STOCHASTIC COMPARISONS OF RANDOM PROCESSES,

WITH APPLICATIONS IN RELIABILITY\(^1\)

by

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Abstract

The usual definition of stochastic comparison of random vectors is extended to stochastic comparison of random processes. We state conditions under which \( \{X(t), t \geq 0\} \) stochastically larger than \( \{Y(t), t \geq 0\} \) implies that \( f(\{X(t), t \geq 0\}) \geq f(\{Y(t), t \geq 0\}) \) for increasing functionals \( f \).

Applications are made to reliability problems, yielding comparisons of the following sort. Assume a system of \( n \) independently operating machines, with an operating period of the \( i \)th machine governed by \( F_i(t) = 1 - e^{-\lambda_i t} \) and a repair period governed by \( G(t) = 1 - e^{-pt} \), \( i = 1, \ldots, n \). Let \( N(t) \) denote the number of machines operating at time \( t \), and let \( N^*(t) \) denote the corresponding number when each \( \lambda_i \) is replaced by \( \lambda = \frac{1}{n} \sum \lambda_i \). Then \( \{N(t), t \geq 0\} \geq \{N^*(t), t \geq 0\} \).

Similar comparisons are made for the process describing the number of operating machines under assumptions such as that of proportional hazards or majorization of machine failure rates. A discrete analogue is also derived.

From these stochastic comparisons we may then deduce similar stochastic comparisons for functionals of practical importance in reliability applications, such as the total machine up-time, the first time that the number of functioning machines drops below a specified number, the total time during which at least a specified number of machines are functioning, etc.
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1. Introduction. In many applied problems, the exact calculation of quantities of interest for the stochastic process under study is a forbidding task. In such cases, the statistician may compute bounds on these parameters by comparing the given stochastic process with a simpler stochastic process. For example, in Pledger and Proschan (1971), the reliability of certain types of systems of unlike components is approximated by computing the reliability of corresponding systems of like components. It is important to note that in making such comparisons, the statistician is obtaining a bound on one or perhaps several parameters.

In the present paper we introduce a considerably stronger type of comparison of two stochastic processes. This permits us to obtain bounds not just on a few parameters, but simultaneously on an uncountably infinite class of functionals of the stochastic process. We show in

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detail how the new notion may be applied in the study of systems of
unlike components undergoing alternating failure and repair; we obtain
bounds simultaneously on quantities such as first time to system
failure, total functioning time in a specified interval, etc.

2. Stochastic Comparison. For random variables $X$ and $Y$, the
statement that $X$ is stochastically larger than $Y$ (written $X \succ Y$) means
that $P[X > t] \geq P[Y > t]$ for all real $t$. An extension to random
vectors is obtained by calling random vector $X$ stochastically larger than
random vector $Y$ (written $X \succ Y$) if $f(X) \succ f(Y)$ for all increasing
real valued functions $f(x)$. See Veinott (1965).

As a further extension of stochastic comparison to stochastic
processes, we call stochastic process $\{X(t), t \geq 0\}$ stochastically
larger than stochastic process $\{Y(t), t \geq 0\}$ (written
$\{X(t), t \geq 0\} \succ \{Y(t), t \geq 0\}$) if

$$
(2.1) \quad (X(t_1), \ldots, X(t_n)) \succ (Y(t_1), \ldots, Y(t_n))
$$

for every choice of $0 \leq t_1 < t_2 < \cdots < t_n$, $n = 1, 2, \ldots$. In this
section we derive some implications of the stochastic comparison of
random processes as defined in (2.1); in particular, we obtain the
stochastic comparison of $f(\{X(t), t \geq 0\})$ with $f(\{Y(t), t \geq 0\})$ for
continuous increasing functionals $f$.

For $M > 0$, let $D[0, M]$ be the space of all real valued functions
on $[0, M]$ which are right continuous and have left hand limits. Let
$S^M$ be the class of stochastic processes $\{Z(t), t \geq 0\}$ such that

$P(\{Z(t), 0 \leq t \leq M\} \in D[0, M]) = 1.$ (We write $Z^M$ for $\{Z(t), 0 \leq t \leq M\}$.)

Convergence in $D[0, M]$ is assumed to be with respect to the Skorohod topology. (See Billingsley, 1968, pp. 111-112.) We may now state:

**Theorem 2.1.** Let $M > 0$, $\{X(t), t \geq 0\} \in S^M$, and $\{Y(t), t \geq 0\} \in S^M$.

Let $\Gamma$ be the collection of functions in $D[0, M]$ which are realizations of $\{X(t), t \geq 0\}$ or of $\{Y(t), t \geq 0\}$, or their approximations

$$X_n(t) = \begin{cases} 
X^M_{\left(\frac{i}{n} M\right)} & \text{for } \frac{i}{n} M < t < \frac{i+1}{n} M, \ 0 \leq i \leq n - 1, \\
X^M(M) & \text{for } t = M,
\end{cases}$$

$n = 1, 2, \ldots$, and similarly for $Y_n(t)$. Let $f: \Gamma \to (-\infty, \infty)$ be continuous and increasing. Then $\{X(t), t \geq 0\} \overset{st}{\geq} \{Y(t), t \geq 0\}$ implies $f(X^M) \overset{st}{\geq} f(Y^M)$.

**Proof.** First note that $P[X_n \to X^M, Y_n \to Y^M$ as $n \to \infty] = 1$.

Since $f$ is continuous on $\Gamma$, it follows that $P[f(X_n) \to f(X^M)]$, $f(Y_n) \to f(Y^M)$ as $n \to \infty] = 1$. Since $\{X(t), t \geq 0\} \overset{st}{\geq} \{Y(t), t \geq 0\}$, it follows that $(X^M(0), X^M(\frac{1}{n} M), \ldots, X^M(M)) \overset{st}{\geq} (Y^M(0), Y^M(\frac{1}{n} M), \ldots, Y^M(M))$.

