MOMENT FORMULAS FOR \( [X/m] \)

by

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1. **Introduction.** Let $X$ be an integer-valued random variable. Let $m$ be a given integer $> 1$ and let $[X/m]$ denote the greatest integer $\leq X/m$. We provide formulas for characteristics of the distribution of $[X/m]$ expressed in terms of characteristics of the distribution of $X$. Exact formulas are given for the characteristic function and moments, and useful approximations are introduced and analyzed.

The motivating application, in which $X$ plays the role of number of events of a Poisson series in a fixed time interval, is dealt with in Section 4. The formulas utilized there are derived in general form in Section 2, and analysis is carried out in both Sections 2 and 3.

**Notation.** For a random variable $\xi$, we shall denote by $\mu_{\xi}^{(k)}$ the $k$-th moment $E_{\xi}^{k}$, by $\psi_{\xi}(z)$ the characteristic function $E_{\exp(i\xi z)}$, and by $\psi_{\xi}^{(k)}$ the $k$-th derivative of $\psi_{\xi}$.
2. General Formulas. The characteristic function of \( [X/m] \), for a fixed integer \( m > 1 \), is expressed in terms of that of \( X \). From this basic result, differentiation yields exact formulas for the moments of \( [X/m] \) in terms of characteristics of \( X \). Besides the exact expressions, convenient approximations are introduced. Cases in which the approximations have favorable asymptotic properties are treated here and in Section 3.

**CHARACTERISTIC FUNCTION OF \([X/m]\).** For all real \( z \),

\[
\begin{align*}
\psi_{[X/m]}(z) &= \frac{1}{m} \sum_{r=0}^{m-1} \sum_{s=0}^{m-1} e^{-\frac{i(z+2\pi s)}{m}} \psi_{X/m} \left( \frac{z+2\pi s}{m} \right).
\end{align*}
\]

**PROOF.** Let \( r_0, r_1 \) be integers, \( 0 \leq r_0, r_1 \leq m-1 \). Let \( \delta_{ij} = 1 \) or 0 according as \( i = j \) or not. We will use

\[
\sum_{r=0}^{m-1} \delta_{r_0 r} e^{\frac{iz}{m} (r_0 - r)} = 1
\]

and

\[
\delta_{r_0 r_1} = \frac{1}{m} \sum_{s=0}^{m-1} e^{-\frac{i2\pi s}{m} (r_0 - r_1)}.
\]

Since \( X \) is integer-valued, the random variable \( R = X - m[X/m] \) takes values 0, 1, ..., \( m-1 \). Hence

\[
\begin{align*}
iz \sum_{r=0}^{m-1} e^{-\frac{i(z+2\pi s)}{m} (X - r)} &= \frac{1}{m} \sum_{r=0}^{m-1} \sum_{s=0}^{m-1} e^{-\frac{i(z+2\pi s)}{m} (R - r)} \\
&= \frac{1}{m} \sum_{r=0}^{m-1} \sum_{s=0}^{m-1} e^{-\frac{i(z+2\pi s)}{m} (X - r)}.
\end{align*}
\]

(2.4)
Taking expectations, (2.1) follows.

Turning now to moments, we note the existence of various sufficient conditions for moments \( \mu_x^{(k)} \) to be given by \( i^{-k} \psi_x^{(k)}(0) \), \( k = 1, 2, \ldots \). An example ([6], p. 487) is

\[
(2.5) \quad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mu_x^{(n)}(n) = L < \infty.
\]

When \( X \) satisfies some such condition, then so does \([X/m] \) and differentiation of (2.1) leads to the following representations.

**Moments of \([X/m]\).** For each \( k = 1, 2, \ldots \),

\[
(2.6) \quad \mu_x^{(k)} = \sum_{m=0}^{\infty} \sum_{r=0}^{m} \sum_{s=0}^{n} \mu_{[X/m]}^{(k)} \frac{i^{-2\pi rs}}{m} \psi_x^{(j)}(2\pi ms).
\]

The terms corresponding to \( s = 0 \) in (2.6) provide an appealing approximation,

\[
(2.7) \quad A_k = \sum_{m=0}^{\infty} \sum_{r=0}^{m} \mu_{[X/m]}^{(k)} \psi_x^{(j)}(2\pi rs).
\]

wherein the distribution of \( X \) is involved solely through its first \( k \) moments. In certain cases, as to be seen below and in Section 3, \( A_k \) is asymptotically equivalent to \( \mu_{[X/m]}^{(k)} \) as an appropriate parameter tends to \( \infty \).

