that the \(V_n\)'s are not identically distributed but are independent. Also \(\|V_n\| \equiv 1\). Let \(\{A_n\}\) be an independent sequence of random variables defined by \(A_n = *1\) each with
probability 1/2. Clearly, $A_k A_n$ and $(V_k, V_n)$ are independent for each $k$ and $n$. Moreover, $\{A_n V_n\}$ is an independent sequence of random elements and thus is weakly uncorrelated. But, for each $w \in \mathcal{W}$ and each $n$

$$\left\| \frac{1}{n} \sum_{k=1}^{n} A_k V_k \right\| = \left\| \frac{1}{n}(\pm 1, \pm 1, ..., \pm 1, 0, ...) \right\| = 1,$$

and hence $\frac{1}{n} \sum_{k=1}^{n} A_k V_k$ cannot converge to $0 = E(A_n V_n)$.

The next result is for product sequences of random variables and random elements. The purpose of product sequences is two-fold. First, it provides a way of weakening the condition of identically distributed random elements in Theorems 3.3 and 3.5. This result is given by Theorem 3.9. Secondly, the product of a random variable and a random element provides an easy method of obtaining a random element with a particular distribution. For example, this method was used to construct the random elements in Example 3.6.

**Theorem 3.7:**

Let $X$ be a separable normed linear space and let $\{V_n\}$ be a sequence of identically distributed random elements in $X$ with $E\|V_1\|^2 < \infty$. Let $\{A_n\}$ be a sequence of uncorrelated random variables with zero means and such that

\[
(3.7.1) \quad \frac{1}{n} \sum_{k=1}^{n} E(A_k^2) \leq \tau \quad \text{for all } n
\]

where $\tau > 0$. If

\[
(3.7.2) \quad E[A_k f(V_k) A_n f(V_n)] = E(A_k A_n) E[f(V_k) f(V_n)]
\]
for each \( x^* \) and for each \( k \) and \( n \),

then,

\[
\frac{1}{n} \sum_{k=1}^{n} A_k V_k \rightarrow 0 \text{ in probability.}
\]

**Proof:**

As before, it is sufficient to prove the theorem when \( X \) has a monotone basis \( \{b_n\} \) or when \( X \) is complete and has a Schauder basis \( \{b_n\} \). Also, by choosing \( \Gamma \) large enough in (3.7.1) it can be assumed that

1. \( \frac{1}{n} \sum_{k=1}^{n} [E(A_k^2)]^k \leq \Gamma \) for all \( n \), and

2. \( \frac{1}{n} \sum_{k=1}^{n} E|A_k| \leq \Gamma \) for all \( n \).

Let \( \{x_1, x_2, \ldots\} = D \) and \( \{TV_n\} \) be the same as in the proof of Theorem 3.3.

Finally, let \( e > 0 \) and \( d > 0 \) be given.

First, \( E\|V_1\|^2 < \infty \) implies that

\[
E\|TV_1\|^2 = \sum_{i=1}^{\infty} \|x_i\|^2 P[TV_1 = x_i] < \infty.
\]

Hence, there exists an integer \( j \geq 1 \) such that

3. \( \sum_{i=1}^{\infty} \|x_i\|^2 P[TV_1 = x_i] < [ed/8\Gamma(m+1)]^2 \).

There exists a continuous partial sum operator \( J \) with \( \|J\| \leq m \) such that \( \{x_1, \ldots, x_j\} \subseteq J(X) \). By construction for each \( n \)
\( \left(4\right) \) \( \left| \frac{1}{n} \sum_{k=1}^{n} A_k (V_k - TV_k) \right| \leq \frac{\lambda}{n} \sum_{k=1}^{n} \left| A_k \right| \) and

\( \left(5\right) \) \( \frac{1}{n} \sum_{k=1}^{n} A_k (JV_k - JTV_k) \leq \frac{1}{n} \sum_{k=1}^{n} \left| A_k \right| \| J \| \| V_k - TV_k \| \leq m \lambda / n \sum_{k=1}^{n} \left| A_k \right| . \)

