A NOTE CONCERNING SOME RESULTS OF
LANDAU, SHEPP AND SATO

by

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Summary. Two alternate and elementary proofs of the Landau-Shepp theorem are presented. An error in the proof of Sato's theorem is rectified.

Introduction. The Landau-Shepp (1970) theorem may be stated as follows:

Theorem 1 (Landau-Shepp).

Let \( \{X_n\}_n \) be a collection of random variables such that every finite subcollection has a normal distribution. Then

\[
(1) \quad \Pr\{\sup_n |X_n| < \infty\} = 0 \text{ or } 1.
\]

If

\[
(2) \quad \Pr\{\sup_n |X_n| < \infty\} = 1
\]

then

\[
(3) \quad \mathbb{E}(e^{\varepsilon \sup_n |X_n|^2}) < \infty
\]

for some \( \varepsilon > 0 \).
Sato (1971) proved the following using Theorem 1:

**Theorem 2 (Sato)**

Let \( m \) be a Gaussian measure on a separable Banach space \( X \).

Then

\[
(4) \quad \int_X e^{\varepsilon |x|^2} \, dm(x) < \infty
\]

for some \( \varepsilon > 0 \).

We present two alternate and simple proofs of (1) of Theorem 1 in this note. In proving Theorem 2 Sato (1971) actually used the assumption (see p. 228) that \( X^\vee \), the dual of \( X \), is separable. Thus he did not prove Theorem 2 as stated. We rectify this error in this note.

**First Proof of (1) of Theorem 1.**

We can construct independent random variables \( \{ Y_n \} \) and constants \( \{ a_{nk}, k = 1, \ldots, n, n = 1, 2, \ldots \} \) by the usual orthogonalization process such that

\[
(5) \quad X_n = \frac{1}{n} \sum_{k=1}^{n} a_{nk} Y_k, \quad n = 1, 2, \ldots
\]

Note that

\[
(6) \quad \{ \sup_n |X_n| < \infty \} = \bigcup_{N=1}^{\infty} \bigcap_{n=1}^{N} \{|X_n| \leq N\}
\]

\[
= \bigcup_{N=1}^{\infty} \bigcap_{n=1}^{N} \{| \sum_{k=1}^{n} a_{nk} Y_k | \leq N\}
\]

\[
= \bigcup_{n=1}^{\infty} \bigcap_{n=1}^{\infty} \{| \sum_{k=m}^{n} a_{nk} Y_k | \leq N\}
\]
for \( m = 1, 2, \ldots \). Thus the event \( \{ \sup_n |X_n| < \infty \} \) belongs to the tail \\
\( \sigma \)-field of \( \{ Y_n \} \) and by the Kolmogorov 0-1 law has probability 0 \\
or 1.

**Second Proof of (1) of Theorem 1**

If \( \{ E(X_n) \} \) is an unbounded sequence then it is easy to check \\
that

\[
P(\sup_n |X_n| < \infty) = 0.
\]

We shall therefore assume that \( \{ E(X_n) \} \) is bounded. By translating \\
the \( X_n \)'s we may assume that \( E(X_n) = 0, n = 1, 2, \ldots \). The joint \\
probability measure \( P \) of \( \{ X_1, X_2, \ldots \} \) may be considered as a Gaussian \\
measure on \( (\mathbb{R}_\infty, \mathcal{B}_\infty) \). Here \( \mathbb{R}_\infty \) is the countable product of the real \\
line and \( \mathcal{B}_\infty \) is the product \( \sigma \)-field. Thus \( P \) is a Gaussian measure \\
on a separable Fréchet space and has mean 0. From a result due to \\
Rajput (1971, Theorem 5.1)

\[
P(C) = 0 \text{ or } 1
\]

for every measurable linear subspace \( C \) of \( \mathbb{R}_\infty \). It is easy to check \\
that \( \{(x_1, x_2, \ldots): \sup_n |X_n| < \infty\} \) is a measurable linear subspace of \\
\( \mathbb{R}_\infty \). The proof of (1) of Theorem 1 is complete.

**Proof of Theorem 2**

We first prove the following lemma.
Lemma 1. Let $X$ be a separable Banach space. There exists a sequence 
\{$f_n$\} in the dual $X^*$ such that 

(7) \[ \|f_n\| \leq 1, \quad n = 1, 2, \ldots \]

and 

(8) \[ \|x\| = \sup_{n} |f_n(x)| \text{ for each } x \in X. \]

Proof of Lemma 1. Let $S^\#$ be the unit ball in $X^\#$. Since $X$ is separable, 
$S^\#$ is compact and metrizable in the $w^\#$-topology of $X^*$ restricted 
to $S^*$ (see Dunford and Schwartz (1958), Part I, p. 424-426). Thus 
there is a subset $D = \{f_n\}$ of $S^\#$ which countable and dense in $S^\#$. 
For each $x$, $f(x)$ is $w^\#$-continuous in $f$. Thus 

\[ \|x\| = \sup_{f \in S^\#} |f(x)| = \sup_{n} |f_n(x)|. \]

Now, Theorem 2 follows immediately from Lemma 1 and Theorem 1.

Remark. It is easy to see that Lemma 1 is true when $X$ is only 
a separable normed space. Thus in Theorem 2 one need only assume 
that $X$ is a separable normed space.
REFERENCES


Rajput, Balram, Gaussian measures on $L_p$ spaces, $1 \leq p \leq \infty$ (1971) Institute of Stat. Mimeo. Series No. 782, Dept. of Stat. Univ. of North Carolina, Chapel Hill.

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13. **Abstract**
    Two alternative and elementary proofs of the Landau–Shepp theorem are presented. An error in the proof of Sato's theorem is rectified.

14. **Key Words**
    Gaussian measures in abstract spaces
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