The remainder, \( R_k = \mu_{[X/m]}^{(k)} - A_k \), will be dealt with in Section 3 for the two cases that the distribution of \( X \) is either compound Poisson or of power series type. The analysis will use moment properties of \( X \). In the present section, without restrictions on
the distribution of \( X \), we shall study \( R_1 \) in some detail because of its relative simplicity and importance. Other properties than moments of \( X \) will be used.

Before proceeding with \( R_1 \), we give a convenient expression for \( R_k \). Using (2.3), the terms in \( R_k \) corresponding to \( j = k \) may be found to sum to zero, so that \( R_k \) may be written

\[
R_k = -\frac{i^{-k}}{m+k+1} \sum_{r=1}^{m-1} \sum_{s=1}^{m-1} \frac{e^{i \frac{2\pi rs}{m}}}{k-1} \sum_{j=0}^{k-1} \frac{(-ir)^k - j \psi_X(j)}{m}. \tag{2.8}
\]

In particular,

\[
R_1 = -\frac{1}{m^2} \sum_{r=1}^{m-1} \sum_{s=1}^{m-1} r \omega_s^{-r} \psi_X \left( \frac{2\pi s}{m} \right), \tag{2.9}
\]

where \( \omega_s = \exp(i2\pi s/m) \), \( s = 0, 1, \ldots, m-1 \), are the \( m \)-th roots of unity. Define \( P_r = P_{r,X} = \sum_{n=-\infty}^{\infty} P(X = mn + r) \). Then either by direct computation or from (2.9) one obtains

\[
R_1 = -\frac{1}{m^2} \sum_{r=0}^{m-1} (r - \frac{m-1}{2}) P_r. \tag{2.10}
\]

Obviously, a sufficient condition for \( R_1 = 0 \) is \( P_r = 1/m \) for \( r = 0, 1, \ldots, m-1 \), which occurs when \( X \) has a uniform distribution on \( q \)m consecutive integers for a positive integer \( q \). More generally, it follows easily from (2.10) that

\[
|R_1| \leq \frac{1}{4} \max_{0 < r < m-1} \left| P_r - \frac{1}{m} \right|. \tag{2.11}
\]

Thus \( R_1 \) is "close" to zero when the \( P_r \) are "nearly equal". For example, suppose that \( X \) is unimodal with mode \( M \), i.e., \( P(X = n)/P(X = n + 1) \)
is < 1 for n < M and ≥ 1 for n ≥ M. Then it can be shown that

\[(2.12) \quad |P_{\lambda} - \frac{1}{m}| \leq P(X = M).\]

Thus, for X Poisson with parameter \( \lambda \), we have

\[(2.13) \quad |R_1| \leq \frac{1}{4} e^{-\lambda} [\lambda]/[\lambda]! \simeq \frac{1}{4\sqrt{2\pi}\lambda}, \lambda \to \infty.\]

However, sharper results than (2.13) will be obtained later.

A more delicate approach to the analysis of \( R_1 \), based on the behavior of \( \psi_X \) on the interval \([0, \pi]\), is now considered. Note that (2.9) yields

\[(2.14) \quad |R_1| \leq \frac{m-1}{2} \max_{1 \leq s \leq m-1} |\psi_X^{(2\pi s)}(\lambda)|.\]

Define \( \psi_X \) to be monotone if \( |\psi_X(z)| \), or equivalently \( \psi_X(z)\overline{\psi}_X(z) \) is nonincreasing for \( z \in [0, \pi] \). When \( \psi_X \) is monotone we have

\[(2.15) \quad |R_1| \leq \frac{m-1}{2} \psi_X^{(2\pi)}(m).\]

Some examples of monotone \( \psi_X \) are as follows:

(i) If \( P(X = 0) = 1 \), then \( \psi_X(z) = e^{iz} \) and \( |\psi_X(z)| = 1 \).

(ii) If \( P(X = 0) = q = 1 - p = 1 - P(X = 1) \), then \( \psi_X(z) = q + p \exp(iz) \)

and \( \psi_X(z)\overline{\psi}_X(z) = q^2 + p^2 + 2pq \cos(z) \), and thus a Bernoulli variate has

monotone \( \psi \).

