TESTING POISSON VERSUS ERLANG, USING COUNT DATA

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0. **Summary.** A central role in modeling traffic flow is played by the Poisson model, which is accurate in the case of light density flow and is easy to manipulate mathematically. For moderately high density flow, the Erlang model is a generalization which has theoretical appeal but is somewhat difficult to manipulate. This report lays foundations supporting greater practical use of the Erlang model as an alternative to the Poisson.

A limitation of current procedures is that headway data is required. The present report develops a procedure utilizing count data, which is usually more convenient to acquire. The statistic receiving primary attention is the sample index of dispersion based on Erlang counts. Its asymptotic distribution theory is obtained and application to estimation of parameters and testing of hypotheses is treated.

1. **Introduction.** The Poisson model is widely used in traffic flow theory, possessing special validity in the case of light density flow and in any case being easy to manipulate. For moderately high density flow, a more general model which arises for consideration in a natural way is the Erlang. In this model, headways (gaps between vehicles) have a probability distribution of gamma form, of which a special case is the negative exponential form corresponding to the Poisson model. In the non-exponential case, less weight is attached to very short headways, thus permitting greater accord with the fact that vehicles actually have lengths and cannot follow at infinitesimal headways, a distinction that
has increased importance in high volume conditions. To be specific, we assume that headways are distributed according to a probability density of the form

\[(1.1) \quad f(x) = \theta^m x^{m-1} e^{-\theta x} / (m-1)!, \quad 0 < x < \infty,\]

where \(\theta > 0\) and \(m\) is a positive integer. For \(m = 1\), this corresponds to a Poisson Model. In general, we shall refer to the model given by (1.1) as Erlang of order \(m\). (Since (1.1) is the density of a sum of \(m\) negative exponential variates, the Erlang of order \(m\) model may be regarded as a series of events consisting of every \(m\)-th event of a Poisson series.) Further background details on the roles of the Poisson and Erlang models may be found in Haight [10] and Drew [9].

A major question that arises in practice is whether a Poisson model fits the given situation well enough to be adopted in lieu of the possibly more valid Erlang type. The latter offers greater potential validity because of its additional parameter, but at the price of considerable loss of mathematical simplicity and ease of manipulations. Thus it is necessary to weigh the gain in accuracy against the loss in convenience. Formulating the question as a statistical hypothesis testing problem, we would assume that the headways are distributed according to (1.1) for some unknown \(\theta > 0\) and some unknown integer \(m \geq 1\) and test the "null" hypothesis

\[H_0: m = 1\]

versus some "alternative" hypothesis such as

\[H_1: m > 1\]

or

\[H_{-1}: m = m_0,\]
for a specified integer \( m_0 > 1 \). (E.g., \( m_0 = 2 \) would represent the nearest competitor to the Poisson model within the class given by (1.1).) In this report we develop a basis for testing \( H_0 \). The results will be relevant also to the matter of estimating the parameters \( \theta \) and \( m \).

In dealing with these objectives, we seek procedures which are oriented toward count data (consisting of the observed numbers of vehicles in selected time periods) rather than headway data. If one were to be guided solely by the mathematical exigencies apropos to gamma distributions, one would unavoidably arrive at a statistical procedure whose implementation would stipulate the use of headway data. Indeed, as may be found in Cox and Lewis [5], e.g., there is already available for our problem ample statistical methodology applying to data of headway type. However, considerations of cost and convenience make it desirable to have procedures implementable with count data. Further, we distinguish two types of count data: synchronous (S), whereby an observation period commences at the moment a vehicle arrives at the site of observation, and asynchronous (AS), whereby an observation period commences at a moment selected independently of the stream of arrivals. (See Haight [10] for discussion.) The asynchronous count data is typically the easiest to acquire (see [15]) and in this report we shall be oriented accordingly.

In order to deal with AS count data for testing \( H_0 \), or for estimating the parameters in (1.1), we pass to the relevant "counting distributions," i.e., the probability distributions of the numbers of arrivals in specified observation periods. The AS counts form a stationary point process (see [7]), so that the count in a specified interval has a distribution depending solely on the length of the interval, not upon its location on the time
axis. We shall denote by $v_n(t)$ the probability of $n$ arrivals in a
specified time interval of length $t$. Corresponding to the Erlang of
order $m$ process, whereby interarrival times have probability density
(1.1), we have ([11] [15])

\[(1.2a)\quad v_0(t) = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) e^{-\theta t}(\theta t)^k/k!,\]

\[(1.2b)\quad v_n(t) = \sum_{k=-m+1}^{m-1} \left(1 - \left|\frac{k}{m}\right|\right) e^{-\theta t}(\theta t)^{nm+k}/(nm+k)!, \quad n \geq 1.

