STATIONARY STOCHASTIC POINT PROCESSES. I

by Klaus Matthes

ON THE STRUCTURE OF STATIONARY POINT PROCESSES

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PREFACE

Following are translations of two papers which stand as elegant and illuminating contributions to the foundations of point process theory. They provide important background and collateral reading relative to recent papers such as Yu. K. Belayev's "Elements of the General Theory of Random Streams of Events" (translated by M. R. Leadbetter, University of North Carolina, Institute of Statistics Mimeo Series No. 703) and M. R. Leadbetter's "On Basic Results of Point Process Theory" (University of North Carolina, Institute of Statistics Mimeo Series No. 686). I am indebted to Mrs. Linda Evans for a fine debut at mathematical typing.

R. J. S.
For a long time stationary stochastic point processes have played a role in the applications of probability calculus. Here will be mentioned only the theory of excitations in electron tubes [20] and the theory of service [4]. The first steps toward a general theory, such as the general formulation of the "Campbell Theorems" [6] and particularly the results established by Palm and Khintchine [4] [23] on the superposition of independent stationary stochastic point processes, have become only in the last year newly taken up and systematically developed [18] [17] [10]. Essentially only the recurrent stochastic point processes [7] [19] [5], especially the Poissonian, have previously been examined in detail.

Although the general theory of stationary stochastic point processes is still very imperfect, there nevertheless already lies before it a series of applications [13] [12] [11]. The following sections shall provide an introduction to this theory.

1. *Fundamentals*

In the introduction of the notion of a stochastic marked point process, let us be guided by the idea of chance events occurring in the points of a random denumerable subset of the time axis $\mathbb{R}^1$. Thus, for example, electrons become emitted with random speeds at random time points. In a telephone

exchange, calls arrive at random time points and possess random potential durations.

In both examples, the random events become described through random real numbers, i.e., through random elements of the measurable space \([\mathbb{R}^1, B_1]\). In the applications, however, other measurable spaces may arise, such as, e.g., if randomly formed impulses, i.e., random real functions, become selected at random instants. In the next section it will become apparent that even simple theoretical inquiries lead to complicated measurable spaces. For this reason we will assume merely that the events may be conceived as random elements of some fixed measurable space \([\mathbb{K}, \mathbb{K}])

Let the fact that at time \(x\) an event occurs, by which the element \(k\) from \(K\) becomes realized, be designated by the statement of an \([x, k]\) from \(\mathbb{R}^1 \times K\). We call \([x, k]\) a marked point with the position \(x\) and the mark \(k\). A denumerable subset \(\phi\) of \(\mathbb{R}^1 \times K\) is called a marked point sequence if it satisfies the following conditions:

a) \(\phi\) contains no two differently marked points with the same position.

b) The positions of the marked points from \(\phi\) possess no limit points in \(\mathbb{R}^1\).

Let us now complete the collection \(M_K\) of all marked point sequences to a measurable space \([M_K, M_K]\), and indeed let us [10] put for \(H_K\) the smallest \(\sigma\)-algebra rendering measurable the mapping

\[
\phi \longrightarrow \text{Card}(\phi \cap Z)
\]

for all direct products \(Z = I \times L\) of bounded intervals \(I\) with sets \(L\) from \(K\) (and accordingly for all \(Z\) from the direct product of the \(\sigma\)-algebra \(B_1\) of Borel sets of \(\mathbb{R}^1\) with the \(\sigma\)-algebra \(\mathbb{K}\)). A probability space of the form \([M_K, M_K, P]\) will be called a stochastic marked point process (=smpp).

We number the elements of a marked point sequence according to the sequence
of their positions, always with the convention that the smallest positive position receives the null index. Thus $\phi = \{(x_i, k_i)\}_{i=-1}^{i<n} \text{ with } x_r < x_s \text{ for } r < s \text{ and } 0 \leq m, m \leq +\infty$. If either $n$ or $m$ is null, then the positions of all the marked points of $\phi$ lie entirely in $(-\infty, 0]$ or entirely in $(0, +\infty)$.

Also, with this representation the marked point sequence satisfies the above-stated definition of $"K"$, on account of

1.1. $"K"$ is the smallest $\sigma$-algebra which, for all natural $r, s$, all finite sequences $I_{-r}, \ldots, I_s$ of intervals as well as all finite sequences $L_{-r}, \ldots, L_s$ of sets from $K$, contains the set

$$\{\phi : n > s, m \geq r \text{ and } (x_i, k_i) \in I_i \times L_i \text{ for } -r \leq i \leq s\}.$$ 

The statement of the mark becomes superfluous when $K$ possesses only one element. In this case we can interpret $\phi$ as a countable subset of $R^1$. The universe of all these $\phi$ we denote by $\mathcal{M}$ and the corresponding $\sigma$-algebra by $\mathcal{M}$. A probability space $[\mathcal{M}, \mathcal{M}, \mathbb{P}]$ will be called a stochastic point process ($\approx$ spp).

To each spp $[\mathcal{M}, \mathcal{M}, \mathbb{P}]$ and each probability law $\mathbb{P}$ on a $\sigma$-algebra $K$ of subsets of $K$ one can associate a spp $[\mathcal{M}_K, \mathcal{M}_K, \mathbb{Q}]$ as follows: Let $\phi = \{x_i\}_{-m}^{n} \text{ be distributed according to } \mathbb{P}$. In addition let a doubly infinite sequence of random elements $k_i$ of $[K, K]$, each distributed according to $\mathbb{P}$, be given. This sequence $(x_i)_{-\infty}^{+\infty} \text{ is independent of } \phi \text{ as well as within itself}$. Then the sequence

$$\psi = \{(x_i, k_i)\}_{-m}^{n} \text{ is } \mathbb{P}$$
depends measurably on \([\phi, (k_i)]\) and consequently defines an smpp. In this case we speak of independent marking. In the example cited at the beginning of this section, it may be assumed with good accuracy that independent marking holds, since the emission of an electron does not alter practically the condition of the cathode. The corresponding is true also for the second example. We now introduce an interpretation of the smpp, in which the independent marking forms a special case: The position \(x_i\) of the element from \(\phi\) gives the timepoint at which an object \(0_i\) with random positive lifetime (such as a lightbulb) dies off and without loss of time becomes replaced by an object \(0_{i+1}\). The mark \(k_i\) should contain statements about the fixed operating characteristics of \(0_{i+1}\). It is clear that in general \(k_i\) will depend upon the lifetime \(x_{i+1} - x_i\) of \(0_{i+1}\).