(iii) If \( \psi_X \) is monotone, then so is \( \psi_X^n \), and thus by (ii) a

Binomial variate has monotone \( \psi \).

(iv) If \( P(X = -1) = P(X = 1) = \alpha \), \( P(X = 0) = 1 - 2\alpha \), where

\( 0 < \alpha \leq \frac{1}{2} \), then \( \psi_X(z) = 1 - 2\alpha + 2\alpha \cos(z) \).

(v) If \( X \) is Poisson \( (\lambda) \), then \( \psi_X(z)\overline{\psi}_X(z) = \exp[2\lambda(\cos(z) - 1)] \).
(vi) If $X$ is given and the real part of $\psi_X$ is nonincreasing on $[0,\pi]$, then $\exp[\lambda(\psi_X(z) - 1)]$ is monotone. That is, the compound Poisson variate corresponding to $X$ has monotone $\psi$.

For the case that $X$ is Poisson $(\lambda)$, we thus have, by (v) above and (2.15), that

\begin{equation}
|R_1| \leq \frac{e^{\lambda(1 - \cos \frac{2\pi}{m})}}{2^{m-1}}
\end{equation}

an improvement over (2.13). Indeed, (2.17) may be seen to be sharp in asymptotic order of magnitude as $\lambda \to \infty$. 
3. Behavior of \( R_k \) for specific types of distribution.

The remainder term \( R_k \), which has been examined above for the case \( k = 1 \), is dealt with now for general \( k \) but with restrictions on the distribution of \( X \). Namely, the cases \textit{compound Poisson} and \textit{power series type} will be treated. Thirdly, a common member of these two cases, the \textit{simple Poisson}, will be treated separately because of its special importance and in order to state more precise results for it.

(i) \textit{compound Poisson case}. We suppose that the characteristic function of \( X \) is of the form

\begin{equation}
(3.1) \quad \psi_X(z) = \exp \lambda [\psi_Z(z) - 1],
\end{equation}

where \( Z \) is a random variable taking value \( j \) with probability \( f_j \), \( j = \ldots, -1, 0, 1, \ldots \) (If \( f_1 = 1 \), then \( X \) is Poisson.)

We will use Faà di Bruno's formula ([1], p. 823),

\begin{equation}
(3.2) \quad \frac{d^n}{dx^n} g(h(x)) = \sum_{j=0}^{n} g^{(j)}(h(x)) \sum_{(n; a_1, \ldots, a_n)^r \neq (n)} \prod_{k=0}^{n} [h^{(k)}(x)]^{a_k},
\end{equation}

where the inner sum is over \( a_1 + 2a_2 + \cdots + na_n = n \) and \( a_1 + a_2 + \cdots + a_n = j \), and \( (n; a_1, \ldots, a_n)^r \) denotes the number of ways of partitioning a set of \( n = a_1 + 2a_2 + \cdots + na_n \) different objects into \( a_j \) subsets containing \( j \) objects for \( j = 1, 2, \ldots, n \).

Summed within the constraints indicated, the numbers \( (n; a_1, \ldots, a_n)^r \) add up to the number of partitions of \( n \) elements into \( j \) nonempty subsets. The latter numbers are known as the Stirling numbers of the 2nd kind and will be denoted by \( S(n, k) \). Applying (3.2) with \( g(z) = \exp z \) and \( h(z) = \psi_Z(z) \), we obtain

\begin{equation}
(3.3) \quad \psi_X^{(n)}(z) = \psi_X(z) \sum_{j=0}^{n} \frac{\lambda^j}{j!} \sum_{(n; a_1, \ldots, a_n)^r \neq (n)} \prod_{k=0}^{n} [\psi_Z^{(k)}(z)]^{a_k}.\]
\end{equation}
Now, if \( E|Z|^n < \infty \), then clearly we have, for \( \ell \leq n \),

\[
\psi_Z^{(\ell)}(z) = \sum_{j=-\infty}^{\infty} f_j(ij)^{\ell} e^{ijz}
\]

and thus

\[
|\psi_Z^{(\ell)}(z)| \leq E|Z|^\ell \leq (E|Z|^n)^{\ell/n} ,
\]

where the last step follows by Jensen's inequality. The use of (3.5) in (3.3) yields

\[
|\psi_X^{(n)}(z)| \leq |\psi_X(z)| \sum_{j=0}^{n} \lambda_j^n \sum_{(n; a_1, \ldots, a_n)} \prod_{k=0}^{n} (E|Z|^n)^{ka_k/n}
\]

\[
= \exp(\lambda[E(\cos z) - 1])(E|Z|^n) \sum_{k=0}^{n} S(n,k)\lambda^k.
\]

