THE ASYMPTOTIC DISTRIBUTION OF THE
NUMBER OF SYSTEM FAILURES WITH
STOCHASTIC HAZARD RATES

by

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1. Introduction. In equipment reliability studies, one is often concerned with systems composed of many components, each of which is subject to failure and replacement, and the system fails with any component failure. Such a system was considered by Drenick [6], when the hazard functions for the different components are deterministic. Under some conditions on the behavior of the hazard functions near zero, Drenick proved that the asymptotic distribution for the initial survival probability of a complex piece of equipment is the exponential distribution. His conditions are quite restrictive and are not, for example, satisfied by the Weibull distribution \( F(t) = 1 - \exp(-at^2) \). Cox and Smith [3] considered a similar model for a piece of equipment made up for a large number of components all alike. They assumed that the common distribution function, \( F(x) \), of the interval between component failures is known, in particular, they assumed that \( F(x) = O(x^\beta) \) as \( x \to 0 \), where \( 0 < \beta < 1 \). They proved that for a large number of components, the asymptotic distribution of the number of failures of a system is Poisson.

The present paper is concerned with the overall pattern of failure for a complex system composed of a large number of components. In particular, as the number of components increase, the asymptotic distribution of a number of system failures in determined when the hazard functions for the components are realizations of stochastic processes. The usefulness of this model is demonstrated by the two examples given at the end of the paper. The first example deals with the situation when the system is operating in random environment caused by a random load on the system. The second example deals with a case when the components are being selected from heterogeneous populations due to variations in their manufacturing processes.
2. Model

Consider a system composed of $N$ components each of which is subject to failure. Given the hazard rates for the components, $\phi_{N,i}(\cdot)$ for component $i$, the probability that component $i$ fails in the time interval $(t, t+\Delta t)$, given that it is in service at time $t$, is $\phi_{N,i}(t)\Delta t + o(\Delta t)$. We assume that $\phi_{N,i}(\cdot)$ is a realization of a stochastic process $\{\phi_{N,i}(t), t \geq 0\}$ for all $i$ ($1 \leq i \leq N$).

We shall consider a system with the following properties:

(i) different components are not, in general, alike and, given their hazard rates, are stochastically independent; (ii) the system fails with any component failure; (iii) each component is immediately replaced upon its failure by a component of the same type; the hazard rate of the new component is a realization of the stochastic process associated with this type of component.

The assumption of stochastic hazard rate is not new. This type of variability has been considered by Antelman and Savage [1], Mercer [10], Harris and Singpurwalla [8] and Gaver [7]. This type of variability can be used when the system is exposed to a random load function and the hazard rates are functions of this load function. It can also account for the case when the parameters of the hazard functions are random variables; this might be due to the heterogeneity of the populations from which the different components are selected. For a discussion of these ideas see Birnbaum and Saunders [2], Cozzalino [4] and McNolty [9].

Throughout this paper we shall assume that $\{\phi_{N,i}(t), t \geq 0\}$ is a non-negative measurable stochastic process for all $N$ and all $i$ ($1 \leq i \leq N$), $m_{N,i}(t) = E \phi_{N,i}(t)$ exists for $t \geq 0$ and if $A$ is a Lebesgue
measurable parameter set with finite measure then

\[ \int_{A} m_{n,1}(x) dx < \infty. \]

These assumptions insure that all sample functions are Lebesgue integrable over any parameter set \( A \) with finite measure, see Doob ([5], p. 62). We shall need the following lemmas to prove the main theorem.

**Lemma 1.** Given two sequences of random variables \( \{X_n\} \) and \( \{Y_n\} \) such that \( X_n \xrightarrow{P} X \) and \( X_n + Y_n \xrightarrow{P} 0 \), then \( Y_n \xrightarrow{P} -X \).

**Lemma 2.** If a Borel function \( f \) has a derivative at \( a \) and \( X_n \xrightarrow{P} a \), then

\[
f(X_n) = f(a) + f'(a) (X_n - a) + (X_n - a) Z_n
\]

where \( Z_n \xrightarrow{P} 0 \).

The proofs of these two lemmas are simple and may be omitted.