Since $f$ is an increasing function of the $n+1$ values, $X^M(0), \ldots, X^M(M)$, it follows from the definition of the stochastic comparison of two vectors, that $f(X_n) \overset{st}{\geq} f(Y_n)$. As a consequence a corresponding statement holds for the limits: $f(X^M) \overset{st}{\geq} f(Y^M)$.

Theorem 2.1 tells us that a "nice" increasing functional of stochastically ordered random processes preserves stochastic ordering.
Some examples of increasing functionals of interest in applications are:

(i) \( f_1(\mathbf{X}^M) = \int_0^M x(t) \, dt \) (the area under the process),

(ii) \( f_2(\mathbf{X}^M) = \int_0^M x(t) e^{-\alpha t} \, dt, \ \alpha > 0 \) (the total discounted value of the process),

(iii) \( f_3(\mathbf{X}^M) = \begin{cases} \inf \{ t : X(t) \leq c \} & \text{if } X(t) \leq c \text{ for some } 0 \leq t \leq M, \\ M & \text{otherwise} \end{cases} \)

(the first time the process falls below a barrier).

In Section 3 we shall apply the concept of stochastic comparison of two random processes and Theorem 2.1 to certain reliability problems. We will need the following result of Esary and Proschan (unpublished).

**Lemma 2.2.** Let \( \mathbf{X}, \mathbf{X}', \mathbf{Y}, \) and \( \mathbf{Y}' \) be \( n \)-dimensional random vectors such that \( \mathbf{X} \stackrel{st}{\geq} \mathbf{Y}, \mathbf{X}' \stackrel{st}{\geq} \mathbf{Y}' \), with \( \mathbf{X} \) and \( \mathbf{X}' \) independent, and \( \mathbf{Y} \) and \( \mathbf{Y}' \) independent. Then

\[
(2.3) \quad (\mathbf{X}, \mathbf{X}') \stackrel{st}{\geq} (\mathbf{Y}, \mathbf{Y}'),
\]

where \( (\mathbf{X}, \mathbf{X}') \) denotes the \( 2n \)-dimensional random vector \( (X_1, \ldots, X_n, X'_1, \ldots, X'_n) \).

**Proof.** Let \( f(\mathbf{x}, \mathbf{x}') \) be a real valued increasing function of \( 2n \) arguments such that \( Ef(\mathbf{X}, \mathbf{X}') \) and \( Ef(\mathbf{Y}, \mathbf{Y}') \) exist. Let \( \mathbf{X}^u \) be an
n-dimensional random vector which is independent of $X$ and of $Y$ and has the same distribution as $X'$. Then $Ef(X, X' | X' = x) = Ef(X, X^* | X^* = x') > Ef(Y, X^* | X^* = x')$, since $f$ is increasing in its first $n$ arguments. It follows that $Ef(X, X') > Ef(Y, X^*)$. By a second application of the same argument, we obtain $Ef(Y, X^*) > Ef(Y, Y')$. Combining the two inequalities, we conclude that $Ef(X, X') > Ef(Y, Y')$. By Theorem 3 of Veinott (1965), the conclusion now follows.$\|$

From Lemma 2.2 we immediately obtain

**Theorem 2.3.** Let $\{X(t), t \geq 0\} \st \geq \{Y(t), t \geq 0\}$, and $\{X'(t), t \geq 0\} \st \geq \{Y'(t), t \geq 0\}$, with $\{X(t), t \geq 0\}$ independent of $\{X'(t), t \geq 0\}$, and $\{Y(t), t \geq 0\}$ independent of $\{Y'(t), t \geq 0\}$. Then $\{X(t) + X'(t), t \geq 0\} \st \geq \{Y(t) + Y'(t), t \geq 0\}$.

**Proof.** Let $f$ be an increasing function and $0 \leq t_1 \leq \ldots \leq t_k$. Then by Lemma 2.2, $f[X(t_1) + X'(t_1), \ldots, X(t_k) + X'(t_k)] \st \geq f[Y(t_1) + Y'(t_1), \ldots, X(t_k) + Y'(t_k)]$. Thus by definition, $\{X(t) + X'(t), t \geq 0\} \st \geq \{Y(t) + Y'(t), t \geq 0\}$.$\|

Another lemma which we use crucially in the applications of Section 3 is due to Veinott (1965); as Veinott has pointed out (unpublished communication), the hypothesis actually required is somewhat weaker than that of his Theorem 4:
Lemma 2.4. Let $X$ and $Y$ be $n$-dimensional random vectors such that (i) $X \overset{st}{\geq} Y_1$, (ii) $P[Y_j > z | Y_1 = a_1, \ldots, Y_{j-1} = a_{j-1}]$ is increasing in $a_1, \ldots, a_{j-1}$ for $2 \leq j \leq n$ and all real $z$, and (iii) $X_j | X_1 = a_1, \ldots, X_{j-1} = a_{j-1} \geq Y_j | Y_1 = a_1, \ldots, Y_{j-1} = a_{j-1}$ for each $a_1, \ldots, a_{j-1}$ and $2 \leq j \leq n$. Then $X \overset{st}{\geq} Y$.

3. Applications in Reliability Theory. In this section we show how the stochastic comparison of random processes discussed in Section 2 can be used to derive bounds in analyzing complicated stochastic processes arising in reliability theory.

