CONTINUOUS TIME PROGRAMMING WITH NONLINEAR
TIME DELAYED CONSTRAINTS

by

William H. Farr and Morgan A. Hanson

FSU Statistics Report No. M248
AFOSR-H Report No. 5

December, 1972
Florida State University
Department of Statistics

This work was supported by the Air Force Office of Scientific
Research, Department of the Air Force, under Grant AFOSR 72-2345.
CONTINUOUS TIME PROGRAMMING WITH NONLINEAR
TIME DELAYED CONSTRAINTS

by

William H. Farr† and Morgan A. Hanson†

1. INTRODUCTION

The work presented here is an extension of our previous work described
in [1], to which the reader is referred for the necessary background inform-
ation and results to be utilized here. In addition to some direct gener-
alizations of the model considered in [1], the effect of time lags is also
introduced, thus extending the results of Larsen and Polak [8].

In Section 3, a typical example of such a model in practice is described
in terms of a large scale water storage-power generating system. The pro-
blem to be considered is

Primal Problem I

Maximize

(1) \[ L(z) = \int_{0}^{T} \phi(z(t))dt \]

subject to

(2) \[ f(z(t),t) \leq c(t) + \sum_{j=0}^{r} \int_{0}^{t} g_{j}(z(s-a_{j}), t, s-a_{j})ds \quad 0 < t < T, \]

†Department of Statistics, Florida State University, Tallahassee, Florida 32306.
(3) \( z(t) \geq 0 \quad 0 \leq t \leq T, \)
and

(4) \( z(t) = 0 \quad t < 0, \)

where \( z(t) \in \mathbb{E}^n; f(z(t), t), g_j(z(s-a_j), t, s-a_j), c(t) \in \mathbb{E}^m; \) the set \( \{a_0, \cdots, a_i\} \) is an arbitrary finite collection of nonnegative numbers representing the time lags of the problem;

\[
f(z(t), t) = \begin{bmatrix}
  f_1(z(t), t) \\
  \vdots \\
  f_m(z(t), t)
\end{bmatrix}
\]

is such that for each \( i, f_i \) is a scalar function, convex and differentiable in its first argument throughout \([0, T]\), for which

(5) \( \frac{\partial f_i}{\partial z_k} \geq 0 \quad k = 1, \cdots, n, \quad 0 \leq t \leq T, \)

and for each \( k \) and \( t \) there exists an \( i \) such that \( \frac{\partial f_i}{\partial z_k} > 0; \)

(6) \( g_j(z(s-a_j), t, s-a_j) = \begin{bmatrix}
  g_{j1}(z(s-a_j), t, s-a_j) \\
  \vdots \\
  g_{jm}(z(s-a_j), t, s-a_j)
\end{bmatrix} \)

is such that for each \( i, g_{ji}(z(s-a_j), t, s-a_j) \) is a scalar function, concave and differentiable in its first argument in the interval \([0, T]\) for which
and $\phi$ is a scalar, concave, continuously twice differentiable function. Notice that if $f(z(t),t) = B(t)z(t)$ then requirements (5) and (6) are that $B(t) \geq 0$ with a positive element in each column for each $t$. Thus if $f$ is a linear function we require that it be positive, but if $f$ is nonlinear, it can be negative provided only that it has a non-negative derivative. It will be assumed that all functions of $t$ are bounded and measurable on $[0,T]$. 

For the primal problem it is seen that if $\phi(z(t)) = a'(t)z(t)$, $f(z(t),t) = B_{m \times n}z(t)$, $g_0(z(s-a_0),t,s-a_0) = c_{m \times n}z(s-a_0)$, and $g_1(z(s-a_1),t,s-a_1) = d_{m \times n}z(s-a_1)$ where $a_0 = 0$, $a_1 = 1$ then this is the problem considered by Larsen and Polak [8]. If $f(z(t),t) = B(t)z(t)$, $g_0(z(s-a_0),t,s-a_0) = k(t,s-a_0)z(s-a_0)$ where $a_0 = 0$, and $\phi$ is of the appropriate form, then this is the problem considered by Hanson [5], Hanson and Mond [6], Tyndall [10], [11], Levinson [9], and Grinold [2], [3].