This distribution has defied exact analysis. While its mean is simply
$\theta t/m$, the simplest expression for its variance appears to be (see [14])

\[(1.3)\quad \sigma^2(t) = \frac{\theta t}{m^2} + \frac{m^2-1}{6m^2} + E_m(\theta t),\]

where $E_m(z)$ is a cumbersome quantity satisfying

\[(1.4)\quad |E_m(z)| \leq \frac{(m-1)(m-1)}{2} e^{-2(\sin \frac{\pi}{m})^2 z}, \quad z \geq 0.

(Higher moments are even more cumbersome.) It is seen from (1.4) that
the term $E_m(\theta t)$ in (1.3) is exponentially negligible as $t \to \infty$, greatly
simplifying analysis. In §2 and §3 of this report we shall develop
results intended for application (as in §4) with regard to counting
periods of length $t$ sufficiently long for $\sigma^2(t)$ to be approximated reason-
ably well by its asymptote $\theta t/m^2$. 
A characteristic of the distribution (1.2) which is of particular interest in traffic flow theory (see Buckley [3] for discussion) is the index of dispersion, i.e., the ratio of variance to mean. Denoting the index by \( I = I(t) \), we have

\[
(1.5) \quad I(t) = \sigma^2(t)/(\theta t/m) = \frac{1}{m} + o(t^{-1}), \quad t \to \infty.
\]

For a Poisson model the index is unity. By contrast, an Erlang model of order \( m, m > 1 \), is thus "underdispersed" for \( t \) sufficiently large.

We note from (1.5) that the hypothesis \( H_0 \) discussed earlier is essentially a hypothesis about the index of dispersion.

On the basis of the preceding motivations, stressing the importance of utilizing AS count data and formulating our hypothesis-testing problem in terms of the parameter \( I \), we consider a sample index of dispersion based on AS count data.

The sample index, \( \hat{I} \), is defined as follows. Let a length \( t \) for the counting periods be specified and fixed. Let \( X_1, \ldots, X_n \) denote the observed counts corresponding to \( n \) distinct observation periods taken sufficiently far apart in time so as to produce statistically independent counts. Set \( \bar{X} = n^{-1} \sum_1^n X_i \), the sample mean, and \( s^2 = n^{-1} \sum_1^n (X_i - \bar{X})^2 \), the sample variance. By analogy with the definition of \( I \), we put \( \hat{I} = s^2/\bar{X} \). In this report we shall assume that each \( X_i \) has the distribution (1.2).

In §2 we present the asymptotic joint distribution of \( \bar{X} \) and \( s^2 \), for \( n \) and \( t \) taken sufficiently large. As a corollary, the asymptotic
distribution of $\hat{I}$ is obtained. Details of proof are given in §3.

Finally, §4 treats applications to testing of hypotheses and estimation of parameters, with some numerical examples.

2. Asymptotic distribution theory for the sample index of dispersion.

Corresponding to a random sample $X_1, \ldots, X_n$ from a distribution $F$, let

$$(2.1) \quad \bar{X} = n^{-1} \sum_{i=1}^{n} X_i, \quad s^2 = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$ 

The asymptotic bivariate distribution of $(\bar{X}, s^2)$ as $n \to \infty$ is easy to describe in principle and depends only on the moments of the distribution $F$. Let us put $\mu = EX_1 = \int_{-\infty}^{\infty} x dF(x)$ and $\mu_k = E(X_1 - \mu)^k$, $k \geq 2$. By standard theory for sampling distributions (see Cramér [6], Ch. 28), the vector $(\bar{X}, s^2)$ is asymptotically bivariate normal with mean vector $(\mu, \mu_2)$ and covariance matrix $n^{-1} \Sigma$, where

$$(2.2) \quad \Sigma = \begin{bmatrix} \mu_2 & \mu_3 \\ \mu_3 & \mu_4 - \mu_2^2 \end{bmatrix}.$$ 

It is assumed here, of course, that the elements of $\Sigma$ are finite.