Assume each of $n \geq 2$ machines runs continuously until it fails, at which time repair is initiated. After repair is completed, the machine resumes operation. The operating period for machine $i$ has distribution $F_i$, while the repair period has distribution $G_i$, $i = 1, \ldots, n$. All periods are mutually independent. Letting $N(t)$ denote the number of machines operating at time $t$, then $\{N(t), t \geq 0\}$ may be a very unwieldy stochastic process since we have not assumed that $F_1 = \cdots = F_n$ nor that $G_1 = \cdots = G_n$. In this section we obtain bounds or distributions of functionals of $\{N(t), t \geq 0\}$ in terms of an analogously defined but simpler process $\{N^*(t), t \geq 0\}$. Let $N^*(t)$ denote the number of machines operating at time $t$ among a second set of $n$ machines under the same assumptions as for the first set, except that now each machine has common life distribution $F$ and common repair distribution $G$. In
our proofs, we shall use the notation that for $i = 1, \ldots, n$,
$X_i(t) = 1$ if the $i^{th}$ machine in the set of unlike machines is
functioning, and $0$ otherwise. Similarly $X_i^*(t) = 1$ if the $i^{th}$
machine in the set of like machines is functioning, and $0$ otherwise.

In the theorems that follow we obtain stochastic comparisons
of $\{N(t), t \geq 0\}$ and $\{N^*(t), t \geq 0\}$ as defined and discussed in
Section 2, under appropriate assumptions on $F_1, \ldots, F_n, G_1, \ldots, G_n, F$
and $G$.

In the first model we consider, no repair is possible and $F_1, \ldots,
F_n$ have proportional hazards. (The hazard function or more simply
hazard, $R(t)$ of a life distribution $F(t)$ is defined by
$R(t) = \log F(t)$, where $F(t) = 1 - F(t)$). Hazards $R_1(t), \ldots, R_n(t)$
are said to be proportional if $R_i(t) = \lambda_i R(t)$ for $\lambda_i > 0$, $i = 1, \ldots, n$
where $R(t)$ is a hazard. See Esary, Marshall, and Proschan, 1970,
and Pledger and Proschan, 1971.) In Theorem 3.2 below we obtain
an extension of Corollary 2.7 of Pledger and Proschan (1971).

For ease of exposition we treat the case of $n = 2$ machines first in

**Lemma 3.1.** For $i = 1, 2$, let $\lambda_i > 0$ and $F_i(t) = 1 - e^{-\lambda_i R(t)}$
where $R(t)$ is a hazard function. Let $F(t) = 1 - e^{-\lambda R(t)}$, where
$\lambda = \frac{1}{2} (\lambda_1 + \lambda_2)$. Assume each machine begins operating at time
t $= 0$ and is not repaired upon failure. Then $\{N(t), t \geq 0\}$
$\leq^*_R \{N^*(t), t \geq 0\}$. 
Proof. Let \(0 = t_1 < t_2 < \cdots < t_k\) for \(k \geq 1\). The assumption that \(t_1 = 0\) is made without loss of generality since

\[(3.1) \quad (N(t_1), \ldots, N(t_k)) \overset{st}{\geq} (N^*(t_1), \ldots, N^*(t_k))\]

implies \((N(t_{i_1}), \ldots, N(t_{i_j})) \overset{st}{\geq} (N^*(t_{i_1}), \ldots, N^*(t_{i_j}))\) for each \(i_1, \ldots, i_j \subset \{1, \ldots, k\}\) such that \(i_1 < \cdots < i_j\). To prove (3.1) it suffices to verify conditions (i), (ii), and (iii) of Lemma 2.4.

(i) holds since \(N(t_1) = 2 = N^*(t_1)\).

(ii) holds since \(P[N^*(t_j) > z | N^*(t_1) = a_1, \ldots, N^*(t_{j-1}) = a_{j-1}] = P[N^*(t_j) > z | N^*(t_{j-1}) = a_{j-1}]\), an increasing function of \(a_{j-1}\).

(iii) holds if \(a_{j-1} = 0\) or 2. Assume then that \(a_{j-1} = 1\). Let

\[(3.2) \quad A = \{N(t_1) = a_1, \ldots, N(t_{j-1}) = a_{j-1}\}, A^* = \{N^*(t_1) = a_1, \ldots, N^*(t_{j-1}) = a_{j-1}\}.

Then \(P[N(t_j) > 2 | A] = 0 = P[N^*(t_j) > 2 | A^*]\) since no repair is possible.

Also \(P[N(t_j) > 0 | A] = 1 = P[N^*(t_j) > 0 | A^*]\) since \(N, N^*\) are \(\geq 0\) a.s.

Finally note that \(P[N(t_j) > 1 | A] = P[N(t_j) = 1 | A]\) and \(P[N^*(t_j) > 1 | A^*] = P[N^*(t_j) = 1 | A^*]\). Assume that \(\lambda_1 \geq \lambda_2\). Define

\[(3.3) \quad \gamma = P[X_2(t_{j-1}) = 1 | A];\]
\[ \gamma \] represents the conditional probability that machine 2 is operating at time \( t_{j-1} \).