The dual for this problem is:

**Dual Problem I**

Minimize

\[
(8) \quad g(u,w) = \int_{0}^{T} \left\{ \phi(u(t)) - u'(t)\phi(u(t)) - w'(t)f(u(t),t) \right. \\
+ w'(t)c(t) + w'(t)[\phi'(u(t),t)]'u(t) + \sum_{j=0}^{r} \int_{0}^{T} w'(s+a_j)g_j(u(t),s+a_j,t)ds - \sum_{j=0}^{r} \int_{0}^{T} w'(s+a_j)g_j(u(t),s+a_j,t)u(t)ds \right\} dt,
\]
subject to

\[(9) \quad u(t), w(t) \geq 0 \quad 0 \leq t \leq T\]

\[(10) \quad u(t) = 0 \quad t > T\]

and

\[(11) \quad [\nabla f'(u(t), t)]u(t) \geq \nabla \phi(u(t)) + \sum_{j=0}^{r} \int_{t}^{T} [\nabla g_j'(u(t), s+\alpha_j, t)]w(s+\alpha_j)ds.\]

The notation \(\nabla f'(u(t), t)\) and \(\nabla g_j'(u(t), s+\alpha_j, t)\) signifies that

\[
\nabla f'(u(t), t) = \begin{bmatrix}
\frac{\partial f_1(u(t), t)}{\partial u_1(t)} & \cdots & \frac{\partial f_m(u(t), t)}{\partial u_1(t)} \\
\frac{\partial f_1(u(t), t)}{\partial u_2(t)} & \cdots & \frac{\partial f_m(u(t), t)}{\partial u_2(t)} \\
\frac{\partial f_1(u(t), t)}{\partial u_n(t)} & \cdots & \frac{\partial f_m(u(t), t)}{\partial u_n(t)} \\
\end{bmatrix}
\]

and

\[
\nabla g_j'(u(t), s+\alpha_j, t) = \begin{bmatrix}
\frac{\partial g_{j_1}(u(t), s+\alpha_j, t)}{\partial u_1(t)} & \cdots & \frac{\partial g_{j_m}(u(t), s+\alpha_j, t)}{\partial u_1(t)} \\
\frac{\partial g_{j_1}(u(t), s+\alpha_j, t)}{\partial u_2(t)} & \cdots & \frac{\partial g_{j_m}(u(t), s+\alpha_j, t)}{\partial u_2(t)} \\
\frac{\partial g_{j_1}(u(t), s+\alpha_j, t)}{\partial u_n(t)} & \cdots & \frac{\partial g_{j_m}(u(t), s+\alpha_j, t)}{\partial u_n(t)} \\
\end{bmatrix}
\]
2. RESULTS

**Theorem 1:** If the Primal Problem is feasible, then there exists an optimal solution \( \bar{z}(t) \).

The following lemma will be used in the proof of Theorem 1.

**Lemma 1 (Gronwall's Lemma):** Let the integrable scalar \( g(t) > 0 \) satisfy

\[
(12) \quad g(t) \leq a + \sum_{i=0}^{p} c_i \int_{0}^{t} g(s-a_i) ds \quad 0 \leq t \leq T,
\]

\[
g(t) = 0 \quad \text{for} \quad t < 0,
\]

where \( a > 0, c_i > 0 \ \forall i \), with strict positivity for at least one \( i \), and \( \{a_i\} \) is a finite collection of nonnegative numbers. Then

\[
g(t) \leq a \exp\left( \sum_{i=0}^{p} c_i t \right) \quad 0 \leq t \leq T.
\]

**Proof:** By a change of variable \( (12) \) becomes

\[
g(t) \leq a + \sum_{i=0}^{p} c_i \int_{0}^{t-a_i} g(s) ds,
\]

\[
(13) \quad \leq a + \sum_{i=0}^{p} c_i \int_{0}^{t} g(s) ds, \quad \text{since} \ g(s) > 0.
\]

Now let \( G(t) = \int_{0}^{t} g(s) ds \). From \( (13) \)

\[
\frac{d}{dt} \left( \exp\left( \sum_{i=0}^{p} c_i t \right) G(t) \right) \leq a \exp\left( - \sum_{i=0}^{p} c_i t \right).
\]

Hence
\[ G(t) = a \sum_{i=0}^{p} c_i \left( \exp \left( \sum_{i=0}^{p} c_i t \right) - 1 \right) \]

which in (13) gives the result.  Q.E.D.