**Lemma 3.** Let \( X_{n_1}, X_{n_2}, \ldots, X_{n_k} \) be a double sequence of non-negative random variables where \( k_n \xrightarrow{P} \infty \) as \( n \rightarrow \infty \) and such that:

(i) \( X_{nj} < 1 \) a.s. for all \( n \) and \( j \),

(ii) \( \hat{X}_n = \max_{1 \leq j \leq k_n} X_{nj} \xrightarrow{P} 0 \) as \( n \rightarrow \infty \),

(iii) \( \sum_{i=1}^{k_n} X_{ni} \xrightarrow{P} X \) as \( n \rightarrow \infty \),

then
\[ \prod_{j=1}^{p} (1+X_{n_j}) \Rightarrow \exp X. \]

Proof. Using the series expansion for \( \log (1+x) \) we can see that

\[ \left| \sum_{i=1}^{k_n} \log(1+X_{n_i}) - \sum_{i=1}^{k_n} X_{n_i} \right| \leq \sum_{i=1}^{k_n} \left( \sum_{j=2}^{\infty} \frac{(X_{n_i})^j}{j} \right) \leq \frac{1}{2} \sum_{i=1}^{k_n} \frac{X_{n_i}^2}{1-X_{n_i}}. \]

From this we can show that

\[ P\left\{ \left| \sum_{i=1}^{k_n} \log(1+X_{n_i}) - \sum_{i=1}^{k_n} X_{n_i} \right| > \varepsilon \right\} \leq P\left\{ \frac{1}{2} \sum_{i=1}^{k_n} \frac{X_{n_i}^2}{1-X_{n_i}} > \varepsilon \right\} \]

\[ \leq P\left\{ \frac{1}{2} \sum_{i=1}^{k_n} \frac{X_{n_i}^2}{1-X_{n_i}} > \varepsilon \mid \hat{X}_n > \frac{1}{2} \right\} P\left\{ \hat{X}_n > \frac{1}{2} \right\} + P\left\{ \frac{1}{2} \sum_{i=1}^{k_n} \frac{X_{n_i}^2}{1-X_{n_i}} > \varepsilon \mid X_n \leq \frac{1}{2} \right\} P\left\{ X_n \leq \frac{1}{2} \right\} \]

\[ \leq P\left\{ \hat{X}_n > \frac{1}{2} \right\} + P\left\{ \sum_{i=1}^{k_n} X_{n_i}^2 > \varepsilon \right\} \]

\[ \leq P\left\{ \hat{X}_n > \frac{1}{2} \right\} + P\left\{ \sum_{i=1}^{k_n} X_{n_i} > \varepsilon \right\}. \]

Now, since \( \hat{X}_n \to 0 \) and \( \sum_{i=1}^{k_n} X_{n_i} \to X \) as \( n \to \infty \), we can see that the right hand
side of the last inequality tends to zero as \( n \to \infty \). Thus

\[
\log \prod_{i=1}^{k} \frac{1 + X_{ni}}{ni} = \sum_{i=1}^{k} \log(1 + \frac{X_{ni}}{ni}) \to 0,
\]

and using Lemma 1, we can show that

\[
\log \prod_{i=1}^{k} (1 + \frac{X_{ni}}{ni}) = X \quad \text{as} \quad n \to \infty.
\]

Using the well known result that if \( f \) is measurable mapping from the reals to the reals which is continuous over a Borel set \( B \) for which \( P(X \in B) = 1 \), and if \( X_{n} \to X \), then \( f(X_{n}) \to f(X) \), the conclusion follows.

With these lemmas we can now prove our main result. Let \( Y_{N}(t) \) denote the number of failures in \([0,t]\) of the original components. Then \( Y_{N}(t) = \sum_{k=1}^{N} V_{k}(t) \), where \( V_{k}(t) \) equals 0 if the \( k^{th} \) component does not fail in \([0,t]\), and equals 1 if the \( k^{th} \) component fails in \([0,t]\). Therefore the conditional probability of the event \( V_{k}(t) = 0 \), given the hazard rates \( \phi_{N,k}(\cdot) \), is given by

\[
Q_{N,k}(t) = \exp \left[ - \int_{0}^{t} \phi_{N,k}(x) \, dx \right] \quad \text{for} \quad k = 1,2,\ldots,N.
\]