We shall apply the foregoing result in the particular case of sampling from the distribution (1.2), the Erlang of order $m$ asynchronous counting distribution. Thus we must have at hand the parameters $\mu$, $\mu_2$, $\mu_3$, and $\mu_4$ for the distribution (1.2). While $\mu = \theta t/m$ is found easily, the other quantities are rather difficult to obtain but nevertheless are accessible via certain recent results [8] [13] [14], which we shall utilize in §3 to obtain suitable approximations to $\mu_2$, $\mu_3$, $\mu_4$ for large
values of $t$, the length of the counting period considered. These approximations, stated in Theorem 3.1 of §3, yield in the present context the following

THEOREM 2.1. For a sample $X_1, \ldots, X_n$ from the distribution (1.2), the vector $(\bar{x}, s^2)$ is asymptotically bivariate normal, as $n \to \infty$, with mean $(\mu, \mu_2)$ and asymptotic covariance matrix $n^{-1} \Sigma$ where

$$
(2.3) \quad \mu = \frac{\theta t}{m}
$$

and the approximations

$$
(2.4) \quad \mu_2 = \frac{\theta t}{m^2}, \quad \Sigma \approx \begin{bmatrix}
\frac{\theta t}{m^2} & \frac{\theta t}{m^3} \\
\frac{\theta t}{m^3} & \frac{2(\theta t)^2}{m^4} + \frac{(2m^2+1)\theta t}{3m^4}
\end{bmatrix}
$$

are valid within an error of order $O(1)$, $t \to \infty$.

We have thus obtained the approximate distribution of $(\bar{x}, s^2)$ for a sufficiently large sample from an Erlang of order $m$ counting distribution based on a sufficiently long counting period. It is possible to deduce somewhat more refined conclusions, if needed, but our objectives in the present report are met by the above-stated theorem.

Turning now to the asymptotic distribution of $\hat{I} = s^2 / \bar{x}$, we apply standard results for functions of asymptotically multivariate normal random variables (see Cramér [6], Ch. 28). Writing $\hat{I} = h(\bar{x}, s^2)$, where
\( h(u,v) = v/u \), we conclude \( \hat{I} \) is asymptotically normal with mean \( h(\mu,\mu_2) \),
which equals the population index of dispersion \( I \), and with asymptotic variance \( \sigma^2/n \), where
\[
(2.5) \quad \sigma^2 = \mu_2 h_1^2 + 2\mu_3 h_1 h_2 + (\mu_4 - \mu_2^2) h_2^2,
\]
and \( h_1 \) and \( h_2 \) are the first order partial derivatives of \( h \) evaluated
at the point \( (u,v) = (\mu,\mu_2) \). With appropriate substitutions in (2.5),
we obtain
\[
(2.6) \quad \sigma^2 = \mu_2^3/\mu - 2\mu_2 \mu_3/\mu^4 + (\mu_4 - \mu_2^2)/\mu^2
\]
and, by analogy with Theorem 2.1, we state

**Theorem 2.2.** For a sample \( X_1, \ldots, X_n \) from the distribution (1.2),
the sample index of dispersion \( \bar{I} = s^2/\bar{x} \) is asymptotically normal, as
\( n \to \infty \), with mean \( \bar{I} = \mu_2/\mu \) and asymptotic variance \( \sigma^2/n \), where \( \sigma^2 \) is
given by (2.6). Further, the approximations
\[
(2.7) \quad \bar{I} \approx \frac{1}{m}, \quad \sigma^2 \approx \frac{2}{m^2}
\]
are valid within an error of order \( O(t^{-1}) \), \( t \to \infty \).

We thus have obtained the approximate distribution of \( \hat{I} = s^2/\bar{x} \) for
a sufficiently large sample from an Erlang of order \( m \) counting distribution
based on a sufficiently long counting period.
3. **Central moments of Erlang of order m counting distribution.** Our object here is to derive for the distribution (1.2) the central moment approximations utilized in the previous section. We shall proceed as follows. Although (1.2) represents an asynchronous mode counting distribution, we shall deal firstly with the associated synchronous mode distribution, for which the characteristic function has been given in closed form by Savage and Serfling [13]. Differentiation of this characteristic function yields the ordinary moments of the synchronous distribution, from which we pass to factorial moments by means of standard identities. From synchronous factorial moments we pass to asynchronous factorial moments by relationships given in Cramér, Leadbetter and Serfling [8], and finally to the desired central moments again calling upon standard identities.