Proof of Theorem 1:

Multiplying constraint (2) by the max vector \((1, \cdots, 1)\)' we have

\[ \sum_{i=1}^{m} f_i(z(t), t) \leq \sum_{i=1}^{m} c_i(t) + \sum_{i=1}^{m} \sum_{j=0}^{r} g_{ji}(z(s-a_j), t, s-a_j) ds, \]

which becomes after a change of variables,

\[ \sum_{i=1}^{m} f_i(z(t), t) \leq \sum_{i=1}^{m} c_i(t) + \sum_{i=1}^{m} \sum_{j=0}^{r} t-a_j g_{ji}(z(s), t, s) ds. \]

Since \(f_i\) and \(g_{ji}\) are convex and concave respectively,

\[ f_i(z(t), t) \geq f_i(0, t) + z'(t) v_{f_i}(0, t) \]

and

\[ g_{ji}(z(s), t, s) \leq g_{ji}(0, t, s) + z'(s) v_{g_{ji}}(0, t, s) \]

\[ \leq C_j + G_j \sum_{k=1}^{n} z_k(s), \]

where \(C_j = \max \{0, g_{j1}(0, t, s), \cdots, g_{jm}(0, t, s), 0 \leq s \leq t \leq T\}\) and

\(G_j = \max \{0, v_{g_{j1}}(0, t, s), \cdots, v_{g_{jm}}(0, t, s), 0 \leq s \leq t \leq T\}\). The quantities
\( \bar{C}_1, \ldots, \bar{C}_r, \bar{G}_1, \ldots, \bar{G}_r \) are all finite since we assumed that all functions of \( t \) are bounded on \([0, T]\). Now

\[
\sum_{i=1}^{m} \bar{r}_i(z(t), t) \geq \sum_{i=1}^{m} \bar{f}_i(0, t) + \sum_{i=1}^{m} z'(t) \bar{g}_i(0, t)
\]

\[
= \sum_{i=1}^{m} \bar{f}_i(0, t) + \sum_{k=1}^{n} z_k(t) \left[ \sum_{i=1}^{m} \frac{\partial \bar{f}_i(z(t), t)}{\partial z_k(t)} \right]_{z(t)=0}.
\]

By assumption (6) for each \( k \) and \( t \) there exists an \( i \) such that

\[
\frac{\partial \bar{f}_i}{\partial z_k} > 0;
\]

therefore for each \( t \) and \( k \) let

\[
a_k(t) = \left[ \frac{\partial \bar{f}_i(z(t), t)}{\partial z_k(t)} \right]_{z(t)=0} > 0.
\]

Hence for each \( t \)

\[
\sum_{k=1}^{n} z_k(t) \left[ \sum_{i=1}^{m} \frac{\partial \bar{f}_i(z(t), t)}{\partial z_k(t)} \right]_{z(t)=0} > \min_{k} a_k(t) \sum_{k=1}^{n} z_k(t),
\]

and thus for all \( t \in [0, T] \)

\[
\sum_{k=1}^{n} z_k(t) \left[ \sum_{i=1}^{m} \frac{\partial \bar{f}_i(z(t), t)}{\partial z_k(t)} \right]_{z(t)=0} > A \sum_{k=1}^{n} z_k(t)
\]

where \( 0 < A = \min_{0 < t < T} \min_{k} a_k(t). \)
Let \( c_0 = \max(0, -\sum_{i=1}^{m} f_i(0, t) \ 0 \leq t \leq T) \) and \( c_1 = \max(0, \sum_{i=1}^{p} c_i(t) \ 0 \leq t \leq T) \).