Let \( P_{N,k}(t) = 1 - Q_{N,k}(t) \) for \( k = 1,2,\ldots,N \). From this we can show that the conditional probability generating function (c.p.g.f.) for \( V_{k}(t) \) is equal to

\[
(1 - P_{N,k}(t)(1-s)).
\]

Since the components are conditionally independent,
the conditional p.g.f. for $Y_{N}(t)$ given the hazard rates is equal to

$$N \prod_{i=1}^{N} (1 - P_{N,i}(t)(1-s)).$$

Thus the unconditional p.g.f. for $Y_{N}(t)$ is equal to

$$G_{N}(s) = E \prod_{i=1}^{N} (1 - P_{N,i}(t)(1-s)).$$

The next proposition gives sufficient conditions on the hazard functions for the convergence in probability, as $N \to \infty$, of $\sum_{i=1}^{N} P_{N,k}(t)$. This proposition has a small amount of independent interest and is used only to prove our main theorem.

Proposition. Let $t > 0$ be arbitrary, but fixed, time. If the stochastic processes $\phi_{N,i}(\cdot)$, $1 \leq i \leq N$, are such that

(i) $\max_{1 \leq i \leq N} \int_{0}^{t} \phi_{N,i}(x)dx \to 0$ as $N \to \infty,$

(ii) $\sum_{i=1}^{N} \int_{0}^{t} \phi_{N,i}(x)dx \to P(t)$ as $N \to \infty,$

then

$$\sum_{i=1}^{N} P_{N,i}(t) \to P(t) \text{ as } N \to \infty.$$ 

Proof. We have shown that $P_{N,i}(t) = 1 - \exp - \int_{0}^{t} \phi_{N,i}(x)dx$. Since $\phi_{N,i}$ is non-negative, then (i) implies that for all $i$ ($1 \leq i \leq N$)

$$\int_{0}^{t} \phi_{N,i}(x)dx \to 0 \text{ as } N \to \infty.$$ 

Since the exponential function has a derivative
at 0, we can use Lemma 2 to write, for any \( k \) \((1 \leq k \leq N)\),

\[
P_{N,k}(t) = 1 - \left(1 - \int_0^t \phi_{N,k}(x) dx + \int_0^t \phi_{N,k}(x) dx\right)
\]

\[
= \int_0^t \phi_{N,k}(x) dx - Z_{N,k} \int_0^t \phi_{N,k}(x) dx
\]

where \( Z_{N,k} \to 0 \) as \( N \to \infty \). We can also show that

\[
Z_{N,k} \int_0^t \phi_{N,k}(x) dx \leq \frac{1}{2} \left( \int_0^t \phi_{N,k}(x) dx \right)^2 .
\]

Thus

\[
\sum_{k=1}^N Z_{N,k} \int_0^t \phi_{N,k}(x) dx \leq \frac{1}{2} \max_{1 \leq i \leq N} \int_0^t \phi_{N,i}(x) dx \sum_{k=1}^N \int_0^t \phi_{N,k}(x) dx
\]

and, from (i) and (ii), we can show that

\[
\sum_{k=1}^N Z_{N,k} \int_0^t \phi_{N,k}(x) dx \to 0 \quad \text{as} \quad N \to \infty.
\]

Hence the conclusion of the proposition.

3. The main result

The purpose of our next theorem is to find the limiting p.g.f. for the total number of failures, of original and replacement components, in an arbitrary time interval \([0,t]\). Let \( X_N(x) \) be the number of failures, of originals or replacements, in \([0,x]\).
Theorem. For an arbitrary, but fixed, t, if the hazard rates $\phi_{N,i}(\cdot)$ are such that:

(i) for $\alpha > 0$ we have $N^\alpha \max_{1 \leq i \leq N} \int_0^t \phi_{N,i}(x)dx \to 0$ as $N \to \infty$.

and

(ii) $\sum_{k=1}^N t \int_0^t \phi_{N,k}(x)dx \to \tau(t)$ as $N \to \infty$,

then

$$\lim_{N \to \infty} G_N(s) = E \exp - (1-s)\tau(t),$$

and

$$P(X_N(t^-) = Y_N(t^-), 0 \leq t^- \leq t) \to 1 \quad \text{as} \quad N \to \infty.$$