For all real \(t\) we define on \(M_K\) the shift operator \(T_t\) by the rule

\[[x, k] \in T_t(\phi) \quad \text{if and only if} \quad [x + t, k] \in \phi.\]

\(T_t\) is always a measurable transformation of \(M_K\) onto itself. An smpp is called stationary if its probability law \(P\) is carried into itself by all \(T_t\). As usual we call a stationary smpp \([\mu_K, \mu'_K, P]\) ergodic if it satisfies the following condition:

e) For any \(S \in M_K\) such that \(P((S \cap T_t(S)) \cup (\overline{S} \cap T_t(S))) = 0\), all \(t,\) \(P(S)\) equals zero or one.

If the even stronger property

m) For all \(S_1, S_2\) from \(M_K\), \(P(S_1 \cap T_t(S_2)) \rightarrow P(S_1)P(S_2)\) as \(t \rightarrow \infty\),

holds, then the stationary smpp is called mixing.
An independently marked smpp is then stationary, stationary and ergodic, or stationary and mixing if and only if the same is true for the associated spp.

The collection of all \( \phi \) for which \( n = m = + \infty \) forms a measurable subset \( M_K^c \) of \( M_K \) which is invariant under all \( T_t \). We put \( M_K^c = M_K^c \cap M_K \). In the study of stationary smpp's, one can often work with the simpler measurable space \( [M_K^c, M_K^c] \). Firstly, in the case of stationary \( P \) with the property \( P(M_K^c) > 0 \), the conditional distribution \( P(\cdot | M_K^c) \) is always stationary also.

Moreover, the following holds [17]:

1.2. For all stationary distributions \( P \) on \( M_K \),

\[
P(\phi \in M_K^c \text{ or } \phi = \emptyset) = 1.
\]

Consider a random vector \([\phi_1, \ldots, \phi_r]\) of marked point sequences with a common mark space \([K, K]\). Suppose that the probability is zero that elements of the \( \phi_i \) possess the same position. Then the mapping of \([\phi_1, \ldots, \phi_r]\)

to \( \bigcup_{i=1}^{r} \phi_i \) is a measurable transformation defined almost everywhere. The superposition \( \bigcup_{i=1}^{r} \phi_i \) of the \( \phi_i \) is thus, in this case, again an smpp.

In the following, the main line of interest is the special case in which the components \( \phi_i \) are independent. Then we call the probability law \( Q \)

do, the convolution of the laws \( P_i \) and write \( Q = P_1 \ast \cdots \ast P_r \). The operation of convolution is associative and commutative. The role of unit element is played by the distribution \( \delta_\emptyset \) characterized by \( \delta_\emptyset(\phi = \emptyset) = 1 \).

Clearly the convolution of stationary \( P_i \) is always well-defined and stationary. The convolution of distributions \( P_i \) is well-defined indeed under the weaker assumption that the \( P_i \) are stationary with at most one exception.
We call a sequence of distributions $P_k$ on $\mathcal{M}_K$ weakly convergent — in symbols $P_k \Rightarrow P$ — if the following condition is satisfied:

k) For each finite sequence $I_1, \ldots, I_L$ of bounded open intervals $I_i = (a_i, b_i)$ with the property

$$P(\phi \cap \{a_1, \ldots, a_L, b_1, \ldots, b_L\} \times K) \neq \emptyset) = 0,$$

and each finite sequence $L_1, \ldots, L_L$ of sets from $K$, the associated $P_k$-induced distribution of

$$[\text{Card}(\phi \cap (I_1 \times L_1)), \ldots, \text{Card}(\phi \cap (I_L \times L_L))]$$

tends to the associated $P$-induced distribution of this vector.

For all $P$ the set of $c$ with the property

$$P(\phi \cap \{c\} \times K) \neq \emptyset) > 0$$

is denumerable. For stationary $P$ it is even empty. Therefore the limit $P$ of a weakly convergent sequence of distributions is uniquely determined. Clearly it is stationary when all $P_k$ are. We have

1.3. If $P_k \Rightarrow P$ and $Q_k \Rightarrow Q$, then $(P_k * Q_k) \Rightarrow P * Q$, when the indicated convolutions are meaningful.

2. Intensity and Palm's Distribution

By the intensity $\rho$ of a stationary spp $\{\cdot, \cdot, F\}$ we mean the expectation of $\text{Card}(\phi \cap (0,1))$. Then, for all Borel sets $B$,

2.1. $E[\text{Card}(\phi \cap B)] = \rho \mu(B)$,
where \( \mu(B) \) denotes the Lebesgue measure of \( B \) and \( 0^-, \omega^*0 \) are set equal to zero.

Now let \( \rho < \infty \). On the basis of theorems of Dobrushin [21] and Korolyuk [4], § II, or [24], there follows:

2.2. \[ \sum_{L=2}^{\infty} \ell P(\text{Card}(\Phi \cap (0,t)) = \ell) = o(t) \quad \text{as } t \to 0^+. \]

Then also

\[
\rho = \lim_{t \to 0^+} \frac{1}{t} E[\text{Card}(\Phi \cap (0,t))]
\]

\[
= \lim_{t \to 0^+} \frac{1}{t} P(\Phi \cap (0,t) \neq \emptyset)
\]

\[
= \lim_{t \to 0^+} \frac{1}{t} P(\text{Card}(\Phi \cap (0,t)) = 1),
\]

so that the intensity \( \rho \) can be interpreted as a "probability density" for the occurrence of an \( x \) from \( \Phi \).