We first show that \( \gamma \geq \frac{1}{2} \). Let \( m = \min(i: a_i = 1) \). Then

\[
\gamma = \left[ e^{-\lambda_2 R(t_{j-1})} e^{-\lambda_1 R(t_{m-1})} \{1 - e^{-\lambda_1 [R(t_m) - R(t_{m-1})]}\} \right] \times \left[ e^{-\lambda_2 R(t_{j-1})} e^{-\lambda_1 R(t_{m-1})} \{1 - e^{-\lambda_1 [R(t_m) - R(t_{m-1})]}\} \right]^{-1}
\]

\[
= \frac{1}{e^{-\lambda_1 - \lambda_2} [R(t_{j-1}) - R(t_{m-1})]} \cdot \frac{\{1 - e^{-\lambda_1 [R(t_m) - R(t_{m-1})]}\}}{1 - e^{-\lambda_1 [R(t_m) - R(t_{m-1})]} - \lambda_2 [R(t_m) - R(t_{m-1})]}
\]

\[ \geq \frac{1}{2} \, . \]

Next we show \( P[N(t_j) = 1|A] \geq P[R*(t_j) = 1|A^*] \). Write

\[
P[N(t_j) = 1|A] = \gamma P[N(t_j) = 1|A, X_2(t_{j-1}) = 1] + (1-\gamma) P[N(t_j) = 1|X_1(t_{j-1}) = 1] = \gamma e^{-\lambda_2 R(t_j) - R(t_{j-1})}
\]

\[
+ (1-\gamma) e^{-\lambda_1 [R(t_j) - R(t_{j-1})]} \geq \frac{1}{2} e^{-\lambda_2 R(t_j) - R(t_{j-1})} + \frac{1}{2} e^{-\lambda_1 [R(t_j) - R(t_{j-1})]}
\]

[since \( \gamma \geq \frac{1}{2} \) and \( \lambda_1 \geq \lambda_2 \)]

\[
\geq e^{-\lambda_2 R(t_j) - R(t_{j-1})} \quad \text{[by Jensen's inequality]}
\]

\[
= P[N^*(t_j) = 1|A^*].
\]
We have thus completed the proof that (iii) of Lemma 2.4 holds.
It follows that \( \{N(t), t \geq 0\} \geq \{N^*(t), t \geq 0\} \).

Next we shall extend Lemma 3.1 to the case \( n > 2 \). The basic idea in carrying out the extension is to apply Lemma 3.1 repeatedly on different pairs of machines to obtain finally the comparison described in

**Theorem 3.2.** For \( i = 1, 2, \ldots, n \), let \( \lambda_i > 0 \) and
\[
F_i(t) = 1 - e^{-\lambda_i R(t)}
\]
\( F(t) = 1 - e^{-\bar{\lambda} R(t)} \), where \( \bar{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i \). Assume each machine begins operating at time \( t = 0 \) and is not repaired upon failure. Then
\( \{N(t), t \geq 0\} \geq \{N^*(t), t \geq 0\} \).

**Proof.** Without loss of generality, reorder indices so that \( \lambda_1 \geq \ldots \geq \lambda_n \). We may write \( N(t) = A(t) + B(t) \), where
\[
A(t) = X_1(t) + X_n(t), \quad \text{while} \quad B(t) = \sum_{i=2}^{n-1} X_i(t).
\]
Let \( \{X_1(t), t \geq 0\} \), \( \{X_n(t), t \geq 0\} \) be mutually independent processes such that
\[
P[X_1(t) = 1] = e^{-L_2(\lambda_1 + \lambda_n) t} = P[X_n(t) = 1].
\]
Let
\[
A(t) = X_1(t) + X_n(t), \quad \text{and} \quad N(t) = A(t) + B(t).
\]
Then by Lemma 3.1, \( \{A(t), t \geq 0\} \geq \{A(t), t \geq 0\} \). Thus by Theorem 2.3,
\[
\{N(t), t \geq 0\} = \{A(t) + B(t), t \geq 0\} \geq \{A(t) + B(t) + B(t)\} = \{N(t), t \geq 0\}.
\]
Note that we have made a stochastic comparison of $N(t)$ based on the original vector $\lambda$, with $N^{(2)}(t)$ based on a vector $\lambda^{(2)}$ in which the largest and smallest pair of $\lambda$'s are averaged; $\lambda^{(2)}$ is a reordering of $(\lambda_2, \ldots, \lambda_{n-1}, \frac{1}{2}(\lambda_1+\lambda_n), \frac{1}{2}(\lambda_1+\lambda_n))$ so that $\lambda^{(2)}_1 \geq \ldots \geq \lambda^{(2)}_n$.

By repeated applications of the argument above, we may generate a sequence of comparisons $\{N(t), t \geq 0\} \triangleright_{st} \{N^{(2)}(t), t \geq 0\} \triangleright_{st} \{N^{(3)}(t), t \geq 0\} \triangleright_{st} \ldots$, with corresponding vectors $\lambda, \lambda^{(2)}, \lambda^{(3)}, \ldots$. Note that $\lambda^{(j+1)}$ is obtained from $\lambda^{(j)}$ by retaining $\lambda^{(j)}_2 \geq \ldots \geq \lambda^{(j)}_{n-1}$, and replacing both the largest, $\lambda^{(j)}_1$, and the smallest, $\lambda^{(j)}_n$, by their average $\frac{1}{2}(\lambda^{(j)}_1 + \lambda^{(j)}_n)$, and then reordering so as to achieve $\lambda^{(j+1)}_1 \geq \ldots \geq \lambda^{(j+1)}_n$.

It follows that for $0 = t_1 < t_2 < \ldots < t_k$ and $j \geq 1$,

$$
(3.4) \quad \{N^{(j)}(t_1), \ldots, N^{(j)}(t_k)\} \triangleright_{st} \{N^{(j+1)}(t_1), \ldots, N^{(j+1)}(t_k)\}.
$$

Note that $\lambda^{(j)}_i \to \bar{\lambda}$ as $j \to \infty$ for $i = 1, \ldots, n$; i.e., $\lambda^{(j)}_i \to (\bar{x}, \ldots, \bar{x})$ as $j \to \infty$. Note also that $P[X_1(t) = b | X_1(u)$, all $u \leq s$, $X(s) = a]$ is a continuous function of $\lambda$ for $s \geq 0$, $t \geq s$, $a = 0$ or 1, and $b = 0$ or 1. Thus $(X_1^{(j)}(t_1), \ldots, X_1^{(j)}(t_k)) \xrightarrow{d} (X_1^a(t_1), \ldots, X_1^a(t_k))$ as $j \to \infty$ ($1 \leq 1 \leq n$), where $\xrightarrow{d}$ denotes convergence in distribution. Let $f$ be a continuous increasing function of $k$ real variables. Then by two successive applications of Corollary 1, page 31, of Billingsley (1968), it follows that $f(N^{(j)}_1(t_1), \ldots, N^{(j)}_j(t_k)) \xrightarrow{d} f(N^a(t_1), \ldots, N^a(t_k))$ as $j \to \infty$. 