For each \( n \)

\[ A_n V_n = J A_n V_n + Q A_n V_n = A_n J V + A_n Q V \]

where \( Q = I - J \) and \( \| Q \| \leq m + 1 \). By \( \left(2\right) \) and \( \left(4\right) \)

\[ P[\left| \frac{1}{n} \sum_{k=1}^{n} A_k V_k \right| > \epsilon] \leq P[\left| \frac{1}{n} \sum_{k=1}^{n} A_k TV_k \right| > \epsilon/2] + 2 \lambda \Gamma / \epsilon \]

\[ \leq P[\left| \frac{1}{n} \sum_{k=1}^{n} A_k JTV_k \right| > \epsilon/4] + P[\left| \frac{1}{n} \sum_{k=1}^{n} A_k QTV_k \right| > \epsilon/4] + 2 \lambda \Gamma / \epsilon . \]

By using \( \left(2\right) \), \( \left(5\right) \), and finally \( \left(1\right) \) and \( \left(3\right) \)

\[ \leq P[\left| \frac{1}{n} \sum_{k=1}^{n} A_k JV_k \right| > \epsilon/8] + \frac{2}{\epsilon n} \sum_{k=1}^{n} B(\| A_k \| \| QTV_k \|) + \frac{(2 + 8m) \lambda \Gamma}{\epsilon} \]

\[ \leq P[\left| \frac{1}{n} \sum_{k=1}^{n} A_k JV_k \right| > \epsilon/8] + \frac{(2 + 8m) \lambda \Gamma}{\epsilon} + d/2 . \]

Using \( \left(3.7.2\right) \) of the hypothesis, the appropriately chosen \( \lambda = \frac{ed}{4(8m + 2)d} \),

and the method of proof in Theorem 3.3, on \( N(e, d) \) can be found such that

\[ P[\left| \frac{1}{n} \sum_{k=1}^{n} A_k JV_k \right| > \epsilon/8] < d/4 \text{ for } n \geq N(e, d) . \]

Hence,

\[ \frac{1}{n} \sum_{k=1}^{n} A_k V_k \to 0 \text{ in probability.} // / \]
Remark:

If \( X \) has a monotone basis or if \( X \) has a Schauder basis and is complete, then it is sufficient in Theorem 3.7 that (3.7.1) hold only when \( f \) is a coordinate functional. On the other hand, by assuming that \( A_n^k \) is independent of \( (V_n^k, V_n^{k'}) \) for all \( k \) and \( n \), other hypotheses in Theorem 3.7 can be weakened. This is formally stated in Theorem 3.8.

**Theorem 3.8:**

Let \( X \) be separable and let \( \{V_n\} \) be a sequence of identically distributed random elements in \( X \) with \( \mathbb{E}\|V_1\| < \infty \). Let \( \{A_n\} \) be a sequence of uncorrelated random variables with zero means.

Also, let

\[
(3.8.1) \quad \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}|A_k| \leq \Gamma \text{ for all } n
\]

where \( \Gamma > 0 \) and let

\[
(3.8.2) \quad \frac{1}{n^2} \sum_{k=1}^{n} \mathbb{E}(A_k^2) \to 0 \text{ as } n \to \infty.
\]

If

\[
(3.8.3) \quad A_n^k \text{ is independent of } (V_n^k, V_n^{k'}) \text{ for all } k \text{ and } n,
\]

then

\[
\frac{1}{n} \sum_{k=1}^{n} A_n^k V_n^k \to 0 \text{ in probability.}
\]

Theorems 3.7 and 3.8 state that the convergence of \( \frac{1}{n} \sum_{k=1}^{n} A_n^k V_n^k \) is dictated by the convergence of \( \frac{1}{n} \sum_{k=1}^{n} A_k V_k \) when the sequences \( \{A_n\} \) and \( \{V_n\} \) are either uncorrelated or independent. The following paragraph is an illustrative example.
Let $X = C[0,1]$ and let $\{V_n\}$ be a sequence of separable Wiener processes on $[0,1]$ with the same parameter $\sigma^2$. With probability 1 $V_n \in C[0,1]$, and $\{V_n\}$ can be regarded as a sequence of identically distributed random elements in $C[0,1]$. It is easy to verify that $\mathbb{E}\|V_n\|^2 < \infty$. Let $\{A_n\}$ be a sequence of uncorrelated variables with zero means and which satisfy conditions (3.7.1) and (3.7.2) of Theorem 3.7 or conditions (3.8.1), (3.8.2) and (3.8.3) in Theorem 3.8. Then,

$$1/n \sum_{k=1}^{n} A_k V_k$$

converges in probability to the zero function on $[0,1]$.