For every stationary smpp \( [M_K, M_K', P] \) and each \( L \) from \( K \), the measurable transformation

\[ \Phi \to \Phi_L = \{ x : \text{ There is an } [x,k] \in \Phi \text{ with } k \in L \} \]

of \( [M_K, M_K'] \) into \( [M, M'] \) produces a stationary distribution on \( M \). The intensity of \( \Phi_L \) will be denoted by \( \lambda(L) \). We have

2.3. \( \lambda \) is a measure on \( K \).

By the intensity \( \rho \) of \( [M_K, M_K', Q] \) we mean the value \( \lambda(K) \). If \( 0 < \rho < +\infty \), then \( \frac{1}{\rho} \lambda \) constitutes a distribution on \( K \): \( \frac{\lambda(L)}{\rho} \) gives the average proportion of marked points of \( \Phi \) whose marks lie in \( L \). Often this number is called the "probability that the mark of a point in \( \Phi \) lies in \( L \)." This characterization.
is justified by the fact that the limit of the relative frequency of events
$k_i \in L$ as $i \to \infty$ equals $\frac{\lambda(L)}{\rho}$ with probability one, if the smpp is ergodic. *

Now consider a stationary smpp with distribution P. We select an $L \in K$
for which $\Phi_L$ lies in $M^*$ with probability one. Then also almost all $\Phi$ belong
to $M^*_K$ so that $P$ may be regarded as a stationary distribution on $M^*_K$.

With the selection of $L$ one may associate the notion that only at the
points of $\Phi$ with marks in $L$ does an object die out and become replaced, in
comparison with which the other points of $\Phi$ indicate merely how the condition
of being the object found in operation at the moment alters haphazardly.

Let $\Phi$ be an element of $M^*_K$ with the property $\Phi_L \in M^*$ and let $x \in \Phi_L$. We
now replace the mark $k$ by $T_X(\Phi)$, i.e., by the statement of the past, present
and future of $[x,k]$ in $\Phi$. We pass thereby to a new mark space $[M^*_K, K^*] = [M^*_K, M^*_K]$.
The mapping to $[(x, T_X(\Phi)); (x) \times L) \cap \Phi \neq \emptyset]$ affects a measurable transformation
of almost all $\Phi$ from $[M^*_K, M^*_K]$ into the measurable space $[M^*_K, M^*_K]$ and
produce thus a new smpp which obviously is also stationary and possesses
the intensity $\rho^* = \lambda(L)$.

* Remark by the referee:

If one assumes only that $t^{-1}\text{Card}(\Phi \cap ((0,t) \times K))$ tends almost surely
with increasing $t$ to a quantity $\eta_K$ positive and finite with probability 1,
then all ratios $t^{-1}\text{Card}(\Phi \cap ((0,t) \times L))$ converge almost surely to quantities
$\eta_L$ nonnegative and finite with probability 1. We then are able to have

$$\nu(L) = \mathbb{E}\left( \frac{N(t)}{\eta_L} \right)$$

a distribution on $K$. If the intensity $\rho$ is finite and $\eta_K$ constant with probability
one, then $\nu$ coincides with the distribution $\frac{1}{\rho}$. In case $\rho$ is finite and $\eta_K$ not
constant, the two distributions are absolutely continuous relative to each other.
However, the distribution $\nu$ can also be meaningful even if $\rho = +\infty$. 
We now wish to assume in addition that $\rho^* < \infty$. Then we can form the distribution $P_L = \frac{1}{\rho^*} \lambda^*$. This is called the Palm distribution of $\Phi$ with respect to $L$ [17] [18] [13]. In the special case $L = K$ as well as for stationary spp’s, we speak of the Palm distribution and denote it by $P_0$. Like all intensities the value $P_L(S)$ for all $S \in M_K^+$ can be characterized locally. For this we denote by $x_L$ the position of the first point of $\Phi$ with nonnegative index and mark lying in $L$ and set $\Phi^L = \text{Tr}_{x_L}(\Phi)$. For the distribution function $p^{t,L}$ of $\Phi^L$ induced by the condition $x_L < t$, we have [13]:

2.4. $\text{Var}(P_L - p^{t,L}) = o(1)$, $t \to 0^+$. 

Here $\text{Var}(P_L - p^{t,L})$ denotes the total variation of $(P_L - p^{t,L})$, i.e.

$$\int_{M_K} |P_L(d\Phi) - p^{t,L}(d\Phi)|.$$ 

On the basis of 2.4 $P_0$ often is interpreted as the 'conditional distribution of $\Phi$ under the condition $\Phi \cap (\{0\} \times K) \neq \emptyset$.' This idea formed the point of departure for the introduction of $P_0$ by Palm. However, it is not precise and conceals the true content and formal meaning of $P_0$. It does not show, apropos to a view of $\xi_0 = \Phi \cap (\{0\} \times K) \neq \emptyset$ as a special hypothesis having zero probability among a range of hypotheses $\xi$, that the relation $P_0 = P(\cdot | \xi = \xi_0)$ holds.

All $\Phi$ of $M_K^+$ for which $\Phi_L$ lies in $M^+$ may be represented by the vector $[x_L, \Phi^L]$. By 2.4 the conditional distribution of the two components under the condition $x_L < t$ is asymptotically $P_L$. Moreover [14] [13]:

2.5. As $t \to 0^+$ the variation of the difference of the conditional distribution of $[x_L, \Phi^L]$ given $x_L < t$ and the direct product of the uniform distribution on $(0, t)$ with $P_L$ tends to 0.
So far we have regarded $P_L$ as a distribution on $\mathcal{M}_K^\infty$. Directly from the definition of $P_L$ it follows that with probability one $\Phi_L \in \mathcal{M}^\infty$ and 0 is the position of a point of $\Phi$ with mark in $L$. We denote by $\mathcal{M}_L^L$ the collection of all $\Phi$ satisfying both requirements and we set $\mathcal{M}_L^L = \mathcal{M}_L^L \cap \mathcal{M}_K^\infty$. $P_L$ can always be regarded as a distribution on $\mathcal{M}_L^L$.

