TESTING WHETHER NEW IS BETTER THAN USED¹

by

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Abstract

A U-statistic $J_n$ is proposed for testing the hypothesis $H_0$ that a new item has stochastically the same life length as a used item of any age (i.e., the life distribution $F$ is exponential), against the alternative hypothesis $H_1$ that a new item has stochastically greater life length ($\bar{F}(x) \bar{F}(y) \geq \bar{F}(x+y)$, for all $x \geq 0$, $y \geq 0$, where $\bar{F} = 1 - F$). $J_n$ is unbiased; in fact, under a partial ordering of $H_1$ distributions, $J_n$ is ordered stochastically in the same way. Consistency against $H_1$ alternatives is shown, and asymptotic relative efficiencies are computed. Small sample null tail probabilities are derived, and critical values are tabulated to permit application of the test.
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1. Introduction and summary. In performing reliability analyses, it has been found very useful to classify life distributions \( F \) (i.e., distributions for which \( F(t) = 0 \) for \( t < 0 \)) according to the monotonicity properties of the failure rate, or alternately, the average failure rate. See Barlow and Proschan (1965), Barlow, Marshall, and Proschan (1963), Birnbaum, Esary, and Marshall (1965), and Esary, Marshall, and Proschan (1970a,b). (Additional references are presented in these papers.)

Recently, several new classes of life distributions have been shown to be fundamental in the study of replacement policies (Marshall and Proschan, 1970). Properties of such life distributions have been treated in Esary, Marshall, and Proschan (1970a, b).

**DEFINITION 1.1** A life distribution \( F \) is new better than used (NBU) if

\[
(1.1) \quad \bar{F}(x+y) \leq \bar{F}(x) \bar{F}(y) \quad \text{for all } x, y \geq 0,
\]

where \( \bar{F} = 1 - F \). The corresponding concept of a new worse than used (NWU) distribution is defined by reversing the inequality in (1.1).

The NBU property defined in (1.1) has also been referred to as

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"positive aging" by Bryson and Siddiqui (1969).

Property (1.1) may be interpreted as stating that the chance \( F(x) \) that a new unit will survive to age \( x \) is greater than the chance \( \frac{F(x+y)}{F(y)} \) that an unfailed unit of age \( y \) will survive an additional time \( x \). That is, a new unit has stochastically greater life than a used unit of any age.

The boundary members of the NBU class, obtained by insisting on equality in (1.1), are of course the exponential distributions for which used items are no worse (and no better) than new items. In this paper we consider the inferential problem of testing

\[
H_0: F(x) = 1 - \exp(-\lambda x), \quad x \geq 0, \quad \lambda > 0 \quad (\lambda \text{ unspecified}),
\]

versus

\[
H_1: \text{ F is NBU (and not exponential)},
\]

on the basis of a random sample \( X_1, X_2, \ldots, X_n \) from the distribution \( F \). In the sequel, unless otherwise stated, \( F \) is assumed continuous.

The testing problem \( H_0 \) vs. \( H_1 \) is analogous to the testing problem of \( H_0^* \) vs. \( H_1^* \) where \( H_1^* \) specifies that \( F \) is an increasing failure rate (IFR) distribution. The distribution \( F \) is said to be IFR if \( -\ln \frac{F(x)}{x} \) is convex. If \( F \) has a density \( f \), this condition is equivalent to the condition that the failure rate \( q(x) = \frac{f(x)}{F(x)} \) is increasing in \( x \) (such that \( F(x) > 0 \)). Tests of \( H_0^* (q(x) = \lambda, \lambda \text{ unspecified}) \) vs. \( H_1^* (q(x) \text{ is monotone increasing but nonconstant}) \) include those considered by Barlow (1968), Barlow (1970), Barlow and Proschan (1969),
Bickel (1969), Bickel and Doksum (1969), and Proschan and Pyke (1967). Since $F$ is NBU if $-\ln \bar{F}(x)$ is superadditive, the IFR class is contained in the NBU class and thus the test of $H_0$ vs. $H_1$ that we propose focuses on a larger class of alternative distributions than do the IFR tests. This will be appropriate, for example, when the underlying physical process suggests that new items are better than used ones but where we can expect the failure rate to fluctuate (and in particular not satisfy $H_1$).

As one example of a practical problem motivating the choice of the null hypothesis $H_0$ and alternative hypothesis $H_1$ above, consider a unit subject to shocks occurring successively in time according to a Poisson process. Since the occurrence of shocks and their effects cannot be directly observed, it is not known whether shocks already experienced by the unit make it more likely to fail under the impact of future shocks or not. However, if $\bar{F}_k$ is the probability that the unit survives the first $k$ shocks, then it is believed that either

$$(a) \quad \bar{F}_k \equiv \frac{\bar{F}_{\ell+k}}{\bar{F}_\ell} \text{ for all } k, \ell \geq 0, \text{ or}$$

$$(b) \quad \bar{F}_k \geq \frac{\bar{F}_{\ell+k}}{\bar{F}_\ell} \text{ for all } k, \ell \geq 0.$$ 

Since under hypothesis (a), the lifelength is exponential, and under hypothesis (b), it is NBU (Esary, Marshall, Proschan, 1970b, Theorem 3.1), a reasonable way to test (a) vs. (b) would be to test $H_0$ vs. $H_1$ above from lifelength observations.