From (15) it then follows that

\begin{align*}
(16) \quad A|z(t)| & \leq -\sum_{i=1}^{m} f_i(0, t) + \sum_{i=1}^{m} c_i(t) + m \sum_{j=0}^{r} t^{-\alpha_j} c_j ds \\
& \quad + m \sum_{j=0}^{r} t^{-\alpha_j} C_j |z(s)| ds \\
& \leq c_0 + c_1 + m \sum_{j=0}^{r} C_j (T^{-\alpha_j}) + \sum_{j=0}^{r} mG_j \int_{0}^{t} t^{-\alpha_j} |z(s)| ds,
\end{align*}

where \( |z(s)| = \sum_{k=1}^{n} z_k(s) \). Expression (16) then becomes after grouping constants together and using a change of variable,

\[ |z(t)| \leq c^* + \sum_{j=0}^{r} c_j^* \int_{0}^{t} |z(s-\alpha_j)| ds. \]

From Lemma 1,

\begin{align*}
(17) \quad |z(t)| & \leq c^* \exp\left( \sum_{j=0}^{r} c_j^* t \right) \leq c^* \exp\left( \sum_{j=0}^{r} c_j^* T \right).
\end{align*}

Since the objective function \( l \) is concave, it follows that for any feasible solutions \( z, z_1 \),

\[ l(z) - l(z_1) \leq T \sup_{[0,T]} |z(t) - z_1(t)| \sup_{[0,T]} |\nu(z_1(t))| < \infty, \]

using result (17). Thus \( l \) is bounded above for all feasible \( z(t) \).

Following the same technique used in the proof of Theorem 1 [1], take
a maximizing sequence, \( z^0(t) \), and use a diagonal process, plus the assumptions of convexity and concavity, to find a maximizing \( z^0 \) satisfying (2) and (3) except on a set of measure zero. Defining \( \bar{z}(t) = 0 \) on this set as well as for \( t < 0 \) and equal to \( z^0 \) on the complement of the set of measure zero we obtain an optimal solution to the primal problem. Constraint (3) is satisfied by \( \bar{z}(t) \) for using the concavity of the functions \( c_j \) and the weak convergence of \( z^{(k_0)}(t) \) to \( z^0(t) \) it can be shown that

\[
\lim_{k_0 \to \infty} \sup_{t \in [0,T]} f(z^{(k_0)}(t), t) \leq c(t) + \sum_{j=0}^{r} \int_{0}^{t} g_j(z^0(s-a_j)),
\]

t, s-a_j)ds holds for all \( t \in [0,T] \). Now since \( f \) is convex

\[
f(z^{(k_0)}(t)) \geq f(z^0(t), t) + [vf'(z^0(t), t)]'(z^{(k_0)}(t) - z^0(t)) \geq 0
\]

By Lemma 2 of reference [1] it follows that

\[
f(z^0(t), t) \leq \lim_{k_0 \to \infty} \sup_{t \in [0,T]} f(z^{(k_0)}(t), t)
\]

except on a set of measure zero since

\[
\lim_{k_0 \to \infty} \sup_{t \in [0,T]} [vf'(z^0(t), t)]'(z^{(k_0)}(t) - z^0(t)) \geq 0
\]

except on this set. On the complement of this set \( \bar{z}(t) \) is defined to be \( z^0(t) \) so that

\[
f(\bar{z}(t), t) \leq \lim_{k_0 \to \infty} \sup_{t \in [0,T]} f(z^{(k_0)}(t), t).
\]
For \( t \) in the set, \( \bar{z}(t) \) is defined to be zero, but then by convexity for any \( k_0 \)

\[
f(z^{(k_0)}(t)) > f(0,t) + \left[ z'f'(0,t) \right]'(t) > f(c,t)
\]

from (3) and (5). Thus

\[
\limsup_{k_0 \to \infty} f(z^{(k_0)}(t)) > f(0,t).
\]

Combining these results it is established that

\[
f(\bar{z}(t),t) \leq \limsup_{k_0 \to \infty} f(z^{(k_0)}(t),t)
\]

\[
\leq c(t) + \sum_{j=0}^{\infty} \int_0^t g_j(z^{0}(s-\alpha_j),t,s-\alpha_j) ds
\]

\[
= c(t) + \sum_{j=0}^{\infty} \int_0^t g_j(\bar{z}(s-\alpha_j),t,s-\alpha_j) ds
\]

Q.E.D.