Over $\Phi \in \mathcal{M}_L^L$ we define the operator $\Theta$ by the equation $\Theta(\Phi) = T_{x_L}(\Phi)$, obtaining thus a measurable transformation of $[\mathcal{M}_L^L, \mathcal{M}_L^L]$ onto itself. We have

2.6. The measurable transformation $\Theta$ carries $P_L$ into itself. The distribution of the past, present and future of a "given" point of $\Phi$ with mark in $L$ agrees exactly with the distribution of these quantities for the points of $\Phi \cap (R_1^L \times L)$ following a "given" point of $\Phi \cap (R_1^L \times L)$: For all positive $t$ and all $V \in \mathcal{M}_L^L$,

$$|\text{Card}\{x: (\{x\} \times L) \cap \Phi \neq \emptyset \text{ and } T_x(\Phi) \in V\} - \text{Card}\{y: (\{y\} \times L) \in \Phi \text{ and } T_z(\Phi) \in V \text{ for points } z \text{ of } P_L \text{ following } y\}| \leq 2.$$

$P$ may be expressed in terms of $P_L$ by a simple formula. Namely, for all $S$ from $\mathcal{M}_K^\infty$, we have [17] [18] [13]:

2.7. $P(S) = \lambda(L) \int \int k_S(T_1(\Phi))d\tau P_L(d\Phi)$.

Here $k_S$ denotes the characteristic function of $S$.

Considering the great importance of this formula, we wish to look into its proof.

Let us denote the points of $\Phi_L$, numbered in the usual way, by $y_1$ or $y_1(\Phi)$, so that in particular $x_L = y_0$. Further let us set $\text{Card}(\Phi_L \cap (0,t)) = n(t)$. On the basis of the definitions of the intensity and the Palm distribution,
we have, for all positive $t$ and all $\phi \in \mathbb{H}_K$,

\begin{equation}
\lambda(L) P_L(\phi) = \int \left[ \frac{1}{t} \sum_{i=0}^{n(t)-1} k_{H_i}(T_{Y_i}(\phi)) \right] P(d\phi).
\end{equation}

Thus each real function $f$ on $\mathbb{H}_K$ which is measurable with respect to $\mathcal{M}_K$ and integrable with respect to $P_L$ satisfies the relation

\begin{equation}
\lambda(L) \int f(\phi) P_L(d\phi) = \int \left[ \frac{1}{t} \sum_{i=0}^{n(t)-1} f(T_{Y_i}(\phi)) \right] P(d\phi).
\end{equation}

Therefore,

\begin{equation}
\lambda(L) \int \left[ \int_0^X k_S(T_{t}(\phi)) d\tau \right] P_L(d\phi)
= \int \left[ \frac{1}{t} \sum_{i=0}^{n(t)-1} \int_{Y_i}^{Y_{i+1}} k_S(T_{t}(\phi)) d\tau \right] P(d\phi)
= \int \left[ \frac{1}{t} \right] \int_0^t k_S(T_{t}(\phi)) d\tau \right] P(d\phi)
+ \int \frac{1}{t} \left[ \int t^{Y_{n(t)}(\phi)} k_S(T_{t}(\phi)) d\tau - \int_0^{Y_{n(t)}(\phi)} k_S(T_{t}(\phi)) d\tau \right] P(d\phi).
\end{equation}

Now

\begin{equation}
\int t^{Y_{n(t)}(\phi)} k_S(T_{t}(\phi)) d\tau = \int_0^{Y_{n(t)}(\phi)} k_S(T_{t}(\phi)) d\tau
\end{equation}

and so also, by stationarity of $P$,

\begin{equation}
\int \int t^{Y_{n(t)}(\phi)} k_S(T_{t}(\phi)) d\tau \right] P(d\phi) = \int \int_0^{Y_{n(t)}(\phi)} k_S(T_{t}(\phi)) d\tau \right] P(d\phi).
\end{equation}

From (3) and (5) we obtain

\begin{equation}
\lambda(L) \int \left[ \int_0^X k_S(T_{t}(\phi)) d\tau \right] P_L(d\phi)
= \int \left[ \frac{1}{t} \int_0^t k_S(T_{t}(\phi)) d\tau \right] P(d\phi).
\end{equation}

The right side of (6) is again $P(S)$. 

Thus we have verified 2.7. For this we required only simple, clearly valid relations which follow routinely from the definition of \( P_L \) given above.

Let us denote by \( F \) the left-continuous distribution function of \( x_L \) relative to \( P_L \). From 2.7 with \( S = M_\mathcal{K} \), it follows that \( \Delta = \int_0^\infty x dF(x) \) agrees with the reciprocal of \( \lambda(L) \).

Let \( z_L \) denote the largest nonpositive element of \( \phi_L \), i.e., the position of the element of \( \phi \) with mark in \( L \) and with largest negative index. Utilizing \( F \) we can relate the distribution of \([z_L, x_L]\) to \( P \) (see [9]): A simple transformation in 2.7 yields

2.8. For all positive \( u, v \)

\[
(x) \quad P(-z_L > u, x_L > v) = \int_0^\infty \frac{1-F(x)}{u+v} dx
\]

\[
(y) \quad P(x_L - z_L > u) = \frac{1}{\Delta} \int_0^\infty x dF(x)
\]

\[
(z) \quad P(-z_L > u) = P(x_L > u) = \int_0^\infty \frac{1-F(x)}{u} dx.
\]

In particular the expected value relative to \( P \) of \( x_L - z_L \) is thus

\[\Delta + \frac{\sigma^2}{\Delta}, \text{ with } \sigma^2 = \int_0^\infty (x-\Delta)^2 dF(x).\]

The foregoing ideas serve also to characterize the Palm distributions, for [17] [18]:

2.9 A distribution \( Q \) on \([M_L, \mathcal{L}]\) is a Palm distribution \( P_L \) if and only if it is invariant with respect to \( \theta \) and \( \Delta = \int x_L(\theta) Q(d\theta) < +\infty \) holds.