Example 3.6 shows that the hypothesis of identically random elements $\{V_n\}$ cannot be replaced by bounds on the moments of $\{\|V_n\|\}$ in any of the theorems in Section 3. The last result (Theorem 3.9) shows that the condition of identically distributed random elements can be slightly weakened in the weak law of large numbers (Theorem 3.5).

**Theorem 3.9:**

Let $X$ be a separable normed linear space and let $\{V_n\}$ be a sequence of identically distributed random elements in $X$ with

$$\text{(3.9.1) } \mathbb{E}\|V_1\|^2 < \infty.$$ 

Let $\{A_n\}$ be a sequence of random variables such that

$$\text{(3.9.2) } 1/n \sum_{k=1}^{n} \mathbb{E}(A_k^2) \leq \Gamma \text{ for all } n$$

where $\Gamma > 0$. If the sequence $\{A_n V_n\}$ is weakly uncorrelated and

$$\text{(3.9.3) } \mathbb{E}(A_n V_n) = 0 \text{ for all } n,$$
then

\[ \frac{1}{n} \sum_{k=1}^{n} A_{k}V_{k} \to 0 \text{ in probability}. \]

**Proof:**

Again it will suffice to prove the theorem when \( X \) has a monotone basis or \( X \) is a Banach space with a Schauder basis. Let \( e > 0 \) and \( d > 0 \) be given. Let \( D = \{x_1, x_2, \ldots\} \) and \( \{TV_n\} \) be the same set and random elements that were defined in the proof of Theorem 3.3 where

\[ \|TV_n - V_n\| \leq \lambda \text{ for each } w \in W \text{ and each } n. \]

Moreover, \( \lambda \) can be chosen arbitrary small with respect to \( e, d, \Gamma, \text{ etc.} \)

Choose \( j \) so that for each \( n \)

\[ (1) \ (m + 1)1/n \sum_{k=1}^{n} \left[ E(A_k^2) \right]^{1/2} \left[ \sum_{i=j+1}^{\infty} \|x_i\| \right]^{2} P[TV_1 = x_i] \leq e d / 16. \]

Let \( J \) be a continuous partial sum operator on \( X \) such that \( \{x_1, \ldots, x_j\} \subseteq \{J(x) \} \)

and \( \|J\| \leq m. \)

\[ (2) \ \|1/n \sum_{k=1}^{n} A_{k}V_{k} - 1/n \sum_{k=1}^{n} A_{k}TV_{k}\| \leq \lambda/n \sum_{k=1}^{n} |A_{k}|. \]

\[ (3) \ \|1/n \sum_{k=1}^{n} A_{k}JV_{k} - 1/n \sum_{k=1}^{n} A_{k}JTV_{k}\| \]

\[ \leq 1/n \sum_{k=1}^{n} |A_{k}| \|J\| \|TV_{k} - V_{k}\| \leq m \lambda/n \sum_{k=1}^{n} |A_{k}|. \]

Using (1), (2), (3), the bounds on \( 1/n \sum_{k=1}^{n} E(A_{k}^2) \) and \( 1/n \sum_{k=1}^{n} E(A_{k}^2) \), and the appropriately chosen \( \lambda > 0, \)

\[ \]
\[ P(\|1/n \sum_{k=1}^{n} A_k V_k\| > e) \leq P(\|1/n \sum_{k=1}^{n} A_k J Y_k\| > e/\delta) + 3d/4. \]

The rest of the proof follows from the proofs of Theorems 3.3 and 3.7. ///

The following discussion indicates how Theorem 3.9 weakens the assumption of identically distributed random elements. The sequence of random elements \( \{Z_n\} \) must be weakly uncorrelated. Moreover, each random element \( Z_n \) must be expressible as \( A_n V_n \) where \( A_n \) is a random variable, \( V_n \) is a random element, and the sequence \( \{V_n\} \) is identically distributed. Additional assumptions are (3.9.1), (3.9.2), and (3.9.3). Thus by Theorem 3.9

\[ 1/n \sum_{k=1}^{n} Z_k \rightarrow 0 \text{ in probability.} \]
REFERENCES


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13. Abstract
Some weak laws of large numbers for random elements in \textit{normed} linear spaces are given in this paper. The main result is a weak law of large numbers for weakly uncorrelated, identically distributed random elements in separable normed linear spaces.

14. Key Works

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- Random Elements
- Normed Linear Spaces
- Pettis Integral
- Schauder Basis
- Coordinate Functionals
- Partial Sum Operators
- Isometric