Proof. From (i) and (ii) and the above proposition, we can show that

$$\sum_{i=1}^N P \phi_{N,i}(t) \to \tau(t) \quad \text{as} \quad N \to \infty.$$

We have shown that

$$G(s) = E \prod_{i=1}^N (1 - (1-s)P_{N,i}(t)).$$

The product on the right hand side is bounded above by 1 for all $N$. Since

$$P_{N,i}(t) = 1 - \exp - \int_0^t \phi_{N,i}(x)dx \leq \int_0^t \phi_{N,i}(x)dx,$$

then condition (i) of the theorem implies that

$$\sum_{i=1}^N P_{N,i}(t) \to 0 \quad \text{as} \quad N \to \infty.$$
Thus Lemma 3 can be used to show that

$$\prod_{i=1}^{N} \frac{(1 - (1-s)^{p_{i}})}{(1-s)^{p_{i}}(t)} \rightarrow \exp - (1-s)^{p}(t) \quad \text{as } N \rightarrow \infty.$$

Then using the bounded convergence theorem we can show that

$$\lim_{N \rightarrow \infty} G_{N}(s) = E \exp - (1-s)^{p}(t).$$

To prove the last part of the theorem we notice that if a component of type i replaced another component, of the same type, at time T(<t), then the conditional probability of its survival up to time t, given the hazard rates, is equal to

$$\exp - \int_{T}^{t} \phi_{N,i}(x)dx.$$

Since $\phi_{N,i}$ is non-negative, this conditional probability is greater than or equal to $\exp - \int_{0}^{t} \phi_{N,i}(x)dx$ a.s. Using this we can show that

$$q(i_1,i_2,\ldots,i_k) = \Pr(\text{no new failures of } i_1,i_2,\ldots,i_k \text{ types in } [0,t])$$

$$V_{i}(t) = 1, \text{ } j = 1,2,\ldots,k \text{ and } V_{i} = 0 \text{ otherwise}$$

$$\geq E \exp - \sum_{j=1}^{k} \int_{0}^{t} \phi_{N,i}(x)dx$$

$$\geq E \exp - k \max_{l<i<N} \int_{0}^{t} \phi_{N,i}(x)dx.$$

Thus for any integer $k \leq N^{\alpha}$, we can show, using condition (i) of the theorem that $q(i_1,\ldots,i_k) + 1$ as $N \rightarrow \infty$. Let $q_{N} = E \exp - N^{\alpha} \max_{l<i<N} \int_{0}^{t} \phi_{N,i}(x)dx$. We can see that $\alpha(i_1,i_2,\ldots,i_k) \geq q_{N}$ for any set of integers $\{i_1,i_2,\ldots,i_k\}$.
where \( k \leq N^\alpha \), and \( q_N \to 1 \) as \( N \to \infty \). Therefore we can show that

\[
P(X_N(t^-) = Y_N(t^-), 0 \leq t^- \leq t) \geq P(Y_N(t) \leq N^\alpha)q_N.
\]

Since \( \{Y_N(t)\} \) converges in distribution, it follows that \( P(Y_N(t) \leq N^\alpha) \to 1 \) as \( N \to \infty \), and thus

\[
P(X_N(t^-) = Y_N(t^-), 0 \leq t^- \leq t) \to 1 \quad \text{as} \quad N \to \infty.
\]

It can also be shown that the limiting process is a stochastic process with independent increments.

4. Examples

We consider next the application of our result to some practical examples.

Example 1. Consider a system consisting of \( N \) components and exposed to a total instantaneous load \( L(t) \), where \( \{L(t), t \geq 0\} \) is a non-negative stochastic process. The load is shared, not necessarily equally, by all the components. We make the assumption, due to Birnbaum and Saunders ([2], p. 153), that the hazard rate of component \( i \), \( \phi_{N,i}(t) \), can be represented as \( \phi_{N,i}(t) = w_i(t)D_{N,i}(t) \). In this representation \( D_{N,i}(t) \) is the instantaneous damage at time \( t \) to component \( i \) and \( w_i(t) \) is the cumulative deterioration of component \( i \) up to time \( t \). The stochastic process \( D_{N,i}(t) \) is a function of \( L(t) \) depending on the structure and interconnections of the system. A simple interpretation of the function \( w_i(t) \) is that it represents the hazard rate of component \( i \) when exposed to a constant non-random load which gives unit instantaneous damage for all \( t \). This is the standard load for which the component is designed.
For further discussion of this assumption see [2], where an empirical example is presented showing remarkable agreement between the experimental data and the predicted result under the above assumptions. Their results are based on the assumption of a non-random load function.