Now let \( F \) be any left-continuous distribution function with the properties \( F(0+) = 0, \Delta = \int_0^\infty x dF(x) < +\infty \) and let \( (\xi_i)_{i=-\infty}^1 < \xi_i < +\infty \) be a doubly infinite sequence of positive random variables independently distributed according to \( F \). Then the spp
\[\{\ldots, - (\xi_n + \ldots + \xi_1), \ldots, - (\xi_2 + \ldots + \xi_1), - \xi_1, 0, \xi_0, (\xi_0 + \xi_1), \ldots, (\xi_0 + \xi_n), \ldots\}\]

satisfies the conditions given in 2.9 and accordingly represents the Palm distribution of a well-defined stationary snpp \([M^+, M^-, P_F]\). The distribution of \(x_0\) under \((P_F)_0\) coincides with \(F\). Thus, in particular, \(P_F\) possesses the intensity \(\lambda^{-1}\). We call \(P_F\) the stationary recurrent snpp belonging to \(F\).

By the one-to-one transformation of \(P\) to \(P_L\), mixtures \(\sum_{i=1}^{n} \alpha_i P_i\) are carried into mixtures \(\sum_{i=1}^{n} \frac{\alpha_i \rho_i^*}{\rho_1^*} (P_i)_L\), where \(\rho_i^*\) is the intensity of \(L\) relative to \(P_i\) and

\[\rho^* = \sum_{i=1}^{n} \alpha_i \rho_i^*\]. Then [17]:

2.10. The Palm distribution \(P_L\) is ergodic relative to the shift operators \(\theta\) if and only if \(P\) is ergodic relative to the \(T_L\).

The Palm distributions \((P_F)_0\) constructed above are each mixing relative to \(\theta\). Therefore all \(P_F\) are ergodic. However the \(P_K\) are mixing only under certain restrictions [14]:

2.11. \(P_F\) is mixing if and only if for all positive \(d\) the inequality

\[\sum_{n=1}^{\infty} (F(nd^+) - F(nd)) < 1\]

is satisfied.

To conclude this section we give another simple relation, concerning the Palm distribution of a convolution.

2.12. Let \(P_1, P_2\) be stationary snpp's with the same mark space \([K,K]\).

If the Palm distributions \((P_1)_L, (P_2)_L\) are defined, then so is \((P_1 \ast P_2)_L\), and we have

\[\frac{\lambda_1(L)}{\lambda_1(L) + \lambda_2(L)} (P_1)_L \ast (P_2)_L + \frac{\lambda_2(L)}{\lambda_1(L) + \lambda_2(L)} (P_1 \ast (P_2)_L)_L\]
3. **Stationary Poisson point processes**

For all nonnegative \( \lambda \) we define the stationary Poisson spp \( \{ M, \lambda, P_\lambda \} \) by setting

\[
P_\lambda = \begin{cases} 
\text{the distribution } P_\lambda \text{ belonging to } F_\lambda(x) = 1 - e^{-\lambda x} (x \geq 0), & \text{if } \lambda > 0, \\
\delta_{\emptyset}, & \text{if } \lambda = 0.
\end{cases}
\]

As is well-known [16], a positive random variable \( \xi \) is distributed according to the distribution \( P_\lambda \) if and only if for all positive \( t \) the conditional distribution of \( (\xi - t) \), given \( \xi > t \), agrees with the unconditional distribution of \( \xi \). The corresponding property of \( P_\lambda \) may be described as follows: For all finite sequences \( B_1, \ldots, B_n \) of bounded pairwise disjoint Borel sets, the random vector \( [\text{Card}(\phi \cap B_1), \ldots, \text{Card}(\phi \cap B_n)] \) has independent components. The distribution of \( \text{Card}(\phi \cap B_1) \) is then Poisson with mean \( \lambda \mu(B_1) \). Thus in particular \( P_\lambda \) has the intensity \( \lambda \).

The independence property formulated above is characteristic* of the \( P_\lambda \):

3.1. A stationary spp is Poisson if and only if for all finite sequences \( I_1, \ldots, I_n \) of bounded pairwise disjoint intervals the random variables \( \text{Card}(\phi \cap I_1), \ldots, \text{Card}(\phi \cap I_n) \) are independent.

The stationary spp consisting of the timepoints at which a cathode emits an electron may with reasonable validity be regarded as Poisson: The emission of an electron leaves (as we have already asserted) the condition of the cathode practically unchanged, so that the independence property formulated in 3.1 is approximately valid.

*This is only one of several formulations of the familiar characterization of the homogeneous Poisson process ([1], chap.
For all finite sequences $B_1, \ldots, B_n$ of bounded Borel sets and all finite sequences $m_1, \ldots, m_n$ of nonnegative integers, the probability $P_\lambda(\text{Card}(\emptyset \cap B_i) = m_i$ for $1 \leq i \leq n$) depends functionally upon $\lambda$. The same then holds also for all subsets of $M$. For every distribution $p$ of a nonnegative random variable we can thus, via

$$P(S) = \int_0^\infty P_\lambda(S)p(d\lambda),$$

introduce the mixture $P = \int_0^\infty P_\lambda(d\lambda)$.

We now consider a characterization of the stationary Poisson spp which makes its extremely intuitive character easy to grasp.