**Lemma 2:**

\[
\int_0^T \int_0^t w'(t) g_j(z(s-\alpha_j),t,s-\alpha_j) ds dt = \int_0^T \int_0^t w'(s) g_j(z(t-\alpha_j),s,t-\alpha_j) ds dt
\]

\[
= \int_0^T \int_0^t w'(s+\alpha_j) g_j(z(s),s+\alpha_j,t) ds dt
\]

and
\[ \int_0^T \int_0^t w'(t) g_j(z(s-t), s-t) \, ds \, dt \]

\[ = \int_0^T \int_0^t w'(s) g_j(z(t), s-t) \, ds \, dt \]

\[ = \int_0^T \int_0^t w'(s+\alpha_j) g_j(z(t), s+\alpha_j, t) \, ds \, dt \]

Proof: Consider the first set of equations. The first equation holds by Fubini's Theorem. Since \( w'(s+\alpha_j) = 0 \) for \( s+\alpha_j > T \) we obtain

\[ \int_0^T \int_0^t w'(s+\alpha_j) g_j(z(t), s+\alpha_j) \, ds \, dt = \int_0^T \int_0^t w'(s+\alpha_j) g_j(z(t), s+\alpha_j, t) \, ds \, dt. \]

Since \( z(t) = 0 \) for \( t < 0 \) we now obtain after renaming the dummy variables

\[ \int_0^T \int_0^t w'(s+\alpha_j) g_j(z(t), s+\alpha_j) \, ds \, dt = \int_{\alpha_j}^T \int_{\alpha_j}^t w'(s) g_j(z(t-\alpha_j), s, t-\alpha_j) \, ds \, dt \]

\[ = \int_0^T \int_0^t w'(s) g_j(z(t-\alpha_j), s, t-\alpha_j) \, ds \, dt. \]

This is the second equality. The same argument is used to establish the second expression.

Q.E.D.
Lemma 3: If \((u^0(t), w^0(t))\) and \(z^0(t)\) are feasible solutions to Dual and Primal Problem I respectively, then

\[
\int_0^T \phi(z^0(t))dt \leq \int_0^T \{\phi(u^0(t)) - u^0 \nabla \phi(u^0(t))

- w^0(t)f(u^0(t), t) + w^0(t)c(t) + w^0(t)[\nabla f'(u^0(t), t)]'u^0(t)

+ \sum_{j=0}^r \int_0^T w^0(j)(s+a_j)g_j(u^0(t), s+a_j, t)ds

- \sum_{j=0}^r \int_0^T w^0'(s+a_j)g_j'(u^0(t), s+a_j, t)u^0(t)ds\}dt.
\]

Proof of Lemma 3:

Since \(\phi\) and \(g\) are concave and \(f\) is convex with respect to their first arguments, we have

\[
\int_0^T \{\phi(z^0(t))dt - \phi(u^0(t)) + u^0 \nabla \phi(u^0(t))

+ w^0(t)f(u^0(t), t) - w^0(t)[\nabla f'(u^0(t), t)]'u^0(t) - w^0(t)c(t)

+ \sum_{j=0}^r \int_0^T w^0'(s+a_j)g_j(u^0(t), s+a_j, t)u^0(t)ds

- \sum_{j=0}^r \int_0^T w^0'(s+a_j)g_j'(u^0(t), s+a_j, t)u^0(t)ds\}dt.
\]
\[
\begin{align*}
&\leq \int_0^T \{z^0(t)\varphi(u^0(t)) + w^0(t)f(z^0(t), t) \\
&- w^0(t)[\varphi'(u^0(t), t)]'z^0(t) - w^0(t)c(t) \\
&+ \sum_{j=0}^r \int_0^T w^0(s + \alpha_j)[\varphi_s(u^0(t), s + \alpha_j, t)]'z^0(t)ds \\
&- \sum_{j=0}^r \int_0^T w^0(s + \alpha_j)g_j(z^0(t), s + \alpha_j, t)ds dt, \\
&= \int_0^T \{z^0(t)\varphi(u^0(t)) - [\varphi'(u^0(t), t)]w^0(t) \\
&+ \sum_{j=0}^r \int_0^T [\varphi_s(u^0(t), s + \alpha_j, t)]w^0(s + \alpha_j)ds dt \\
&- \int_0^T w^0(t)c(t) - w^0(t)f(z^0(t), t) \\
&+ \sum_{j=0}^r \int_0^T w^0(t)g_j(z^0(s - \alpha_j), t, s - \alpha_j)ds dt, \\
\end{align*}
\]
by Lemma 2,