Our test statistic is motivated by consideration of
Viewing the parameter $\gamma(F)$ as a measure of the deviation of $F$ from $H_0$, the classical nonparametric approach of replacing $F$ by the empirical distribution function $F_n$ suggests rejecting $H_0$ in favor of $H_1$ if $\int \left[ F(x+y) dF_n(x) - F(x+y) dF_n(y) \right]$ is too small. We find it more convenient to reject for small values of the asymptotically equivalent U-statistic
(1.5) \( J_n = 2[n(n-1)(n-2)]^{-1/2} \psi(x_{\alpha_1}, x_{\alpha_2} + x_{\alpha_3}), \)

where

(1.6) \( \psi(a, b) = \begin{cases} 1 & \text{if } a > b \\ 0 & \text{if } a \leq b, \end{cases} \)

and the \( \frac{1}{n} \) is over all \( n(n-1)(n-2)/2 \) triples \( (\alpha_1, \alpha_2, \alpha_3) \) of three integers such that \( 1 \leq \alpha_1 \leq n, \alpha_1 \neq \alpha_2, \alpha_1 \neq \alpha_3, \) and \( \alpha_2 < \alpha_3. \) In the sequel, the test which rejects for small \( J_n \) values is referred to as the NBU test.

Section 2 gives the asymptotic normality of \( J_n, \) the result being a direct consequence of Hoeffding's (1948) U-statistic theory. Section 3 demonstrates the unbiasedness of the NBU test for NBU alternatives. In fact, a stronger result (Theorem 3.1) is established, namely, that when \( F \) is superadditive with respect to \( G \) (Definition 3.1),

\[ J_n(X) \leq J_n(Y), \]

where \( Y = (y_1, y_2, \ldots, y_n) \) is a random sample from \( G. \) Consistency is also considered in section 3. The NBU test is consistent if and only if \( \Delta(F) \), defined by (1.4), is strictly less than \( 1/4 \), the latter being the value of \( \Delta \) when \( F \) is exponential. In a result that parallels Theorem 3.1, we show (Theorem 3.2) that \( \Delta(F) \leq \Delta(G) \) when \( F \) is superadditive with respect to \( G. \) Also, consistency against NBU alternatives is established.

Section 4 considers the asymptotic relative efficiency of the NBU test. To the authors' knowledge, other tests for NBU alternatives have not yet been proposed. Thus, we take as competitors, tests designed for ETN alternatives. Since the
NBU class contains the IFR class, one should expect the efficiencies (under IFR alternatives) to favor the IFR tests, and indeed this is the case. On the other hand, as is also to be expected, there are many NBU alternative distributions for which the IFR tests do not perform as well as the NBU test. The class of NBU alternatives \( F_{a,b} \), defined in Example 3.2, for which the NBU test is shown to have power equal to 1 (when \( n \geq 3 \) and \( a \geq \left( \frac{2n-2}{n} \right)^{-1} \)) illustrates this point vividly. This is discussed in Section 4.

The small sample null distribution of the statistic
\[
T_n = n(n-1)(n-2)J_n/2
\]
is considered in Section 5. Exact probabilities are computed in special cases, and in Table 5.1 lower and upper percentile points based on Monte Carlo sampling are given in the .01, .025, .05, .075, and .10 regions for \( n = 4(1)20(5)50 \).

2. **Asymptotic normality.** The results of this section are obtained by applying Hoeffding's (1948) U-statistic theory to the statistic \( J_n \). Let
(2.1) \( \phi(x_1, x_2, x_3) = J_3 = 3^{-1}(\psi(x_1, x_2 + x_3) + \psi(x_2, x_1 + x_3) + \psi(x_3, x_1 + x_2)) \),

and set,

\( \phi_1(x_1) = E\phi(x_1, x_2, x_3), \phi_2(x_1) = E\phi(x_1, x_2, x_3), \)

(2.2)

\( \phi_3(x_1, x_2, x_3) = \phi(x_1, x_2, x_3), \)

(2.3) \( \xi_k = E\phi_k^2(X_1, \ldots, X_k) - \Delta_k, k = 1, 2, 3, \)

where \( \Delta(F) \) is defined by (1.4). Then

(2.4) \( \text{Var}(J_n) = \left( \frac{1}{3} \right)^{-1} \sum_{k=1}^{3} \left( \frac{3}{3-k} \right) \xi_k. \)

(2.5) \( \lim_{n} \text{Var}(J_n) = 9\xi_1. \)

Furthermore we may state

THEOREM 2.1. If \( F \) is such that \( \xi_1(F) > 0 \), then the limiting

distribution of \( \frac{J_n}{\Delta(F)} \) is normal with mean 0 and variance \( 9\xi_1. \)

Since \( \psi(ca, cb) = \psi(a, b) \) for all \( c > 0 \), the statistic \( J_n \) is

scale invariant, and hence in all null computations we may take the

scale parameter of the exponential to be \( \lambda = 1 \). Straightforward

calculations yield the hypothesis values \( \Delta = 1/4 \) and

(2.6) \( \xi_1 = \frac{5}{3888}, \xi_2 = \frac{7}{1296}, \xi_3 = \frac{1}{48}. \)

From Theorem 2.1, we immediately obtain
COROLLARY 2.1. Under $H_0$, the limiting distribution of $n^{3/4}(J_n - 1/4)$ is normal with mean 0 and variance 5/432.