For each natural $n$ and each bounded nonempty interval $I$ let us introduce on $M$ a distribution which corresponds to the idea of a "purely random $n$-element sequence of points from $I$": We start from an independent sequence $\omega_1, \ldots, \omega_n$ of identically distributed random variables on $I$ and to each realization $\varrho_1, \ldots, \varrho_n$ we associate the element $\{\varrho_1, \ldots, \varrho_n\}$ from $M$. Analogously we call a stationary spp $[M, M, P]$ purely random if for all $I$ and all nonnegative $n$ satisfying the condition $P(\text{Card}(\emptyset \cap I) = n) > 0$, the conditional distribution of $\emptyset \cap I$ given that $\text{Card}(\emptyset \cap I) = n$ is purely random. In [15] it is shown that:

3.2. A stationary distribution $P$ on $M$ is purely random if and only if it has the form $\int_0^\infty P_\lambda p(d\lambda)$.

In particular a stationary spp possesses the distribution $P_\lambda$ if and only if it is purely random and $\frac{1}{t}\text{Card}(\emptyset \cap (0, t))$ tends in probability to $\lambda$ with increasing $t$.

The purely random stationary spp can also be characterized by other "mixture properties."
Let $P$ be any stationary distribution on $M$ and $\alpha \in (0,1)$. We now pass to the associated independently marked stationary smpp with mark space $K = \{0,1\}$ and distribution $p = \alpha \delta_1 + (1-\alpha)\delta_0$ for the marks. To each $\Phi$ from $\mathcal{M}_K$ we associate $\Phi_\{1\}$, i.e., we strike out all elements with the mark 0. Next we contract the time axis and transform to $\Phi^* = \{y : \frac{y}{\alpha} \in \Phi_\{1\}\}$. We denote the stationary distribution of $\Phi^*$ by $[P]_\alpha$. Using 3.2 one finds easily that for all purely random $P$ the equation $P = [P]_\alpha$ is satisfied. But also the converse holds [15] [3]:

3.3. A stationary smpp $[M, M, P]$ is purely random if and only if for each $\alpha \in (0,1)$ the equation $P = [P]_\alpha$ is satisfied.

We introduce yet another fundamental property of the purely random $P$.

Again let $P$ be a stationary distribution on $M$ and $p$ the distribution of any random variable. We transform to the corresponding independently marked smpp with mark space $[\mathbb{R}^1, \mathbb{R}^1]$. By this let us imagine that the positions of the elements of $\phi = \{[x_i, k_i]\}$ give the instantaneous positions of vehicles on a highway infinite in both directions and that the corresponding marks give the directions and speeds of these vehicles. We now consider the positions of the vehicles at timepoint 1, that is, the point sequence $\phi^* = \{[x_i, +k_i]\}$. Denote by $[P]^P$ the stationary distribution of $\phi^*$. For all purely random $P$, we have ([8], Chap. VII, § 5) $P = [P]^P$. Conversely, from the relation $P = [P]^P$ one can infer that $P$ is purely random if the speed distribution $p$ satisfies certain conditions [22] [2].
REFERENCES


ON THE STRUCTURE OF STATIONARY POINT PROCESSES*

Jacques Neveu

It is shown by methods of ergodic theory that the determination of the probability law of a stationary point process on \( R \) is equivalent to the determination of the law of its increments. Thus is generalized a theorem of Ryll-Nardzewski on the existence of certain conditional laws of this process.

Let \( \Omega \) be the space of nonempty subsets \( \omega \) of \( R \) which are locally finite and such that \( \inf a = -\infty \), \( \sup a = +\infty \). Let us put \( N_{\omega}(\omega) = \text{Card}(A \Delta \omega) \) for all Borel sets \( A \) from \( R \) and all \( \omega \in \Omega \). Let us endow the space \( \Omega \) with the smallest \( \sigma \)-algebra \( A \) rendering measurable the transformations \( N_{\omega} \).

Let us designate by \( \phi = (\phi_t, t \in R) \) the measurable family of translations on \( \Omega \) [the set \( \phi_t(\omega) \) is that which runs through \( a + t \) as \( a \) runs through \( \omega \)] and let us propose to study the collection of measures on \((\Omega, A)\) invariant under the family \( \phi \).

1. We denote by \( X \) the subspace of \( R^Z \) consisting of doubly infinite sequences \( x = (x_n, n \in Z) \) of real numbers strictly increasing and without finite accumulation points; we endow \( X \) with the \( \sigma \)-algebra \( X \) which is the projection on \( X \) of the Borel \( \sigma \)-algebra of \( R^Z \). The translations of coordinates, say \( \theta_k[(x_n, n \in Z)] = (x_{n+k}, n \in Z) \) \((k \in Z)\), constitute a measurable family \( (\theta_k, k \in Z) \) on \((X, X)\).


-19-
By the measurable transformation $\alpha$ from $X$ onto $\Omega$ which associates
with the sequence $x = (x_n, n \in \mathbb{Z}) \in X$ the set $\alpha(x) = \{x_n, n \in \mathbb{Z}\} \in \Omega$, the space
$\Omega$ becomes identified as the space of orbits of the points of $X$ under the
family $\Theta$; the subset $\alpha^{-1}[\alpha(x)]$ of $X$ coincides in fact with the orbit
$\{\Theta_kx, k \in \mathbb{Z}\}$ of the point $x(x \in X)$. For every measurable positive real-valued
function $f$ on $X$, there exists a unique measurable positive real-valued
function $f_\Omega$ on $\Omega$ such that
\[ f_\Omega \circ \alpha = \sum_k f \circ \Theta_k \] on $X$;
in particular, for each Borel $A$ from $\mathbb{R}$, we have
\[ f_\Omega = N_A \text{ if } f(x) = I_A(x_0). \]
In addition, one may find at least one measurable transformation from $X$ onto $\mathbb{Z}$, say
$\rho$, such that
\[ \rho \circ \Theta_k = k + \rho \text{ on } X \]
and such that the transformation $\rho \times \alpha$ from $X$ onto $\mathbb{Z} \times \Omega$ is an isomorphism
of measurable structure; one such transformation, for example, is defined by the
formula
\[ \rho[(x_n, n \in \mathbb{Z})] = \ell \text{ if } x_{\ell + 1} \geq 0 > x_{\ell}. \]
Finally, the measurable family $\phi = (\phi_t, t \in \mathbb{R})$ of translations on $X$ defined
by $\phi_t[(x_n, n \in \mathbb{Z})] = (x_n + t, n \in \mathbb{Z})$ induces the family $\phi$ on $\Omega$ through
the intermediary of the transformation $\alpha$. Thus the following lemma is easy to
establish:

**Lemma 1.** The formula $\int_{\Omega} f d\lambda = \int_X f d\mu$, where $f$ runs through the measurable
positive real-valued functions on $X$, establishes a one-to-one correspondence
between the positive $\sigma$-finite measures $\lambda$ on $\Omega$ and the positive $\sigma$-finite
measures \( \mu \) on \( X \) which are invariant under the family \( \Theta \). Further, in this correspondence, \( \lambda \) is invariant under the family \( \phi \) if and only if \( \mu \) is invariant under the family \( \psi \).