The synchronous counting distributions associated with the headway density (1.1) are usefully described in terms of the Poisson model. For \( m = 1 \), the density (1.1) is negative exponential, \( f(x) = \theta \exp(-\theta x) \), and the corresponding counting process is Poisson. Now, in general, (1.1) is the density of the sum of \( m \) independent equidistributed negative exponential gaps. It follows that a series of events having gap density (1.1) may be viewed as a process consisting of every \( m \)-th event in a Poisson series. This characterization is of use as follows. For a series of events having gap density (1.1), the random count \( Z \) in a synchronous mode counting period of specified length \( t \) may thus be expressed as \( Z = \lfloor Y/m \rfloor \), where \( \lfloor x \rfloor \) denotes the integer part of \( x \) and \( Y \) is the random count in a similar counting period for a Poisson series with parameter \( \theta \). Now it is shown in [13],
for two counting variables related as $Z = [Y/m]$, that the respective characteristic functions, $\psi_Z(s) = E \exp iZs$ and $\psi_Y(s) = E \exp iYs$, are related as follows:

$$
\psi_Z(s) = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{r=0}^{m-1} e^{-ir(s+2\pi k)/m} \psi_Y(s+2\pi k/m).
$$

(3.1)

Applying this result to our case, we have

$$
\psi_Y(s) = e^{\lambda(e^{is} - 1)},
$$

where $\lambda = \theta t$, and thus

$$
\psi_Z(s) = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{r=0}^{m-1} e^{-ir(s+2\pi k)/m} e^{\lambda(e^{is} - 1)} e^{i(s+2\pi k)/m}.
$$

(3.3)

Now the $n$-th moment of $Z$ is given by $i^{-n}\psi_Z^{(n)}(0)$, where $\psi_Z^{(n)}(0)$ denotes the $n$-th derivative of $\psi_Z(\cdot)$ at $s = 0$. Hence we obtain from (3.3), for $n \geq 1$,

$$
EZ^n = \frac{1}{m^{n+1}} \sum_{\ell=0}^{n} \binom{n}{\ell}(-1)^{n-\ell} \sum_{r=0}^{m-1} \sum_{k=0}^{m-1} e^{-i2\pi kr/m} \left[i^{-\ell} \psi_Y(\ell)(2\pi k/m)\right].
$$

(3.4)

The cases of interest to us here are $n = 1, 2, 3$. Recall now that $EZ^n$ is a function of $t$, denote it by $A_n(t)$, and write $A_n(t) = B_n(t) + R_n(t)$, where

$$
B_n(t) = \frac{1}{m^{n+1}} \sum_{\ell=0}^{n} \binom{n}{\ell}(-1)^{n-\ell} \left( \sum_{r=0}^{m-1} \sum_{k=0}^{m-1} e^{-i2\pi kr/m} \right) \psi_Y^{\ell},
$$

(3.5)
which is the contribution to $A_n(t)$ from the terms with $k = 0$ in (3.4).

In particular, using the well-known [2] identities

\[
\begin{align*}
(3.6) & \sum_{r=0}^{m-1} r^0 = m, \quad \sum_{r=0}^{m-1} r = \frac{(m-1)m}{2}, \quad \sum_{r=0}^{m-1} r^2 = \frac{(m-1)m(2m-1)}{6}, \\
& \sum_{r=0}^{m-1} r^3 = \frac{(m-1)^2 m^2}{4},
\end{align*}
\]

we have

\[
(3.7a) \quad B_1(t) = \frac{1}{m} EY - \frac{m-1}{2m},
\]

\[
(3.7b) \quad B_2(t) = -\frac{1}{2} \frac{EY^2}{m^2} + \frac{m-1}{m} \frac{EY}{m} - \frac{(m-1)(2m-1)}{6m^2},
\]

\[
(3.7c) \quad B_3(t) = \frac{1}{3} \frac{EY^3}{m} - \frac{3(m-1)}{3} \frac{EY^2}{2m} + \frac{(m-1)(2m-1)}{3} \frac{EY}{2m} - \frac{(m-1)^2}{4m}.
\]