Define

\[
\Delta = \max_{1 \leq s \leq m-1} E(\cos \frac{2\pi s}{m} Z).
\]

Then (3.6) in conjunction with (2.8) gives

\[
|R_k| \leq \frac{m-1}{m^{k+1}} e^{\lambda(n-1)} \sum_{j=0}^{k-1} \binom{k}{j} \sum_{r=1}^{m-1} \sum_{j=0}^{r-k-1} (E|Z|^j)^{\frac{1}{\ell}} S(\ell,\lambda^\ell).
\]

Thus the asymptotic behavior of \( R_k \) as \( \lambda \to \infty \) is characterized by

\[
|R_k| = O(\lambda^{k-1} e^{(n-1)\lambda}) , \lambda \to \infty.
\]

If \( Z \) is such that \( \Delta < 1 \), then as \( \lambda \to \infty \) the remainder term \( R_k \to 0 \) at an exponential rate while the approximation \( A_k \to \infty \) and is thus valid in the asymptotic sense. The condition \( \Delta < 1 \) is satisfied, e.g., if \( Z = 1 \), but not for all choices of \( Z \).
(ii) **power series case.** Following [10], we say that \( X \) has a power series distribution if

\[
P(X = j) = A_j \theta^j / f(\theta), \quad j = \ldots -1, 0, 1, \ldots,
\]

where \( \theta > 0 \), \( A_j \geq 0 \) and \( f(\theta) = \sum A_j \theta^j \). (Besides the Poisson distribution, which corresponds to the case \( A_j = 0 \) if \( j < 0 \) and \( = 1/j! \) if \( j \geq 0 \), this class includes the binomial, negative binomial and logarithmic distributions.) The characteristic function of the distribution in (3.10) is

\[
\psi_X(z) = \sum A_j (\theta e^{iz})^j / f(\theta),
\]

or simply

\[
\psi_X(z) = f(\theta e^{iz}) / f(\theta).
\]

Differentiation of (3.11a) and the use of (3.11b) leads to

\[
(\psi_X^n)(z) = i^n \psi_X(z) \sum j^m A_j (\theta e^{iz})^j / f(\theta e^{iz}).
\]

Let us denote by \( \mu_X^n(z) \) the quantity \( EX^n \) when the parameter \( \theta \) takes the value \( z \). Then (3.12) becomes

\[
(\psi_X^n)(z) = i^n \psi_X(z) \mu_X^n(\theta e^{iz}),
\]

a convenient formula for dealing with specific distributions of power series type.

For example, consider \( X \) **negative binomial:**

\[
P(X = j) = \binom{N-1+j}{N-1} \left( \frac{1}{1+P} \right)^N \left( \frac{P}{1+P} \right)^j, \quad j = 0, 1, \ldots,
\]

where \( N \) is a positive integer and \( 0 < P < \infty \). We have \( A_j = \binom{N-1+j}{N-1} \),

\[
\theta = P/(1 + P) \quad \text{and} \quad f(\theta) = (1 - \theta)^{-N}.
\]

Thus

\[
\psi_X(z) = \left( \frac{1 - \theta}{1 - \theta e^{iz}} \right)^N = (1 + P - Pe^{iz})^{-N}.
\]
For $z$ not a multiple of $2\pi$, we have $|\psi_X(z)| = O(P^{-N})$ as $P \to \infty$, $N$ fixed.

Now the $n$-th moment of $X$ is a polynomial of order $n$ in $P$, so that 

$\mu_X^{(n)}(\theta \exp(iz))$ is a polynomial of order $n$ in the variable 

$\theta \exp(iz)/(1 - \theta \exp(iz)) = P \exp(iz)/(1 + P - P \exp(iz))$. Hence, again for $z$ not a multiple of $2\pi$, $|\mu_X^{(n)}(\exp(iz))| = O(P^n)$ as $P \to \infty$, $N$ fixed.