Now assume that the load function \( L(x) \) is such that, for all finite \( t, D_{N,i}(x) \leq D_N \) a.s. for all \( x \in [0,t] \) and all \( i (1 \leq i \leq N) \). This can be the case when \( \sup_{0 \leq x \leq t} L(x) \) is a.s. within the elastic region for each of the components. Let \( D_N = o_p(N^{-\alpha}) \) for some \( \alpha > 0 \), and \( D_N \) might depend on \( t \). Also assume that, for any finite \( t, \int_0^t w_i(x)dx < A(t) \) for all \( i \), where \( A(t) \) is some finite function of \( t \). We notice that this condition on \( w_i(\cdot) \) does not exclude any of the hazard rates that are commonly used. If we assume further that \( \sum_{i=1}^{N} \int_0^t w_i(x)D_{N,i}(x)dx \rightarrow P(t) \), we can see that the conditions of the theorem are satisfied and the limiting distribution will have a p.g.f. given by \( E \exp - (1-s)P(t) \).

We observe that if we have identical components sharing equally the load, then \( D_{N,i}(t) = \frac{\Delta(t)}{N} \), where \( \Delta(t) \) is the total instantaneous damage at time \( t \). Also

\[
\sum_{i=1}^{N} \int_0^t \phi_{N,i}(x)dx = \int_0^t w(x)\Delta(x)dx
\]

where \( w(\cdot) \) is the cumulative deterioration for any of the components. The conditions of the theorem are satisfied and \( G_N(\cdot) \rightarrow E \exp - (1-s)\int_0^t w(x)\Delta(x)dx \). We notice that in case \( w(x)\Delta(x) = \varepsilon \), where \( \varepsilon \) is a constant, the limiting p.g.f. is the one corresponding to a homogeneous Poisson process. Thus we can have an exponential reliability curve for a system with redundancy even in the presence of a random stress and even when the components do not have an exponential failure law.
Example 2. Due to heterogeneity of the populations from which the different components are selected and the interconnections of the components of the system we may assume that

$$\phi_{N,i}(t) = L_{N,i} + N_{N,i}(\theta_{N,i} \cdot t) N_{N,i} \theta_{N,i},$$

where the shape parameters $\theta_{N,i}$ are non-random and $\theta_{N,i} > -1$ for all $i$, $L_{N,i}$ and the scale parameters $M_{N,i}$ are non-negative random variables. We are assuming that the failure of component $i$ is due to two factors; one independent of age and the other increasing (for $\theta_{N,i} > 0$) or decreasing (for $\theta_{N,i} < 0$) with increasing age. The assumption of non-random known shape parameters seems to be reasonable for components that have an efficient quality control during their production. We make this assumption to avoid unnecessary complications in our exposition. Let us assume further that $-1 < \theta_{N,i} < A < \infty$ for all $i$ and large $N$. With these assumptions we can see that if $t \leq 1$

$$\int_{0}^{t} \phi_{N,i}(x) dx \leq L_{N,i} t + M_{N,i}$$

and if $t > 1$

$$\int_{0}^{t} \phi_{N,i}(x) dx \leq L_{N,i} t + M_{N,i} A + l.$$ 

Thus if $\sup_{i} L_{N,i}$ and $\sup_{i} M_{N,i}$ are both $o_p(N^{-\alpha})$ the condition (i) of the theorem follows. If we further assume that $\sum_{i=1}^{N} L_{N,i}$ and $\sum_{i=1}^{N} M_{N,i} t^{\theta_{N,i}}$ both converge in probability, then condition (ii) of the theorem follows and the limiting p.g.f. can be determined.
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REFERENCES