It will be convenient at this point to convert to factorial moments. Let

\[
\alpha_n = \alpha_n(t) \text{ denote the n-th factorial moment of } Z, \text{ i.e.,}
\]

\[
(3.8) \quad \alpha_n = \alpha_n(t) = E(Z-1) \cdots (Z-n+1), \quad n \geq 1,
\]

and let $a_n$ denote the n-th factorial moment of $Y$. It is well-known, and easily established, that

\[
(3.9) \quad a_n = (\delta t)^n
\]

\[
(3.10) \quad EY = a_1, \quad EY^2 = a_2 + a_1, \quad EY^3 = a_3 + 3a_2 + a_1,
\]
and

\[(3.11) \quad \alpha_1 = A_1, \quad \alpha_2 = A_2 - A_1, \quad \alpha_3 = A_3 - 3A_2 + 2A_1,\]

Hence we readily find

\[(3.12) \quad \alpha_1(t) = \frac{\theta t}{m} - \frac{m-1}{2m} + R_1(t),\]

\[(3.13) \quad \alpha_2(t) = \frac{(\theta t)^2}{m^2} - \frac{2(m-1)\theta t}{m^2} + \frac{(m-1)(3m-1)}{6m} + R_2(t) - R_1(t),\]

and

\[(3.14) \quad \alpha_3(t) = \frac{(\theta t)^3}{\nu} - \frac{9(m-1)(\theta t)^2}{2m^3} + \frac{3(m-1)(2m-1)\theta t}{m^3}

- \frac{3(m-1)(3m-1)}{4m^2} + R_3(t) - 3R_2(t) + 2R_1(t).\]

Our next step is to pass to the factorial moments of the asynchronous counts, that is, of the counting distribution (1.2). Denoting the \(n\)-th factorial moment of this distribution by \(\beta_n(t)\), we have by Theorem 2.5 of [8] that

\[(3.15) \quad \beta_n(t) = n(\frac{\theta}{m}) \int_0^t \alpha_{n-1}(u)du, \quad n \geq 1.\]

Thus, by (3.12)-(3.15), we have

\[(3.16) \quad \beta_1(t) = \frac{\theta t}{m},\]

\[(3.17) \quad \beta_2(t) = \frac{(\theta t)^2}{m^2} - \frac{(m-1)\theta t}{m} + \frac{2\theta}{m} \int_0^t R_1(u)du,\]
(3.18) \[ \beta_3(t) = \frac{(\theta t)^3}{m^3} - \frac{3(m-1)(\theta t)^2}{m^3} + \frac{(m-1)(5m-1)(\theta t)}{2m^3} + \frac{3\theta}{m} \int_0^t [R_2(u) - R_1(u)]du, \]

and

(3.19) \[ \beta_4(t) = \frac{(\theta t)^4}{m^4} - \frac{6(m-1)(\theta t)^3}{m^4} + \frac{6(m-1)(2m-1)(\theta t)^2}{m^4} - \frac{3(m-1)(3m-1)(\theta t)}{m^3} + \frac{4\theta}{m} \int_0^t [R_3(u) - 3R_2(u) + 2R_1(u)]du. \]

We will pass to our final objective, the central moments \( \mu_2, \mu_3 \) and \( \mu_4 \) of the distribution (1.2), by means of the following easily verified relations:

(3.20) \[ \mu_2 = \beta_2 + \beta_1 - \beta_1^2, \]

(3.21) \[ \mu_3 = \beta_3 + 3\beta_2 + \beta_1 - 3\beta_2\beta_1 - 3\beta_1^2 + 2\beta_1^3, \]

(3.22) \[ \mu_4 = \beta_4 + 6\beta_3 + 7\beta_2 + \beta_1 - 4\beta_2\beta_1 - 12\beta_1^2 \]

\[ - 4\beta_1^2 + 6\beta_2\beta_1^2 + 6\beta_1^3 - 3\beta_1^4. \]

For the purposes of the present report, terms of order \( o(t) \) may be neglected in expressing \( \mu_2, \mu_3 \) and \( \mu_4 \) explicitly as functions of \( t \). With this in mind, inspection of formulas (3.20)-(3.22) reveals how accurately it is necessary to deal with \( R_1(t), R_2(t) \) and \( R_3(t) \). Because of the terms \( \beta_1\beta_2, \beta_1^2\beta_2, \beta_1\beta_3 \) in (3.21) and (3.22), we need \( \int_0^t R_1(u)du \) within error of order \( o(t^{-1}) \) and
\[ \int_0^t R_2(u) du \] within an error or order \( o(1) \). Finally, due to \( \beta_4 \) in (3.22), we need \( \int_0^t R_3(u) du \) within an error of order \( o(t) \).