It follows from (2.8) and (3.13) that, for $k < N$ fixed, the remainder $R_k \to 0$ as $P \to \infty$,

(3.16) \[ |R_k| = O(P^{k-1-N}), \quad P \to \infty, \]

so that $A_k$ is an asymptotic equivalent to $\mu^{(k)}[X/m]$.

As a second example, consider $X$ logarithmic:

(3.17) \[ P(X = j) = \theta^j / j[-\log(1-\theta)], \quad j = 0, 1, \ldots, \]

where $0 < \theta < 1$. Here $A_j = 1/j$ and $f(\theta) = -\log(1 - \theta)$. Let us again consider asymptotics as $P \to \infty$, where $\theta = P/(1 + P)$. As in the previous example, the function $\mu_X^{(n)}(\theta \exp(iz))$ is found to be a polynomial of order $n$ in the variable $P \exp(iz)/(1 + P - P \exp(iz))$, so that its exact order of magnitude is $O(P^n)$ as $P \to \infty$, for $z$ not a multiple of $2\pi$.

However, we find that the magnitude of $\psi_X(z) \to 0$ at the exact rate $O(1/\log P)$ as $P \to \infty$. Consequently, by (3.13), the magnitude of $\psi_X^{(n)}(z)$ does not $\to 0$ as $P \to \infty$, so that the remainder $R_k$ is not negligible in the present example.

(iii) simple Poisson case. The relevant characteristic function is

(3.18) \[ \psi_X(z) = \exp[\exp(iz) - 1]. \]

Appropriate substitutions in (3.8) yield

(3.19) \[ |R_k| \leq \frac{m-l}{m+1} e^{-\lambda(1-\cos2\pi/m)} \sum_{j=0}^{k-1} \sum_{r=1}^{m-1} \sum_{\ell=0}^{j} S(j,\ell)\lambda^\ell, \]

where $S(j,\ell)$ is a certain function of $j$ and $\ell$. This bound is useful for estimating the contribution of the large $k$ terms in the sum (3.8) for large $P$.
so that the approximation $A_k$ is asymptotically valid as $\lambda \to \infty$ and the error satisfies

$$|R_k| = o(\lambda^{k-1} e^{-\lambda (1 - \cos \frac{2\pi}{m})}), \quad \lambda \to \infty.$$  

More precise results may be obtained by making substitutions in (3.3) instead of (3.8). We obtain

$$\psi_X^{(n)}(x) = i^n e^{i \lambda (e^{i \lambda} - 1)} \sum_{j=0}^{n} S(n,j) \lambda^j e^{i j \lambda},$$

which in conjunction with (2.8) leads to the following exact expression:

$$R_k = \sum_{m=1}^{\infty} \sum_{s=1}^{m} \sum_{j=0}^{k-1} \sum_{l=0}^{l} \frac{1}{k+l+1} S(k-r,j,s) S(j,l) \lambda^r \lambda^s \cos \left[ \frac{2\pi m}{m} \frac{2\pi s}{m} (\ell - r) \right].$$

In particular, we have that the first moment of $[X/m]$ is approximated by

$$A_1 = \frac{1}{m} \lambda - \frac{m-1}{2m},$$

with error bounded, according to (3.19), by

$$\frac{(m-1)^2}{2m} e^{-\lambda (1 - \cos \frac{2\pi}{m})},$$

and that the second moment of $[X/m]$ is approximated by

$$A_2 = \frac{1}{m^2} \lambda^2 - \frac{(m-1)^2}{m^3} \lambda + \frac{(m-1)(2m-1)}{6m^2},$$

with error bounded by

$$e^{-\lambda (1 - \cos \frac{2\pi}{m})} \frac{(m-1)}{m} \left[ \frac{(m-1)(2m+5)}{6} \lambda + 1 \right].$$

We have obtained our results for the Poisson distribution as a special case of the development for the compound Poisson case. Although the Poisson distribution is also of power series type, it appears that nothing additional is found by pursuing that avenue.
4. Application to point process theory. The motivating application for the considerations of the previous sections is the point process consisting of every m-th point of a Poisson series (in time t, say). Such a process, called Erlang of order m, has been much studied as a special type of renewal process (see Cox [2]) and in connection with various practical questions in reliability theory (see Mercer [11] and Cox and Lewis [3]), in queueing theory and operations research (see Morse [1]) and Jewell [9]), and in traffic flow theory (see Haight [7], [8], Whittlesey and Haight [17] and Serfling [13]).