3. Unbiasedness and consistency. In this section we first show that the test which rejects $H_0$ if $J_n \leq J_{n,\alpha}$, where $J_{n,\alpha}$ satisfies $P_0[J_n \leq J_{n,\alpha}] = \alpha$, is unbiased. That is, $P_1[J_n \leq J_{n,\alpha}] > \alpha$, where $P_1(P_0)$ indicates the probability is computed for an $F$ satisfying $H_1(H_0)$.

DEFINITION 3.1. Let $F$ and $G$ be continuous distributions, $G$ be strictly increasing on its support, and $F(0) = 0 = G(0)$. Then $F$ is said to be superadditive with respect to $G$ if $G^{-1}F$ is superadditive, that is,

$$G^{-1}F(x_1 + x_2) \geq G^{-1}F(x_1) + G^{-1}F(x_2), \text{ for all } x_1, x_2 \geq 0.$$  \hspace{1cm} (3.1)

When the inequality in (3.1) is reversed, $F$ is said to be subadditive with respect to $G$.

THEOREM 3.1. Let $F$ be superadditive with respect to $G$.

Then $J_n(X) \leq J_n(Y)$, where $X = (X_1, \ldots, X_n)$ is a random sample from $F$ and $Y = (Y_1, \ldots, Y_n)$ is a random sample from $G$.

PROOF. Let $Y'_i = G^{-1}F(X_i)$, $i = 1, \ldots, n$. Then $(Y'_{1}, \ldots, Y'_{n}) = (Y'_1, \ldots, Y'_n)$. Now we show

$$x_3 \geq x_1 + x_2 \Rightarrow y'_3 \geq y'_1 + y'_2.$$ \hspace{1cm} (3.2)

The implication given by (3.2) can be seen as follows:
\[ X_3 \geq X_1 + X_2 \implies F(X_3) \geq F(X_1 + X_2) \]

\[ \implies G^{-1}F(X_3) \geq G^{-1}F(X_1 + X_2) \geq G^{-1}F(X_1) + G^{-1}F(X_2), \]

where the last inequality is a consequence of the superadditivity of \( G^{-1}F \). Equivalently, \( Y_3^* \geq Y_1^* + Y_2^* \). From (3.2) we have \( \psi(Y_3^*, Y_1^* + Y_2^*) \geq \psi(x_3, x_1 + x_2) \) and thus

\[
J_n(Z) \leq J_n(Y^*) = J_n(Y) . 
\]

**COROLLARY 3.1.** The NBU test is unbiased against NBU alternatives.

**PROOF.** By taking \( G \) to be exponential, and noting that \( F \) is NBU if and only if \( F \) is superadditive with respect to the exponential (i.e., \( -\ln F(x) \) is a superadditive function for \( x \geq 0 \)), the result is a direct consequence of Theorem 3.1.

Some examples of parametric families of life distributions which are increasingly superadditive as the parameter \( \theta \) increases are:

(a) **Weibull.** \( F_\theta(t) = 1 - \exp(-\lambda t^\theta), \ t \geq 0, \lambda > 0. \)

(b) **Gamma.** \( F_\theta(t) = \frac{\Gamma(\theta + 1)}{\Gamma(\theta)} \int_0^t \exp(-\lambda x) \, dx, \ t \geq 0, \lambda > 0. \)

In each case, for fixed \( \lambda > 0 \) and \( 0 < \theta_1 < \theta_2 < \infty \), \( F_{\theta_2} \) is superadditive with respect to \( F_{\theta_1} \). It follows that the power function is an increasing function of the parameter \( \theta \).
We next turn to consistency. From Theorem 2.1 it is easily seen that the NBU test is consistent if and only if \( \Delta(F) < \frac{1}{4} \). We now prove

**Theorem 3.2.** Let \( F \) be superadditive with respect to \( G \). Then \( \Delta(F) \leq \Delta(G) \).

**Proof.** Make the transformation \( F(x_i) = G(y_i), i = 1, 2. \)

Then

\[
\Delta(G) = \int \int \frac{G}{y_1 + y_2} \, dG(y_1) \, dG(y_2) = \int \int [G^{-1}(x_1) + G^{-1}(x_2)] \, dF(x_1) \, dF(x_2).
\]

Since \( G^{-1} \) is superadditive, then

\[
G^{-1}(x_1) + G^{-1}(x_2) \leq G^{-1}(x_1 + x_2).
\]
Combining (3.3) and (3.4) gives

\[ \Delta(G) \geq \int \int G^{-\frac{1}{2}} \overline{F}(x_1 + x_2) \, dF(x_1) \, dF(x_2) = \Delta(F). \]  