We now deal with \( \int_0^t R_n(u) du \) somewhat generally. First, it is easily verified (e.g., see [13]) that

\[ i^{-\ell} \psi_Y(\ell)(s) = \sum_{j=0}^{\ell} S(\ell,j) e^{ijs(\ell t)} j! t e^{is-1}, \]

where \( S(\cdot, \cdot) \) are the Stirling numbers of the 2nd kind (see [1]), defined by the relation

\[ n^n = \sum_{r=0}^{n} S(n,r) x(r), \quad n > 0, \]

where \( x(r) = x(x-1) (x+r+1), \quad r > 0, \) and \( x(0) = 1. \) (It so happens that \( S(n,r) \) is the number of ways to partition \( n \) distinguishable objects into \( r \) distinct subsets.) Making use of (3.23) in the definition of \( R_n(t) \), via (3.4) and (3.5), we may write \( R_n(t) \) as

\[ \frac{1}{m+1} \sum_{k=0}^{n} (-1)^{n-k} k^{m-1} \sum_{r=0}^{n-k} S(\ell,j) \sum_{j=0}^{m-1} e^{\frac{-i\pi k}{m} (r-j)} \theta t \left( e^{\frac{m}{m} - 1} \right) \]

Now, for any complex number \( c \),

\[ \int_0^t u^n e^{cu} du = \frac{e^{cu}}{c^{n+1}} \left( (cu)^n - n(cu)^{n-1} + n(n-1)(cu)^{n-2} + \ldots \right. \]

\[ + (-1)^{n-1} n! (cu) + (-1)^n n! \right) \bigg|_0^t \]
and hence

\[ (3.27) \quad \int_0^t u^n e^{cu} du = \frac{(-1)^{n+1} n!}{c^{n+1}} + O(t^n e^{R(c)t}), \quad t \to \infty, \]

where \( R(c) \) is the real part of \( c \). Applying (3.27) in conjunction with (3.25), we obtain

\[ (3.28) \quad \int_0^t R_n(u) du = \frac{1}{m^{n+1}} \sum_{\ell=0}^n \binom{n}{\ell} (-1)^{n-\ell} \sum_{r=0}^{m-\ell} r^{n-\ell} \sum_{j=0}^2 S(\ell,j) \frac{(-1)^{j+1} j!}{\ell!} D_{r,j,k} \]

\[ + O(t^n e^{-\Delta t}), \quad t \to \infty, \]

where

\[ D_{r,j,k} = \sum_{k=1}^{m-1} e^{\frac{i 2\pi k}{m} (r-j)} \frac{\frac{2\pi k}{m}}{(e^{\frac{2\pi k}{m}} - 1)^{-j-1}} \]

and \( \Delta = \theta(1 - \cos \frac{2\pi}{m}) \). Thus we have represented \( \int_0^t R_n(u) du \) as a constant plus a term of order \( O(t^n \exp(-\Delta t)) \), where \( \Delta > 0 \) if \( m > 1 \). (If \( m = 1 \), \( R_n(\cdot) \) is zero.) Hence our previously stated requirements on the error in approximating \( t \int_0^t R_n(u) du \) for \( n = 1, 2, 3 \) are met if we completely ignore \( \int_0^t R_n(u) du \) in the case \( n = 3 \) and approximate it by the appropriate constant term in the cases \( n = 1, 2 \). So we now evaluate the constants involved. Using

\[ (3.29) \quad S(0,0) = 1, S(1,0) = 0, S(1,1) = 1, S(2,0) = 0, S(2,1) = 1, S(2,2) = 1 \]

and basic trigonometric manipulations, we find
\[ (3.30) \quad \int_0^t R_1(u)du = \frac{1}{2\theta m} \sum_{k=1}^{m-1} \frac{1}{1 - \cos \frac{2\pi k}{m}} + O(te^{-\Delta t}) \]

and

\[ (3.31) \quad \int_0^t R_2(u)du = -\frac{1}{2\theta m} \sum_{k=1}^{m-1} \frac{1}{1 - \cos \frac{2\pi k}{m}} + O(t^2 e^{-\Delta t}), \]

for \( t \to \infty \)