This, in conjunction with (3.4), implies that

\[ f(N(t_1), \ldots, N(t_k)) \overset{\text{st}}{\geq} f(N^*(t_1), \ldots, N^*(t_k)). \]

Because each increasing functional can be approximated arbitrarily well by a continuous increasing function, the desired result now follows from definition 2.1. \|

Next we consider models in which repair does occur. In our first model, time is measured in integer values; e.g., failure occurs after a random number of cycles of operation, and similarly for repair. Let \( P_p \) denote the geometric distribution which places probability \((1-p)p^{j-1}\) at \( j = 1, 2, \ldots \).

**Theorem 3.3.** Assume either that \( N(0) = N^*(0) = n \) or that \( N(0) = N^*(0) = 0 \).

(a) Let \( F_i = \mathcal{G}_{p_i} \) for \( 1 \leq i \leq n \), \( F = \mathcal{G}_p \), where \( p = (\Pi p_i)^{1/n} \), and \( G_1 = \cdots = G_n = \mathcal{G}_{1-r} \), where \( 0 < r \leq \min(p_1, \ldots, p_n) \). Then \( (N(j), j = 0, 1, \ldots) \overset{\text{st}}{\leq} (N^*(j), j = 0, 1, \ldots) \).

(b) Let \( G_i = \mathcal{G}_{r_i} \) for \( 1 \leq i \leq n \), \( G = \mathcal{G}_r \), where \( r = (\Pi r_i)^{1/n} \), and \( F_1 = \cdots = F_n = F = \mathcal{G}_{1-p} \), where \( 0 < p \leq \min(r_1, \ldots, r_n) \). Then \( (N^*(j), j = 0, 1, \ldots) \overset{\text{st}}{\geq} (N(j), j = 0, 1, \ldots) \).

**Proof.** (a) As in the proof of Lemma 3.1, we first let \( n = 2 \) and show that the three conditions of Lemma 2.4 hold.
(i) \( N(0) = N^*(0) \) by hypothesis.

(ii) \( P[N^*(j) = 2| N^*(1) = a_1, \ldots, N^*(j-1) = a_{j-1}] \)

\[ = P[N^*(j) = 2 | N^*(j-1) = a_{j-1}] \]

\[
\begin{cases} 
  r^2 & \text{if } a_{j-1} = 0 \\
  pr & \text{if } a_{j-1} = 1 \\
  p^2 & \text{if } a_{j-1} = 2.
\end{cases}
\]

Similarly,

\[
P[N^*(j) = 0 | N^*(j-1) = a_{j-1}] = \begin{cases} 
  (1-r)^2 & \text{if } a_{j-1} = 0 \\
  (1-p)(1-r) & \text{if } a_{j-1} = 1 \\
  (1-p)^2 & \text{if } a_{j-1} = 2.
\end{cases}
\]

Since \( r \leq p \), (ii) of Lemma 2.4 is satisfied.

(iii) Define \( A = \{ N(1) = a_1, \ldots, N(j-1) = a_{j-1} \} \), and \( A^* = \{ N^*(1) = a_1, \ldots, N^*(j-1) = a_{j-1} \} \). If \( a_{j-1} = 0 \), then \( P[N(j) = z | A] = P[N^*(j) = z | A^*] \)

for \( z = 0, 1, 2 \). If \( a_{j-1} = 2 \), then \( P[N(j) = 2 | A] = p_1 p_2 = p^2 \)

\[ = P[N^*(j) = 2 | A^*], \text{ while } P[N(j) = 0 | A] = (1-p_1)(1-p_2) \leq (1-p)^2 \]

\[ = P[N^*(j) = 0 | A^*]. \]

The remaining case is more complicated. Assume \( a_{j-1} = 1 \). Let \( \gamma = P(X_2(j-1) = 1 | A) \), and assume \( p_1 \leq p_2 \). As in Lemma 3.1, we can
prove that $P[\mathcal{N}(j) > i | \mathcal{A}] > P[\mathcal{N}^*(j) > i | \mathcal{A}^*]$ for all $i$ if we can prove that $\gamma \geq \frac{1}{2}$, as shown by the following argument. If $\gamma \geq \frac{1}{2}$, then $P[\mathcal{N}(j) = 2 | \mathcal{A}] = \gamma p_2 r + (1-\gamma)p_1 r \geq pr = P[\mathcal{N}^*(j) = 2 | \mathcal{A}^*]$, while $P[\mathcal{N}(j) = 0 | \mathcal{A}] = \gamma (1-p_2)(1-r) + (1-\gamma)(1-p_1)(1-r) \leq (1-p)(1-r) = P[\mathcal{N}^*(j) = 0 | \mathcal{A}^*]$. Therefore, $P[B_1] \leq P[B_2]$.

Thus the verification of (iii) of Lemma 2.4 will be complete if we show that $\gamma \geq \frac{1}{2}$. To this end, let $B_1 = \mathcal{A} \cap \{x_1(j-1) = 1\}$ and $B_2 = \mathcal{A} \cap \{x_2(j-1) = 1\}$. It will suffice to show that $P[B_1] \leq P[B_2]$.