2. We denote by \( Y \) the subspace of \( R_+^Z \) (where \( R_+ = ]0, \infty[ \) in \( R \)) formed by the doubly infinite sequences \( y = (y_n, n \in \mathbb{Z}) \) of strictly positive real numbers such that \( \sum_{\infty}^{0} y_n = \infty = \sum_{0}^{\infty} y_n \); this space \( Y \) is endowed with the \( \sigma \)-algebra \( \mathcal{V} \) which is the image on \( Y \) of the Borel \( \sigma \)-algebra over \( R_+^Z \).

Through the measurable transformation \( \beta \) from \( X \) onto \( Y \) defined by the formula \( \beta[(x_n, n \in \mathbb{Z})] = (x_{n+1} - x_n, n \in \mathbb{Z}) \), the space \( Y \) becomes identified as the space of orbits of the points of \( X \) relative to the family \( \phi \); the subset \( \beta^{-1}[\beta(x)] \) of \( X \) coincides indeed with the orbit \( \{\phi_t(x), t \in R\} \) of the point \( x \) (\( x \in X \)). For every measurable positive real-valued function \( f \) on \( X \), there exists a unique measurable positive real-valued function \( f_Y \) on \( Y \) such that

\[
f_Y \circ \beta = \int_R f \circ \phi dt \quad \text{on } X;
\]

in particular, for each Borel \( A \) from \( R \), we have

\[
f_Y = \int_A dt \quad \text{if } f(x) = I_A(x_0).
\]

Moreover, there exists at least one measurable transformation \( \sigma \) from \( X \) onto \( R \) such that

\[
\sigma \circ \phi_t = t + \sigma \quad \text{on } X
\]

and such that the transformation \( \sigma \times \beta \) from \( X \) onto \( R \times Y \) is an isomorphism of measurable spaces; one such transformation \( \sigma \) is defined, for example, by

\[
\sigma[(x_n, n \in \mathbb{Z})] = x_0.
\]
Finally, the family \( \Theta \) on \( X \) (which commutes with the family \( \Phi \)) induces on \( Y \), via the transformation \( \beta \), the family \( \Theta = (\Theta_k, k \in \mathbb{Z}) \) of translations of coordinates of \( Y \); we have, therefore,

\[
\Theta_k[(y_n, n \in \mathbb{Z})] = (y_{n+k}, n \in \mathbb{Z}) \quad (k \in \mathbb{Z})
\]

and

\[
\beta \circ \Theta_k = \Theta_k \circ \beta \quad (k \in \mathbb{Z}).
\]

The following lemma is thus analogous to Lemma 1:

**Lemma 2.** The formula \( \int_X f d\mu = \int_Y f d\nu \), where \( f \) runs through the measurable positive real-valued functions on \( X \), establishes a one-to-one correspondence between the positive \( \sigma \)-finite measures \( \mu \) on \( X \) which are invariant under the family \( \Phi \) and the positive \( \sigma \)-finite measures \( \nu \) on \( Y \). Further, in this correspondence, the measure \( \mu \) is invariant under the family \( \Theta \) if and only if the measure \( \nu \) is invariant under the family \( \Theta \).

3. **Theorem 3.** The formulas of Lemmas 1 and 2 establish a one-to-one correspondence between the positive \( \sigma \)-finite measures \( \lambda \) on \( \Omega \) invariant under the family \( \Phi = (\Phi_t, t \in \mathbb{R}) \) of translations on \( \Omega \) and the positive \( \sigma \)-finite measures \( \nu \) on \( Y \) invariant under the family \( \Theta = (\Theta_k, k \in \mathbb{Z}) \) of translations of coordinates of \( Y \).

In writing successively the formulas of the lemmas for the functions \( f \) equal to

\[
f(x) = I_{\{x_1 \geq 0 > x_0\}},
\]

\[
f(x) = I_{\{t \geq x_1 \geq 0 > x_0\}}.
\]
and

\[ f(x) = I_A(x_0) \quad (A: \text{Borel set in } \mathbb{R}) , \]

we obtain from the correspondence of the theorem between \( \lambda \) and \( \nu \) that

\[ \lambda(\Omega) = \int_{Y} y_0 \, d\nu(y) , \]

\[ \lambda([N[0,t] > 0]) = \int_{Y} \min(y_0, t) \, d\nu(y) , \]

and

\[ \int_{\Omega} N_{\Omega}(\omega) \, d\lambda(\omega) = (\int_{A} dt) \cdot \nu(Y) . \]

The first formula thus serves to show that \( \lambda \) is a probability on \( \Omega \) if and only if the expectation under \( \nu \) of one (= each) of the coordinates of \( Y \) has the value 1. The other formulas yield then the equivalence of the following three conditions:

\[ \begin{align*}
&\text{a. } \int_{\Omega} N_{\Omega} \, d\lambda < \infty \text{ for an open interval } A \text{ (or for any Borel } A \text{ having finite Lebesgue measure);} \\
&\text{b. } \lim_{t \downarrow 0} t^{-1} \lambda([N_{[0,t]} > 0]) < \infty; \\
&\text{c. } \text{the measure } \nu \text{ is finite.}
\end{align*} \]

(The existence of the limit (b), which here results immediately from the decrease of the function \( t^{-1} \int_{Y} \min(y_0, t) \, d\nu(y) \) with \( t \), has been established previously in [1].)