**LEMMA.** For any positive integer \( m \),

\[ (3.32) \quad \sum_{k=1}^{m-1} \frac{1}{1 - \cos \frac{2\pi k}{m}} = \frac{m^2 - 1}{6}. \]

**Proof.** One way to obtain (3.32) is by a straightforward application of the note [14]. We are indebted to Professor Morgan Hanson for this insight. Alternatively, we here deduce it by combining

\[ (3.33) \quad u_2(t) = \frac{\theta t}{m^2} + \frac{\theta m^2 - 1}{2m} + O(e^{-\Delta t}), \quad t \to \infty, \]

which was proved in [14], with (3.17) and (3.20) above, to obtain

\[ (3.34) \quad \frac{2\theta}{m} \int_0^t R_1(u)du = \frac{\theta m^2 - 1}{6m} + O(e^{-\Delta t}), \quad t \to \infty. \]

Then (3.32) follows from (3.30) and (3.34). \[ \square \]

With \( \lambda = \theta t \) and \( \Delta = \theta (1 - \cos \frac{2\pi}{m}) \), we thus have the following formulas for \( \beta_1, \beta_2, \beta_3, \beta_4 \):
For $t \to \infty$. Substitution of (3.35)-(3.38) into (3.20)-(3.22) yields

**THEOREM 3.1.** Let $Z$ be a random variable having the distribution (1.2). Then the central moments of $Z$, $\mu_k(t) = \mathbb{E}(Z-EZ)^k$, for the cases $k = 2, 3$ and 4, satisfy

(3.39) $\mu_2(t) = \frac{\theta t}{m^2} + \frac{m^2-1}{6m} + o(1), t \to \infty,$

(3.40) $\mu_3(t) = \frac{\theta t}{m^3} + o(1), t \to \infty,$

and

(3.41) $\mu_4(t) = \frac{3(\theta t)^2}{m^4} + \frac{\theta t}{m^2} + o(t), t \to \infty.$

4. **Applications.** The problem of major interest in this report is that of testing, for an Erlang of order $m$ model, the hypothesis that "$m=1"$ (the Poisson case) against some alternative, such as "$m>1"$, "$m=2"$, or the like.
We shall suppose that our (as count) data consists of a sample \(X_1, \ldots, X_n\) from distribution (1.2). It shall be assumed that \(n\) and \(t\) are both sufficiently large for the approximations given in Theorems 2.1 and 2.2 to apply with reasonable accuracy. A guideline for selection of \(n\) and \(t\) will be discussed later.

In terms of the index of dispersion and its sample estimate, our hypothesis is

\[ H_0: I = 1, \]

with alternatives "I<1", "I=1\(\frac{1}{2}\)", etc., and the chosen test statistic, \(\hat{I}\), is approximately normally distributed with mean \(I\) and standard deviation \(I^{\frac{1}{2}}/\sqrt{n}\). Clearly, then, \(H_0\) is to be rejected if the observed value of \(\hat{I}\) is sufficiently small. A test of given size \(\alpha\) is thus set up as follows. Let

\[ \phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \]

the standard normal cumulative distribution function, and define \(z_\alpha\) by \(\phi(-z_\alpha) = \alpha, 0 < \alpha < 1\). Then our test procedure is

\[ (4.1) \quad \text{"Accept or Reject } H_0 \text{ according as } \hat{I} \text{ is greater or less than } 1 - z_\alpha \sqrt{2}/\sqrt{n}.\]

The Type I error (rejecting \(H_0\) when it is true) has probability

\[ P_{H_0}[\hat{I} < 1 - z_\alpha \sqrt{2}/\sqrt{n}], \]

or equivalently

\[ (4.2) \quad P_{H_0}\left[\frac{\hat{I} - 1}{\sqrt{2}/\sqrt{n}} < -z_\alpha\right], \]
which by the asymptotic normality is approximately $\Phi(-z_\alpha) = \alpha$, so that the test has approximate size $\alpha$.

For example, suppose that $n = 50$ and that a test with Type I error probability of $\alpha = .05$ is desired. From Table 1 of [6] we find $z_{.05} \approx 1.6$, so that the test (4.1) gives the rejection region "$\hat{I} < .68$".