We will say that at time $t$ the system is in state $\bar{a} = (a_1, a_2)$ if $x_1(t) = a_1, x_2(t) = a_2$, where of course $a_1 = 0$ or $1$, $a_2 = 0$ or $1$. Let a path be a sequence $\bar{a} = \{a_1, \ldots, a_{j-1}\}$, where $a^i = a$ means that the system is in state $a$ at time $i$. Then $P[B_1]$ is the sum of the probabilities of the paths $\bar{a}$ which are consistent with $N(1) = a_1, \ldots, N(j-1) = a_{j-1}$ and have $a^{j-1} = (1,0)$. Let $C$ be the set of all such paths. For each $\bar{a} \in C$, let $f(\{a\}) = \{\bar{a}\}$, say, be the path such that $\bar{a}^i = a^i$ for $i \leq \max(s:a^s \neq (1,0))$ and $\bar{a}^i = (0,1)$ for $i > \max(s:a^s \neq (1,0))$. Note that $f$ is a one-to-one function.

Fix $\{a\} \in C$, and let $m = \max(s:a^s \neq (1,0))$. That $P[\{a\}] \leq P[f(\{a\})]$ follows from

\begin{align*}
(3.5) & \quad r(1-p_2)[p_1(1-r)] \cdots [p_1(1-r)] \leq p_2(1-r)[p_2(1-r)] \cdots [p_2(1-r)] \\
& \text{if } a^m = (0,1); \\
(3.6) & \quad p_1(1-p_2)[p_1(1-r)] \cdots [p_1(1-r)] \leq p_2(1-p_1)[p_2(1-r)] \cdots [p_2(1-r)] \\
& \text{if } a^m = (1,1); \\
(3.7) & \quad r(1-r)[p_1(1-r)] \cdots [p_1(1-r)] \leq r(1-r)[p_2(1-r)] \cdots [p_2(1-r)] \\
& \text{if } a^m = (0,0). 
\end{align*}
Therefore, \( P[B_1] = \sum P\{a\} \leq \sum P[f\{a\}] = P[B_2] \), where the summation is over all \( \{a\} \in \mathcal{C} \). Thus we have completed the proof that (iii) of Lemma 2.4 holds.

It follows by Lemma 2.4 that the desired conclusion holds for \( n = 2 \). The extension to \( n > 2 \) can be accomplished by applying the stepwise argument of Theorem 3.2 to \( \log p_1, \ldots, \log p_n \).

(b) The dual result may be proved in a similar fashion. ||

Theorem 3.6 below is the continuous analogue of Theorem 3.3.

To prove it, we will need the following lemma which provides a stochastic comparison between random variables \( N(t) \) and \( N^*(t) \) for each fixed value \( t \) (and not between stochastic processes \( \{N(t), t \geq 0\} \) and \( \{N^*(t), t \geq 0\} \)).

**Lemma 3.4.** Assume either that \( N(0) = N^*(0) = n \), or that \( N(0) = N^*(0) = 0 \)

(a) For \( t \geq 0 \), let \( \lambda_i > 0 \), \( F_i(t) = 1 - e^{-\lambda_i t} \) for \( i = 1, \ldots, n \), let \( F(t) = 1 - e^{-\bar{\lambda} t} \), where \( \bar{\lambda} = \frac{1}{n} \sum \lambda_i \), and let \( G_i(t) = \cdots = G_n(t) \) be \( G(t) = 1 - e^{-\rho t} \), where \( \rho > 0 \). Then

\[
(3.8) \quad N(t) \geq N^*(t) \text{ for each } t \geq 0
\]

(b) For \( t \geq 0 \), let \( \rho_i > 0 \), \( G_i(t) = 1 - e^{-\rho_i t} \) for \( i = 1, \ldots, n \), let \( G(t) = 1 - e^{-\bar{\rho} t} \), where \( \bar{\rho} = \frac{1}{n} \sum \rho_i \), and let \( F_i(t) = \cdots = F_n(t) \) be \( F(t) = 1 - e^{-\lambda t} \), where \( \lambda > 0 \). Then

\[
(3.9) \quad N^*(t) \geq N(t) \text{ for each } t \geq 0.
\]

**Sketch of proof of (a).** For \( s < t \), let \( p_i(s,t) = P[X_i(t) = 1|X_i(s) = 1] \) and \( q_i(s,t) = P[X_i(t) = 0|X_i(s) = 0] \). The following expressions are
derived in Barlow and Proschan (1965), p.78,

\begin{align*}
(3.10) \quad p_i(s,t) &= (\lambda_i + r)^{-1} \{ 1 + \lambda_i \exp[-(\lambda_i + r)(t-s)] \}, \\
(3.11) \quad q_i(s,t) &= (\lambda_i + r)^{-1} \{ 1 - \exp[-(\lambda_i + r)(t-s)] \}.
\end{align*}

Assume without loss of generality that \( \lambda_1 \geq \cdots \geq \lambda_n \). It follows from the proof of Theorem 2.2 of Pledger and Proschan (1971) that if \( N(0) = N^*(0) = n \) (alternately, \( N(0) = N^*(0) = 0 \)), then

\[ P[N(t) \geq k] \text{ is a Schur function of } (-\log p_1(0,t), \cdots, -\log p_n(0,t)) \]

(alternately, of \((-\log[1-q_1(0,t)], \cdots, -\log[1-q_n(0,t)]\)) . (See Pledger and Proschan (1971) or Mitrinovich (1971), pp.162-170, for a discussion of Schur functions and related notions.) An inspection of the proof of Theorem 2.2 of Pledger and Proschan (1971), together with Lemma 2.4 of Pledger and Proschan (1971), reveals that (3.8) holds if \(-\log p_1(0,t)\) and \(-\log[1-q_1(0,t)]\) are increasing and concave in \( \lambda_i \) for each \( t \). The proof of these facts is routine but tedious, requiring several differentiations. We omit it.