**Theorem 3.3.** If \( F \) is a continuous, NBU, and not exponential, then the NBU test is consistent.

**Proof.** We need only show that the hypotheses imply \( \Delta(F) < \frac{1}{4} \).

Since \( F \) is assumed continuous, \( \gamma(F) = \frac{1}{4} - \Delta(F) \) and we may equivalently prove \( \gamma(F) > 0 \). Set \( D(x_1, x_2) = \overline{F}(x_1) \overline{F}(x_2) - \overline{F}(x_1 + x_2) \). Then \( D(x_1, x_2) > 0 \) for all \( x_1, x_2 \geq 0 \) since \( F \) is NBU and \( D(x_1, x_2) \neq 0 \) since \( F \) is not exponential.

Assume that \( x_0^1, x_0^2 \) are such that \( D(x_0^1, x_0^2) > 0 \). Let

\[ x_i^* = \sup\{ x : x \geq x_i^0 \text{ and } \overline{F}(x) = \overline{F}(x_i^0) \}, \quad i = 1, 2. \]

Then

\[ D(x_1^*, x_2^*) = \overline{F}(x_1^*) \overline{F}(x_2^*) - \overline{F}(x_1^0 + x_2^0) = \overline{F}(x_1^0) \overline{F}(x_2^0) - \overline{F}(x_1^0 + x_2^0) = D(x_1^0, x_2^0) > 0. \]

Since \( F \) is continuous, \( D \) is continuous and there exist \( \delta_1 > 0, \delta_2 > 0 \), such that \( D(x_1^0 + \delta_1, x_2^0 + \delta_2) > 0 \). Also \( F(x_1^0 + \delta_1) - F(x_1^0) > 0, i = 1, 2 \), since \( x_1^* \) and \( x_2^* \) are points of increase of \( F \). Thus \( \gamma(F) > 0 \).

The NBU distribution used in the following example illustrates that the middle equality of (1.4), namely \( \gamma(F) = \frac{1}{4} - \Delta(F) \), need not hold when the continuity assumption is removed. This example thus emphasizes that \( \Delta(F) \), rather than \( \gamma(F) \), is the basic consistency parameter of the NBU test, since the NBU test is consistent if and only if \( \Delta(F) \) is less than \( 1/4 \) (rather than if and only \( \gamma(F) \) is greater than zero.)
EXAMPLE 3.1. Let $\tilde{F}(x) = \exp(-[\lambda x])$ for $x \geq 0$, where $[x]$ denotes
the largest integer less than or equal to $x$. We now show that $\gamma(F) = 0$
and $\Delta(F) = (e+1)^{-2} \approx 0.072$. Since $\gamma(F)$ and $\Delta(F)$ are scale
invariant (i.e., if $F(x, \beta) = F(\frac{x}{\beta}, 1)$ for every $\beta > 0$, then $\gamma(F_\beta)$ and $\Delta(F_\beta)$ are
constant in $\beta$) we may take $\lambda = 1$. Since $\tilde{F}(i) \tilde{F}(j) - \tilde{F}(i+j) = e^{-i}e^{-j}e^{-(i+j)} = 0$,
we have

$$\gamma(F) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \{\tilde{F}(i) \tilde{F}(j) - \tilde{F}(i+j)\} \ dF(i) \ dF(j) = 0.$$  

We next determine $\Delta(F)$. Since $\gamma(F) = 0$, we have

$$\Delta(F) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{F}(i) \tilde{F}(j) \ dF(i) \ dF(j) = \{\sum_{i=1}^{\infty} \tilde{F}(i) \ dF(i)\}^2.$$  

But,

$$\sum_{i=1}^{\infty} \tilde{F}(i) \ dF(i) = \sum_{i=1}^{\infty} e^{-i}e^{-(i-1)} = (e-1) \sum_{i=1}^{\infty} e^{-2i} = (e+1)^{-1}.$$  

Since $\Delta(F) = (e+1)^{-2} < \frac{1}{4}$, the NDU test is consistent against this
alternative even though $\gamma(F) = 0$. 
Example 3.2 below provides a class of NBU alternatives for which the NBU test is not only consistent, but for which the NBU test has power identically equal to 1 for every n. (We make the minor restriction that $a_n$, the level of the test based on small $J_n$ values, exceed $P_o[J_n = 0] = (2^{n-2})^{-1}$ (see (5.3)), so that when $J_n = 0$ we reject $H_o$ with probability 1.)