There are numerous simple examples where \( \lambda \) is a probability and \( \nu \) is not finite; in the following example the measure \( \lambda \) weights only the periodic sets. Let \( m \) be a positive measure infinite on the open \( R_+ \) such
that $\int_0^\infty a \, dm(a) = 1$ and let $u$ be a mapping of $\mathbb{R} \times \mathbb{R}_+$ into $X$ defined by $u[(t,a)] = (t+na, n \in \mathbb{Z})$. The measure $\mu$ on $X$, the image under the mapping $u$ of the product $dt \times m$ of the Lebesgue measure on $\mathbb{R}$ and the measure $m$ on $\mathbb{R}_+$, is clearly invariant on $X$ under the families $\Theta$ and $\Phi$. The measures $\lambda$ on $\Omega$ and $\nu$ on $Y$ which correspond to the measure $\mu$ through the formulas of lemmas 1 and 2 thus satisfy

$$\lambda(\Omega) = \mu(\{x_1 \geq 0 > x_0\}) = \int_0^\infty a \, dm(a) = 1$$

and

$$\nu(Y) = \mu(\{0 < x_0 \leq 1\}) = \int_0^\infty d(a) = \infty.$$  

Observe, moreover, that every measure finite and positive on $\mathbb{R}_+$ invariant under the family of translations of coordinates is already supported by the set $Y$. If then one limits consideration in the preceding theorem to finite measures $\nu$, one may substitute the space $\mathbb{R}_+$ for the space $Y$ in the statement of the theorem.

4. In the correspondence established in the above theorem, the measure $\lambda$ on $\Omega$ is ergodic relative to the family $\Phi$ if and only if the measure $\nu$ on $Y$ is ergodic relative to the family $\Theta$.

5. In the particular case that $\lambda$ is a probability satisfying condition (a) of paragraph 3 above, the following proposition has been demonstrated in [2] by another method.

**PROPOSITION 4.** To each positive $\sigma$-finite measure $\lambda$ on $\Omega$ invariant under the family $\Phi = (\Phi_t, t \in \mathbb{R})$ of translations of $\Omega$ is associated a unique positive $\sigma$-finite measure $\nu'$ on $\Omega$ which is supported by the subset $\Omega_o = \{\omega: 0 \leq \omega\}$.
of $\Omega$ and which is such that

$$N_A \cdot \lambda = \int_A d\phi_t (\nu^\tau) \quad (A: \text{Borel set in } R).$$

In order to demonstrate this proposition, let us consider the measurable transformations from $Y$ onto the subset $X_o = \{x: x_0 = 0\}$ of $X$ defined by

$$s[(y_n, n \in Z)] = (x_n, n \in Z)$$

with

$$x_n = \sum_{m=0}^{n-1} y_m, \quad -1 < n \leq 0,$$

and

$$n > 0, = 0 \text{ or } < 0.$$  Two measures $\mu$ and $\nu$ defined on $X$ and $Y$ respectively and corresponding to each other as in lemma 2 satisfy the equality $\mu = \int_R dt \phi_t \circ s(\nu)$; consequently, $[\phi_t \circ s(\nu), t \in R]$ is a decomposition of $\mu$ relative to the partition $[\phi_t (X_o), t \in R]$ of $X$ and we have

$$I_A (x_o) \cdot \mu = \int_A dt \phi_t \circ s(\nu) \quad \text{on } X.$$

Now let $\tau$ be the unique measurable transformation of $\Omega_o$ onto $X_o$ such that $x_o [\tau(\omega)] = 0 (\omega \in \Omega_o)$ and such that $\alpha \circ \tau$ is the identity on $\Omega_o$. It is easily verified then that the image $\nu^\tau$ of $s(\nu)$ under $\tau$, $\nu^\tau = \tau \circ s(\nu)$, satisfies the required condition.

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TRANSLATOR'S APPENDIX

A. Concerning the formulas in §3. The first formula,

\[ \lambda(\Omega) = \int_{Y} y \, dv(y), \]

is obtained by applying the formulas of lemmas 1 and 2 to the function

\[ f(x) = I_{\{x_1 \geq 0 > x_o\}}. \]

Now

\[ f^{\circ} \Phi_t = I_{\{x_1 + t \geq 0 > x_o + t\}}, \]

so that

\[ \int_{R} f^{\circ} \Phi_t \, dt = \int_{-x_o}^{-x_1} dt = x_1 - x_o = y_o \]

and hence

\[ f_Y = y_o. \]

Also,

\[ f^{\circ} \Theta_k = I_{\{x_{k+1} \geq 0 > x_k\}}, \]

so that

\[ \sum_{k} f^{\circ} \Theta_k = 1 \]

and hence

\[ f_\Omega = 1. \]
Therefore,

\[ \lambda(\Omega) = \int_{\Omega} f_\lambda d\lambda = \int_X f d\mu = \int_Y f d\nu = \int_Y \gamma d\nu(y) . \]

Likewise, the second formula is obtained by showing that for the function

\[ f(x) = I_{\{t \geq x \geq 0 \geq x_0\}} , \]

we have

\[ f_\gamma = \min(t, y_0) , \]

\[ f_\Omega = I_{\{N_{[0,t]} > 0\}} . \]

The third formula follows from the cited formulas

\[ f_\gamma = \int_A dt , \quad f_\Omega = N_A \]

when \( f(x) = I_A(x_0) \).

B. Concerning the notation in §4. The notation "s(v)" is sometimes expressed "\( v_s^{-1} \)." Thus

\[ s(v) [B] = v[s^{-1}(B)], \quad B \in X, \]

and

\[ v^\tau[E] = \tau \circ s(v)[E] = v[s^{-1}(\tau(E))], \quad E \in A . \]