(b) The proof is similar.||

Actually we can extend Lemma 3.4(a) to the case that the \( N^* \) system is heterogeneous, with failure rates \( \lambda_1^*, \cdots, \lambda_n^* \) respectively, majorized by the failure rates \( \lambda_1, \cdots, \lambda_n \) of the \( N \) system. We say that \((\lambda_1, \cdots, \lambda_n)\) majorizes \((\lambda_1^*, \cdots, \lambda_n^*)\) if \( \lambda_1 \geq \cdots \geq \lambda_n \), \( \sum_{i=1}^n \lambda_i = \sum_{i=1}^n \lambda_i^* \), and \( \sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \lambda_i^* \) for \( k = 1, \cdots, n-1 \) and \( k = n \).

**Theorem 3.5.** Assume either that \( N(0) = N^*(0) = n \); or that \( N(0) = N^*(0) = 0 \).
(a) For $t \geq 0$, let $(\lambda_1, \ldots, \lambda_n)$ majorize $(\lambda_1^*, \ldots, \lambda_n^*)$, $F_i(t) = 1 - e^{-\lambda_i t}$, $F_i^*(t) = 1 - e^{-\lambda_i^* t}$ for $i = 1, \ldots, n$, and $G_1(t) = \cdots = G_n(t) = G(t) = 1 - e^{-\rho t}$, where $\rho > 0$. Then

$$N(t)^{st} \geq N^*(t) \text{ for each } t \geq 0.$$ 

(b) For $t \geq 0$, let $(\rho_1, \ldots, \rho_n)$ majorize $(\rho_1^*, \ldots, \rho_n^*)$, $G_i(t) = 1 - e^{-\rho_i t}$, $G_i^*(t) = 1 - e^{-\rho_i^* t}$ for $i = 1, \ldots, n$, and $F_1(t) = \cdots = F_n(t) = F(t) = 1 - e^{-\lambda t}$, where $\lambda > 0$. Then

$$N^*(t) \geq N(t) \text{ for each } t \geq 0.$$  

**Proof.** The arguments used in the proof of Theorem 2.2 of Pledger and Proschan (1971) in conjunction with the arguments of Lemma 3.4 above may be used to obtain the desired conclusions. We omit the details. ||

We now extend the comparisons of Lemma 3.4 for individual time points to comparisons of entire processes.

**Theorem 3.6 (a)** Under the hypotheses of Lemma 3.4(a),

$$\{N(t), t \geq 0\}^{st} \geq \{N^*(t), t \geq 0\}.$$  

(b) Under the hypotheses of Lemma 3.4(b), \(\{N^*(t), t \geq 0\}^{st} \geq \{N(t), t \geq 0\}.\)

**Proof of (a).** Let $n = 2$. It suffices to verify the conditions of Lemma 2.4 for $0 = t_1 < t_2 < \cdots < t_m$ such that $\max_{2 \leq k \leq m} |t_k - t_{k-1}| < \delta$, where $\delta$, depending only on $\lambda_1, \lambda_2$, and $\rho$, is chosen in a manner to be specified below.

(i) $N(t_1) = N^*(t_1)$ by hypothesis.

(ii) To verify condition (ii) of Lemma 2.4, we need to show that
\[ P[N^*(t_j) \geq a | N^*(t_{j-1}) = z] \text{ is increasing in } z \text{ for each } a. \] Let \( u_j = (\lambda + p)^{-1} \{ \rho + \lambda \exp[-(\lambda + p)(t_j - t_{j-1})] \} \) and \( v_j = (\lambda + p)^{-1} \{ \rho + p \exp[-(\lambda + p)(t_j - t_{j-1})] \}. \]

Then

\[
P[N^*(t_j) = 2 | N^*(t_{j-1}) = z] = \begin{cases} 
(1-v_j)^2 & \text{for } z = 0 \\
u_j(1-v_j) & \text{for } z = 1 \\
u_j^2 & \text{for } z = 2,
\end{cases}
\]

and

\[
P[N^*(t_j) = 0 | N^*(t_{j-1}) = z] = \begin{cases} 
v_j^2 & \text{for } z = 0 \\
v_j(1-u_j) & \text{for } z = 1 \\
(1-u_j)^2 & \text{for } z = 2.
\end{cases}
\]

which are, respectively, increasing and decreasing functions of \( z \).

(iii) Define \( A \) and \( A^* \) as in (3.2). If \( a_{j-1} = 0 \) or \( 2 \), then

\[ N(t_{j}) | A^{st} \geq N^*(t_j) | A^* \text{ by Lemma 3.4(a)}. \]

Suppose \( a_{j-1} = 1 \) and let \( \gamma \) be defined as in (3.3). If \( \lambda_1 = \lambda_2 \), then (iii) of Lemma 2.4 holds trivially. Assume then that \( \lambda_1 > \lambda_2 \).