**EXAMPLE 3.2.** Let $F_{a,b}$ denote the class of distributions with support $[a, b]$ where $b < 2a$. Then, by considering the three cases i) $x < a$; $y < a$, ii) $x > a$; $y > a$, and iii) $x < a$; $y > a$, and substituting in (1.1) one directly verifies that every $F \in F_{a,b}$ is NBU. But for every $F \in F_{a,b}$, $P_F[J_n = 0] = P_F[X(n) < X(1) + X(2)] = 1$, and thus $P_F[\text{Rej } H_o] > P_F[J_n = 0] = 1$.

b. **Asymptotic relative efficiency and power.** As far as we know, no other tests have as yet been proposed for testing against NBU alternatives. Thus in this section we compare the proposed NBU test with tests designed for a smaller class of alternatives, the IFR class. When the underlying distribution is actually IFR, it is to be expected that an IFR test will in general perform better than the NBU test. Switching the comparison to grounds where the NBU test should excel, we exhibit a class of NBU distributions for which the NBU test performs distinctly better then the IFR tests.

IFR tests that have been proposed include:

1. Proschon and Pyke (1967): Define the normalized spacings
\[ S_i = (n-i+1)(X(i) - X(i-1)), \] where \( X(1) \leq \ldots \leq X(n) \) are the ordered \( X \)'s with \( X_o \overset{\text{def}}{=} 0 \). The Proschan-Fyke test rejects \( H_o \) in favor of \( H_1^* \) for large values of

\[(4.1) \quad V_n = \sum_{i<j} V_{ij}, \]

where \( V_{ij} = 1 \) if \( S_i > S_j \), 0 otherwise.

(ii) Total time on test (cf. Epstein (1960), Barlow (1968), Bickel and Doksum (1969), Bickel (1969), Barlow and Proschan (1969), Barlow (1970)): Reject \( H_o \) in favor of \( H_1^* \) for large values of the cumulative total time on test statistic

\[(4.2) \quad K_n = \frac{n-1}{\sum_{i=1}^{n-1} S_i / \sum_{t=1}^{n} S_t}. \]

(iii) Bickel and Doksum (1969): Reject \( H_o \) in favor of \( H_1^* \) for large values of

\[(4.3) \quad W_n = \sum_{i=1}^{n} \left( -\ln(1 + (n+1) \left[ -\ln(1 - R_i/(n+1)) \right] \right), \]

where \( R_i \) is the rank of \( S_i \) in the joint ranking of \( S_1, \ldots, S_n \).

Let \( \{F_n^{\theta_o}\} \) be a sequence of alternatives with \( \theta_n \rightarrow \theta_o \), where \( F_n^{\theta_o} \) is exponential. From the results of Proschan and Pyke (1967), Bickel and Doksum (1969), and Theorem 2.1, we find the Pitman asymptotic relative efficiency of the NEU test with respect to the Proschan-Fyke test to be

\[(4.4) \quad e_F(J,V) = (12/5)(A^*(\theta_0)/\mu^*(\theta_0))^2, \]

where
\[ (4.5) \quad \Delta(\theta) = \iint_{\mathbb{R}^2} f_{\theta}(x+y) \, dF_{\theta}(x) \, dF_{\theta}(y), \]

\[ (4.6) \quad \mu(\theta) = \int \int_{\mathbb{R}^2} q_{\theta}(y)[q_{\theta}(x)+q_{\theta}(y)]^{-1} f_{\theta}(x) \, f_{\theta}(y) \, dy \, dx, \]

are the asymptotic means of \( J_n \) and \( V_n \) respectively for the alternative \( F_{\theta} \), the factor \((12/5)\) in \((4.4)\) equals \( \lim_n \{ \text{Var}_o(V_n)/\text{Var}_o(J_n) \} \), and \( \Delta'_{\theta}(\theta_o)(\mu'_{\theta}(\theta_o)) \) is the derivative of \( \Delta(\theta)(\mu(\theta)) \) with respect to \( \theta \), evaluated at \( \theta = \theta_o \). For simplicity, we have used the \( V_n \) test in our efficiency calculations. Bickel and Doksum (1969) have shown \( e_{F}(V_n) = \frac{3}{4} \) and \( e_{F}(K_n) = 1 \), for all \( F \), and thus \( e_{F}(J_n) = e_{F}(J_n) = (3/4) e_{F}(J_n) \). We now calculate \( e_{F}(J_n) \) for Weibull and linear failure rate alternatives.

**WEIBULL ALTERNATIVES:** Consider the IFR alternatives

\[ (4.7) \quad F_1(x) = 1 - \exp(-\lambda \theta x^\theta), \lambda > 0, \theta > 1, x \geq 0, \]

where \( q_{\theta}(x) = \theta \lambda x^{\theta-1} \) and \( H_o \) is achieved at \( \theta = \theta_o = 1 \). Then

\[ (4.8) \quad \Delta(\theta) = \iint \theta^2 \lambda^2 (xy)^{\theta-1} \exp(-\lambda \theta (x+y)^\theta + x^\theta + y^\theta) \, dx \, dy, \]

and

\[ (4.9) \quad \Delta'(1) = \lambda^2 \int \left[ 2 + \ln(xy) - \lambda(x+y) \ln(x+y) - \lambda x \ln x - \lambda y \ln y \right] \exp(-2\lambda(x+y)) \, dx \, dy = -\frac{1}{8}. \]