In this section we utilize formulas of the previous sections to provide new results concerning the counting distributions of an Erlang of order m process. These are the distributions pertaining to the random counts in fixed intervals of time. Two cases are distinguished, the synchronous and the asynchronous, according as the counting period commences with the occurrence of an event or at a moment selected independently of the stream of events.

It is assumed that the underlying Poisson process is stationary with parameter \( \theta \). In this case the count in any fixed time interval of length t is Poisson distributed with parameter \( \theta t \), and the counts in disjoint time intervals are mutually independent random variables.

For the corresponding Erlang of order m process, let us denote by \( \mu_k(t) \) the k-th moment of the count in a synchronous counting period of length t. Specifically, \( \mu_k(t) \) is the k-th moment of \( [X/m] \), where X is Poisson with parameter \( \theta t \). Therefore, appropriate substitutions for \( \mu_X^{(k)} \) in (2.7) yield convenient approximations for the functions \( \mu_1(t), \mu_2(t), \ldots \). Likewise, formulas (3.19) and (3.22) with \( \lambda = \theta t \)
give complete information about the errors of the approximations. It is seen that these approximations are asymptotically equivalent as the length \( t \) of the counting period tends to \( \infty \), with the error of the \( k \)-th approximation being

\[
0(t^{k-1} e^{-\frac{2\pi}{m} \theta t}), \ t \to \infty.
\]

Of special interest are the first and second moment functions, for which the respective approximations are

\[
\frac{1}{m} \theta t - \frac{m-1}{2m}
\]

and

\[
\frac{1}{m^2} (\theta t)^2 - \frac{(m-1)^2}{m^3} \theta t + \frac{(m-1)(2m-1)}{6m^2}.
\]

Accordingly, the synchronous variance function is approximated by

\[
\frac{(m-1)}{m^3} \theta t + \frac{m^2 - 1}{12m^2}.
\]

Noting that we are dealing in the present instance with a particular renewal process, it should be added that approximation formulas such as given here have been established already in that context. See Smith [15] [16] and Cox [2]. However, the present results provide more precise information about the errors of approximation. Such a degree of precision was found necessary in [13], e.g., in characterizing the correlation structure of the asynchronous counts. Also, the present results provide a basis for determining what length of counting period suffices for a specified level of accuracy in the approximations, a matter of importance in estimating the parameters \( \theta \) and \( m \).

Above we have dealt with the moments \( \mu_1(t), \mu_2(t), \ldots \) of the
synchronous counting distributions, considered as functions of \( t \). From these we may pass to the corresponding \textit{asynchronous} quantities by means of relationships given by Cramér, Leadbetter and Serfling [5]. In particular, the first \( k \) synchronous moment functions are equivalent, in a stationary point process, to the first \( k+1 \) asynchronous moment functions. These transitions are of considerable practical importance, since count data is typically easier to collect in the asynchronous mode than in the synchronous mode. Thus, in developing a certain test procedure to be implemented with count data, Serfling and Wood [14] make use of the results of the present report in conjunction with results in [5] to approximate the first 4 synchronous moment functions of the Erlang or order \( m \) process. We remark that there, as in general, the transition from synchronous to asynchronous moment functions requires use of the exact formula, (2.6), as a starting point, even if the goal is an approximation to asynchronous moments. It is after the relationships in [5] have been applied that a suitable "remainder" term can be distinguished and neglected.
5. Remarks. (i) Other applications. By analogy with the
treatment in Section 4, other types of "m-th event" processes may be
analyzed. Moreover, the results of Sections 2 and 3 are clearly ap-
plicable to nonstationary point processes.

(ii) non-rational values of m. It is not difficult to extend the
results of this report to allow \( m = \frac{N}{D} \), where \( N \) and \( D \) are positive
integers. We simply write \( \lceil \frac{X}{m} \rceil = \lceil \frac{Y}{N} \rceil \), where \( Y = DX \). It would also
be of interest, but probably not easy, to allow \( m \) to take any positive
value.

(iii) Sheppard's corrections. There appears to be a connection
between the distribution of \( \lceil mx \rceil \) for rational \( m \) and integer-valued \( X \n\) and the distribution of \( \lceil cX \rceil \) for \( c > 0 \) and \( X \) having a continuous
distribution. Presumably this connection may be explored with the
use of Sheppard's corrections [4].
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