Let \( I = t_{j} - t_{j-1} \), and for \( i = 1, 2 \) let \( p_i = p_i(t_{j-1}, t_{j}) \) and \( q_i = q_i(t_{j-1}, t_{j}) \) be as in (3.10) and (3.11). If \( \gamma > \frac{1}{2} \), then for 

\[ \delta < \delta_1 \text{ say}, \]

\[
P[N(t_j) = 2 | A] = \gamma p_2(1-q_1) + (1-\gamma)p_1(1-q_2)
\]

\[ = \gamma[1 - \lambda_2 I + 0(I^2)][\rho I - \frac{1}{2} \rho (\lambda_1 + p) I^2 + 0(I^3)]
\]

\[ + (1-\gamma)[1 - \lambda_1 I + 0(I^2)][\rho I - \frac{1}{2} \rho (\lambda_2 + p) I^2 + 0(I^3)]
\]

\[ = \rho I - \frac{1}{2} \rho^2 I^2 - \frac{1}{2} \rho I^2(\lambda_1 + \lambda_2) - \frac{1}{2} \rho^2 I(\gamma \lambda_2 + (1-\gamma) \lambda_1) + 0(I^3)
\]

\[ > \rho I - \frac{1}{2} \rho^2 I^2 - \frac{1}{2} \rho I^2(\lambda + \lambda) - \frac{1}{2} \rho^2 I^2 + 0(I^3)
\]

\[ = P[N^*(t_j) = 2 | A^*]. \]
Similarly, for $\delta < \delta_2$, $P[N(t_j) = 0 | A] < P[N^*(t_j) = 0 | A^*]$. $\delta_1$ and $\delta_2$ depend only on $\lambda_1, \lambda_2, \rho$.

The case $n = 2$ is completed by an argument similar to that used to show $\gamma > \frac{1}{2}$ in the proof of Theorem 3.3, with expressions (3.5), (3.6), and (3.7) replaced by

\begin{align*}
(3.5') \quad & (1-p_2)(1-q_2)(p_1q_2)\cdots(p_1q_2) < p_2q_1(p_2q_1)\cdots(p_2q_1), \\
(3.6') \quad & p_1(1-p_2)(p_1q_2)\cdots(p_1q_2) < p_2(1-p_1)(p_2q_1)\cdots(p_2q_1), \\
(3.7') \quad & (1-q_1)q_2(p_1q_2)\cdots(p_1q_2) < q_1(1-q_2)(p_2q_1)\cdots(p_2q_1).
\end{align*}

Inequalities (3.5'), (3.6'), and (3.7') are satisfied for $\delta < \delta_3$, a function of $\lambda_1, \lambda_2, \rho$. Clearly, we choose $\delta < \min(\delta_1, \delta_2, \delta_3)$.

The extension to the case $n \times 2$ is by the stepwise procedure of the proof of Theorem 3.2.||

We may now apply Theorem 2.1. Let $M > 0$ and let $\Gamma$ be the set of all $x \in D[0,M]$ such that $x(t) \in \{0, \cdots, n\}$ for each $t \in [0,M]$. Let $N(t)$ and $N^*(t)$ be as in any of Theorems 3.2, 3.3, or 3.6. Define $X(t) = N(t), Y(t) = N^*(t)$; note that $\{X(t), t \geq 0\}, \{Y(t), t \geq 0\}$, and $\Gamma$ satisfy the conditions of Theorem 2.1. Thus for any continuous, increasing functional $f$ on $\Gamma$, we conclude that

$$f(X^M) \geq f(Y^M).$$

Some examples of functionals $f$ of interest are:

1. $f_1(X^M) = \int_0^M N(t) \, dt$, the total up-time during $[0,M]$, \ldots
(ii) \( f_2(X^M) = \int_0^M N(t)e^{-\alpha t} \, dt \) for \( \alpha > 0 \), the total discounted up-time during \([0,M]\).

(iii) \( f_3(X^M) = \min\{\inf\{t:N(t) \leq k\}, M\} \), the first time during \([0,M]\) that the number of functioning machines drops to \( k \) (arbitrarily taken as \( M \), if it never does in \([0,M]\)). Note that with probability one, as \( M \to \infty \), \( f_3(X^M) \to \inf\{t:N(t) \leq k\} \), the first time the number of functioning machines drops to \( k \).

(iv) \( f_4(X^M) = \int_0^M \mathbb{I}_{\{t:N(t) \geq k\}}(s) \, ds \),

where \( \mathbb{I}_A \) is the indicator function of the set \( A \). Thus \( f_4(X^M) \) is the total time in \([0,M]\) during which at least \( k \) machines are operating.
REFERENCES


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13. Abstract  
The usual definition of stochastic comparison of random vectors is extended to stochastic comparison of random processes. We state conditions under which \( \{X(t), t \geq 0\} \) stochastically larger than \( \{Y(t), t \geq 0\} \) implies that \( f(\{X(t), t \geq 0\}) \geq f(\{Y(t), t \geq 0\}) \) for increasing functionals \( f \).

Applications are made to reliability problems, yielding comparisons of the following sort. Assume a system of \( n \) independently operating machines, with an operating period of the \( i^{th} \) machine governed by \( F_i(t) = 1 - e^{-\lambda_i t} \) and a repair period governed by \( G(t) = 1 - e^{-\beta t}, i=1,...,n \). Let \( N(t) \) denote the number of machines oper-
13. Abstract - continued

-ating at time t, and let \( N^*(t) \) denote the corresponding number when each \( \lambda_i \) is replaced by \( \bar{\lambda} = \frac{1}{n} \sum \lambda_i \). Then \( \{N(t), t \geq 0\} \overset{st}{\geq} \{N^*(t), t \geq 0\} \). Similar comparisons are made for the process describing the number of operating machines under assumptions such as that of proportional hazards or majorization of machine failure rates. A discrete analogue is also derived.

From these stochastic comparisons we may then deduce similar stochastic comparisons for functionals of practical importance in reliability applications, such as the total machine up-time, the first time that the number of functioning machines drops below a specified number, the total time during which at least a specified number of machines are functioning, etc.

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<thead>
<tr>
<th>14. Key Words</th>
<th>Link A</th>
<th>Link B</th>
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<td>Role Wt.</td>
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<td>Stochastic comparison</td>
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<td>Random processes</td>
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