Prochan and Pyke (1967) obtained \( \mu'(1) = (1/4) \ln 2 \), and thus from \((4.4)\) we have

\[ (4.10) \quad e_{F_1}(J_n) = \frac{3}{5(\ln 2)^2} = 1.25. \]
LINEAR FAILURE RATE ALTERNATIVES: Consider the IFR alternatives

\[(4.11) \quad F_2(x) = 1 - \exp(-(x+(\theta x^2/2))), \quad \theta \geq 0, \quad x \geq 0,\]

where \(q_\theta(x) = 1 + \theta x\) and \(H_0\) is achieved at \(\theta = \theta_0 = 0\). Then

\[(4.12) \quad \Delta(\theta) = \iint (1+\theta x)(1+y) \exp(-\{2x+2y+\theta x^2+\theta y^2+\theta xy\}) \, dx \, dy,\]

and

\[(4.13) \quad \Delta^*(0) = \iint (-x^2-y^2-xy+y+x) \exp(-\{2x+2y\}) \, dx \, dy = -1/16.\]

Also,

\[(4.14) \quad \mu(\theta) = \int_0^\infty \int_0^\infty (1+\theta y)^2(1+\theta x)[2+\theta(x+y)]^{-1} \exp(-\{x+y+(\theta/2)(x^2+y^2)\}) \, dy \, dx,\]

and

\[(4.15) \quad \mu^*(0) = (1/4) \int_0^\infty (-x^2-y^2+3y+x) \exp(-\{x+y\}) \, dy \, dx = 1/8.\]

Then from (4,4) we have

\[(4.16) \quad e_{F_2}^*(J,V) = .60.\]
Next we compare the power of the NBU test $J_n$ with the power of the IFR tests $V_n$ and $W_n$ for the class $F_{a,b}$ of NBU distributions introduced in Example 3.2. We showed there that the $J_n$ test has power 1 for every $n \geq 3$, for every $F \in F_{a,b}$, as long as $\alpha > \left(\frac{2n-2}{n}\right)^{1}$. Consider the $V_n$ and $W_n$ tests based on the normalized spacings. For simplicity, take $n = 3$ and $\alpha = 1/6$. Then both $V_n$ and $W_n$ reject $H_0$ when $A = [S_1 > S_2 > S_3]$ occurs. It is easily seen that for every $F \in F_{a,b}$, $P_F[S_1 > S_2, S_1 > S_3] = 1$, but for many distributions in $F_{a,b}$, $P_F[A] < 1$, implying that for these distributions the power of the $V_n$ (and $W_n$) test is less than 1. Here the $\alpha = 1/4$ test based on $J_n$ has power 1. The case $n = 3$ was chosen for convenience. It is clear that for larger $n$ we can exhibit $F's \in F_{a,b}$ for which the powers of the $V_n$ and $W_n$ tests are less than 1 (the latter value being the power of the $J_n$ test) and at the same time where the corresponding Type I errors of $V_n$ and $W_n$ exceed that of $J_n$.

Of course, the class $F_{a,b}$ contains $F's$ that are IFR (e.g. uniform on $[a,b]$) and ones that are not. Thus this class simultaneously provides 1) $F's$ which are NBU but not IFR for which the NBU test is better (as is to be expected) and 2) $F's$ which are IFR for which the NBU test is better!
5. **The null distribution.** Define

\[(5.1) \quad T_n = n(n-1)(n-2) J_n^2 = \sum \psi(X_{a_1}, X_{a_2} + X_{a_3}).\]

Let \(X(1) \leq \ldots \leq X(n)\) denote the ordered \(X\)'s. Since \(i \leq \max(j,k)\)
implies \(\psi(X(i), X(j) + X(k)) = 0\), we can rewrite \(T_n\) as

\[(5.2) \quad T_n = \sum_{i>j>k} \psi(X(i), X(j) + X(k)).\]

Note that \(T_n\) has possible values \(0, 1, \ldots, n(n-1)(n-2)/6\). Exact percentile points for the NBU test can be obtained from the distribution of \(T_n\), calculated under the assumption that the \(X\)'s are exponential. For even moderate \(n\), these calculations are prohibitive. We have obtained exact probabilities in some special cases. Where exact probabilities are available, they show excellent agreement with the Monte Carlo values given in Table 5.1.

a) \(P_0[T_n = 0], n \geq 3\): Define the spacings \(A_i = X(i)-X(i-1), i = 1, \ldots, n\). Then

\[P_0[T_n = 0] = P_0[X(n) < X(1) + X(2)]\]

\[= P_0[\sum_{i=1}^{n} A_i < 2A_1 + A_2] = P_0[\sum_{i=3}^{n} A_i < A_1]\]

\[= n! \int \ldots \int \prod_{i=1}^{n} \exp(-(n-i+1)a_i) \, da_i.\]