Besides the Type I error probability of a test procedure, we should examine its Type II error probability of $H_0$ when it is not true. At the alternative $H_1$: "$m = m_0$", the test (4.1) has Type II error probability $P_{H_1}[\hat{I} > 1 - z_\alpha \sqrt{2}/\sqrt{n}]$, or equivalently

$$P_{H_1}[\frac{\hat{I} - 1/m_0}{\sqrt{2/m_0}\sqrt{n}} > \sqrt{n} \frac{m_0 - 1}{\sqrt{2}} - m_0 z_\alpha].$$

The normal approximation in the case of (4.3) is

$$\Phi(-\sqrt{n} \frac{m_0 - 1}{\sqrt{2}} + m_0 z_\alpha),$$

which we note tends to 0 as $n \to \infty$. Because of this, (4.4) may not be regarded as an asymptotic equivalent to (4.3). However, for values of $z_\alpha$, $m_0$ and $n$ such that (4.4) is not too small, we may reasonable use it as a crude approximation to (4.3). And in any case it suggests that (4.3) tends to 0 rapidly as $n \to \infty$.

In the example considered previously ($n=50, \alpha=.05$), let us consider $m_0 = 2$. Then (4.4) becomes $\Phi(-1.8) \approx .04$.

As a further example, take $n=32, \alpha=.05$ and $m_0=2$. Then the test (4.1) corresponds to the rejection region "$\hat{I} < .6$" and has Type I error probability
.05, and the Type II error probability at the alternative "m=2" is approximately 
\( \Phi(-0.8) = .21 \).

Comparison of these examples illustrates how the effectiveness of a test based on the sample index of dispersion increases with the sample size \( n \).

We now discuss the related estimation problem. For the Erlang model of order \( m \) as formulated according to (1.1) and (1.2), the parameters are \( \theta \) and \( m \) and three important parametric functions relative to the distribution (1.2) are its mean \( \mu(t) = \theta t/m \), its variance \( \sigma^2(t) \), and the index of dispersion \( I(t) = \sigma^2(t)/\mu(t) \). Conventional estimators of \( \mu(t) \) and \( \sigma^2(t) \) are, of course,

\[
(4.5) \quad \hat{\mu}(t) = \bar{x}
\]

and

\[
(4.6) \quad \hat{\sigma}^2(t) = s^2,
\]

and so it is natural to consider

\[
(4.7) \quad \hat{I} = s^2/\bar{x},
\]

\[
(4.8) \quad \hat{m} = [\bar{x}/s^2]
\]

and

\[
(4.9) \quad \hat{\theta} = \bar{x}^2/\bar{s}^2
\]

as estimators for \( I \), \( m \) and \( \theta \), respectively. (The latter two estimators are based on the approximation \( \sigma^2(t) \approx \theta t/m^2 \). Recall the notation \([\cdot] \) for greatest integer part.) The asymptotic distributions of these estimators may be derived from Theorem 2.1. The estimator \( \hat{I} \) has already been treated, in Theorem 2.2, and the others may be handled similarly. Thus all of the estimators (4.5)-(4.9)
are asymptotically normal in distribution, with means equal to the parameters being estimated and with asymptotic variances of the form \( C/n \) where \( C \) is constant. Thus each of (4.5)-(4.9) is a consistent estimator (see [12]) and, for fixed \( \gamma(0 < \gamma < 1) \), 100\(\gamma\)% confidence intervals based on these estimators have lengths proportional to \( 1/\sqrt{n} \).

We conclude with some remarks concerning the selection of the length \( t \) of counting periods and the number of \( n \) of counting periods. Since the approximations given in Theorems 2.1 and 2.2 are similar in character, we feel that they are similar in validity and thus that it suffices to choose \( t \) large enough for validity of the approximation of \( \sigma^2(t) \) by \( \theta t/m^2 \). A basis for deciding the latter question is provided by (1.3) and (1.4). (Some crude prior information about the unknown \( \theta \) and \( m \) would have to be available and utilized.) Finally, the number \( n \) is selected so as to achieve suitably small error probabilities in testing, or suitably short confidence intervals in estimation, within guidelines provided by the normal approximations and within the constraints of cost and convenience. For example, one might select \( n \) so that the asymptotic variance of \( \hat{I} \) is \( \leq .04 \). Thus, by Theorem 2.2, we would require \( 2/m \cdot n \leq .04 \). This is satisfied for all values of \( m = 1, 2, \ldots \), if \( n \geq 50 \).
REFERENCES