Evaluating the integral we obtain,
(5.3) \[ P_0[T_n = 0] = (n-2)(n-3) \cdots 1 \prod_{i=3}^{n} (2n-i+1)^{-1} = \left(\frac{2n-2}{n}\right)^{-1}. \]

b) \( P_0[T_n \leq 1], n \geq 4: \)

\[ P_0[T_n \leq 1] = P_0[X(n) < X(1)^+X(3)^+, X(n-1) < X(1)^+X(2)] \]

\[ = P_0[\sum_{i=4}^{n} A_i < A_1, \sum_{i=3}^{n-1} A_i < A_1] \]

\[ = n! \int \cdots \int \Pi \exp(-(n-i+1)a_i)da_i \]

\[ = n! \int \cdots \int \Pi \exp(-(n-i+1)a_i)da_i \]

Evaluating these integrals we obtain,

\[ P_0[T_n \leq 1] = (n-3)! \left[ \prod_{i=4}^{n-1} \left(\frac{2n-i+1}{(2n-2)(2n-1)} \right)^{-1} \left(\frac{(n-2)}{(2n-2)(2n-1)} - \frac{1}{(2n-1)} + \frac{1}{(n+1)} \right) \right] \]

\[ = \left(\frac{2n-3}{n-3}\right)^{-1} \left(\frac{(3n-1)(n-2)}{(2n-2)(2n-1)} \right), n \geq 4. \]

(Note that the factor \( \prod_{i=4}^{n} \) is defined to be 1 for \( n = 4 \). For \( n = 3, \)

\[ P_0[T_3 \leq 1] = 1. \]

c) \( P_0[T_n = n(n-1)(n-2)/6], n \geq 3: \)

\[ P_0[T_n = n(n-1)(n-2)/6] = P_0[X(3)^+X(1)^+X(2), X(4)^+X(2)^+X(3), \ldots, X(n)^+X(n-1)^+X(n-2)] \]
\[ = n! \int \int \int \ldots \int_{0}^{\infty} x_1 x_2 x_{-2}^+ x_{-3} x_{n-1}^+ x_{n-2} \prod_{i=1}^{n} \exp(-x_{n-i+1}) dx_{n-i+1}. \]

To evaluate this \( n \)-fold integral, let

\[ (5.5) \quad c_1^{(j)} = \text{coefficient of } -x_{n-j} \text{ (in the exponent) after the } j^{th} \text{ integration}, \]

\[ (5.6) \quad c_2^{(j)} = \text{coefficient of } -x_{n-j-1} \text{ (in the exponent) after the } j^{th} \text{ integration}. \]

We then obtain

\[ (5.7) \quad c_1^{(j)} = c_1^{(j-1)} + c_2^{(j-1)}, \]

\[ (5.8) \quad c_2^{(j)} = c_1^{(j-1)} + 1, \]

with \( c_1^{(0)} = c_2^{(0)} = 1. \) Thus

\[ (5.9) \quad P_0[T_n = n(n-1)(n-2)/6] = n![\prod_{j=1}^{n-1} (c_1^{(j-1)})^{-1}\prod_{j=1}^{n} (c_1^{(j-1)} + c_2^{(j-2)})^{-1}. \]

Explicit expressions for \( c_1^{(j)}, c_2^{(j)} \) may be derived as follows.

Define the \( y(\cdot) \) sequence by \( y(j+2) = c_2^{(j)} \) with the initial values

\( y(0) = 0, y(1) = 1. \) Then the \( y(\cdot) \) sequence satisfies

\[ (5.10) \quad y(j+2) = y(j+1) + y(j); \]

this is the famous Fibonacci sequence for which (cf. Brand (1966), p. 381)
(5.11) \[ y(n) = (5)^{-\frac{1}{2}} \left\{ \left[ (1+\sqrt{5})/2 \right]^n - \left[ (1-\sqrt{5})/2 \right]^n \right\}. \]

To obtain a general expression for the terms of the sequence, define the \( z(\cdot) \) sequence by \( c(0) = z(0) = 0 \), \( z(1) = 0 \), and use (5.8) and (5.11) to find.

(5.12) \[ z(n-1) = (5)^{-\frac{1}{2}} \left\{ \left[ (1+\sqrt{5})/2 \right]^n - \left[ (1-\sqrt{5})/2 \right]^n \right\} - 1. \]

Using (5.11) and (5.12) we can rewrite (5.9) as

\[
P_o[T_n = n(n-1)(n-2)/6] =
\]

\[
(5.13) \quad \frac{n!}{\prod_{j=1}^{n-1} \left\{ (-1+(5)^{-\frac{1}{2}} \left\{ (1+\sqrt{5})/2 \right)^{j+2} - (1-\sqrt{5})/2 \right\}^{j+2} \right\}^{-1}} \]

\[
\left[ -1+(5)^{-\frac{1}{2}} \left\{ (1+\sqrt{5})/2 \right]^n - (1-\sqrt{5})/2 \right]{n+1} \right\} + (5)^{-\frac{1}{2}} \left\{ (1+\sqrt{5})/2 \right]^n - (1-\sqrt{5})/2 \right\}^n \}
\]

\[ \]

\[ d) \quad P_o[T_4 = a], \quad a = 0, \ldots, 4: \quad \text{Here we give the complete } H_o \text{ distribution of } T_4. \quad \text{(Note, we have already found, via (5.3), the complete } H_o \text{ distribution of } T_3. \quad \text{Namely, } P_o[T_3 = 0] = \frac{1}{4}, \quad P_o[T_3 = 1] = \frac{3}{4}. \]

\[ P_o[T_4 = 0] = P_o[X(4) < X(1) + X(2)] = 7/105, \]

\[ P_o[T_4 = 1] = P_o[X(1) + X(2) > X(3), X(1) + X(2) < X(4), X(1) + X(3) > X(4)] = 4/105, \]

\[ P_o[T_4 = 2] = P_o[X(1) + X(2) < X(3), X(1) + X(3) > X(4)] \]

\[ + P_o[X(1) + X(2) > X(3), X(1) + X(3) < X(4), X(2) + X(3) > X(4)] = 16/105, \]

\[ P_o[T_4 = 3] = P_o[X(1) + X(2) < X(3), X(1) + X(3) < X(4), X(2) + X(3) > X(4)] \]

\[ + P_o[X(1) + X(2) > X(3), X(2) + X(3) < X(4)] = 33/105 \]
\[ P_0[T_n = 4] = P_0[X(4) > X(2) + X(3), X(3) > X(1) + X(2)] = \frac{45}{105}. \]

To make the NBU test practical, we need more tail probabilities than those available via calculations a) - d). Table 5.1, based on Monte Carlo sampling, gives lower and upper critical points of \( T_n \) in the \( \alpha = .01, .025, .05, .075, \) and \( .10 \) regions for \( n = 4(1)20(5)50. \) For \( n \leq 19, \) the values are based on 100,000 replications, for \( n > 19, \) on 10,000 replications. In Table 5.1, the lower tail should be used for tests of \( H_0 \) versus \( F \) NBU, the upper tail for tests of \( H_0 \) versus \( F \) NWU. The lower tail values are integers \( C^L_\alpha \) for which the estimated probabilities \( P_0[T_n \leq C^L_\alpha] \) are closest to \( \alpha, \) and similarly the upper tail values are integers \( C^U_\alpha \) for which the estimated probabilities \( P_0[T_n \geq C^U_\alpha] \) are closest to \( \alpha. \) Parenthetical entries adjacent to critical points give the Monte Carlo estimated tail probabilities.

For a given tail, these entries are omitted when \( n \) is such that all estimated probabilities are within .002 of the nominal \( \alpha. \) For \( n \geq 25, \) all estimated probabilities agree with the nominal \( \alpha \) (to three decimal places). For \( n > 50, \) use the normal approximation (see Section 2) keeping in mind that i) lower tail probabilities of events of the form \( [T_n \leq a] \) are underestimated using the normal approximation and upper tail probabilities of events \( [T_n \geq b] \) are overestimated, and ii) for fixed \( n, \) the approximation improves as \( \alpha \) increases, \( 0 \leq \alpha \leq 1/2. \)

Table 5.2 compares some Monte Carlo probability estimates with exact probabilities obtained in parts a) - d) of this section. The very close agreement inspires confidence that the entries in Table 5.1 are accurate.
|       | 15.05 | 16.90 | 18.97 | 21.10 | 23.39 | 25.81 | 28.37 | 31.00 | 33.71 | 36.50 | 39.36 | 42.31 | 45.36 | 48.52 | 51.79 | 55.15 | 58.61 | 62.17 | 65.83 | 71.60 | 77.49 | 83.51 | 90.70 | 100.00 |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|      1 |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       | 100.00 |

**Critical Values of the Z Statistic**

**Table 5.1**
TABLE 5.2

A comparison of some exact probabilities and the corresponding Monte Carlo estimates

<table>
<thead>
<tr>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
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<tbody>
<tr>
<td>( P_0 [T_n = 0] ):</td>
<td>.067</td>
<td>.018</td>
<td>.005</td>
<td>.001</td>
</tr>
<tr>
<td>( \hat{P}_0 [T_n = 0] ):</td>
<td>.066</td>
<td>.018</td>
<td>.005</td>
<td>.001</td>
</tr>
<tr>
<td>( P_0 [T_n \leq 1] ):</td>
<td>.105</td>
<td>.028</td>
<td>.007</td>
<td>.002</td>
</tr>
<tr>
<td>( \hat{P}_0 [T_n \leq 1] ):</td>
<td>.103</td>
<td>.028</td>
<td>.007</td>
<td>.002</td>
</tr>
<tr>
<td>( P_0 [T_n = n(n-1)(n-2)/6] ):</td>
<td>.429</td>
<td>.179</td>
<td>.054</td>
<td>.011</td>
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<tr>
<td>( \hat{P}_0 [T_n = n(n-1)(n-2)/6] ):</td>
<td>.431</td>
<td>.177</td>
<td>.053</td>
<td>.011</td>
</tr>
</tbody>
</table>

Acknowledgments. We are indebted to Douglas E. Whitten for programing the Monte Carlo calculations which led to Table 5.1, and to Ramesh M. Korwar for checking the efficiency calculations.
REFERENCES