\[\leq 0\quad \text{by (2), (3), (9), and (11).}\]

Q.E.D.
From this lemma it follows that if there exist feasible solutions, \((u^0(t), w^0(t))\) and \(z^0(t)\), for the dual and primal problems, and if the corresponding dual and primal objective functions are equal, then these solutions are optimal for their respective problems. In the theorems which follow we now make the additional assumptions that

\[
(18) \quad \sum_{j=0}^{r} g_j(z(s), t, s) > 0 \quad 0 < s < t < T
\]

and

\[
(19) \quad \sum_{j=0}^{r} v g_j'(z(s), t, s) > 0 \quad 0 < s < t < T.
\]

**THEOREM 2.** If

\[
(c(t) - f(\tilde{z}(t), t) + [\gamma f'(\tilde{z}(t), t)]') \tilde{z}(t)
\]

\[
+ \sum_{j=0}^{t} \int_{0}^{t} g_j(\tilde{z}(s), t, s) ds - \sum_{j=0}^{r} \int_{0}^{t} [v g_j'(\tilde{z}(s), t, s)]' \tilde{z}(s) ds
\]

is nonnegative, where \(\tilde{z}\) is the optimal solution to Primal Problem I, and 0 is a feasible solution for that problem, then there exists a minimizing solution \((\tilde{u}(t), \tilde{w}(t))\) for Dual Problem I such that \(\tilde{u}(t) = \tilde{z}(t)\) and the objective functions, evaluated at these optima, are equal.

**Proof of Theorem 2:**

Consider the following linearized primal problem:
Primal Problem II

Maximize

\[ F(z) = \int_0^T \phi(\tilde{z}(t)) + (z(t) - \tilde{z}(t))' \psi(\tilde{z}(t)) \, dt \]

subject to

\[ z(t) \geq 0 \]

\[ 0 \leq t \leq T, \]

\[ z(t) \leq \tilde{z}(t), \]

and

\[ f(\tilde{z}(t), t) + [\nabla f'(\tilde{z}(t), t)]'(z(t) - \tilde{z}(t)) \]

\[ \leq c(t) + \sum_{j=0}^r \int_0^t g_j(\tilde{z}(s), t, s) \, ds - \sum_{j=0}^r \int_0^t [\nabla g_j'(\tilde{z}(s), t, s)]'z(s) \, ds \]

\[ + \sum_{j=0}^r \int_0^t [\nabla g_j'(\tilde{z}(s), t, s)]'z(s) \, ds \quad 0 \leq t \leq T, \]

where \( \tilde{z}(t) \) is optimal for Primal Problem I. Firstly, we show that \( \tilde{z}(t) \) is also optimal for this problem. Obviously \( \tilde{z}(t) \) satisfies the first two constraints. We now show that it also satisfies (20). Putting \( \tilde{z}(t) \) in (20) we have

\[ f(\tilde{z}(t), t) \leq c(t) + \sum_{j=0}^r \int_0^t g_j(\tilde{z}(s), t, s) \, ds. \]

Since \( \tilde{z}(t) \) is optimal for Primal Problem I from (2)
\[ f(\tilde{z}(t), t) \leq c(t) + \sum_{j=0}^{r} \int_{0}^{t} g_j(\tilde{z}(s), t, s) ds, \]

which becomes after a change of variable,

\[ f(\tilde{z}(t), t) \leq c(t) + \sum_{j=0}^{r} \int_{0}^{t-a_j} g_j(\tilde{z}(s), t, s) ds. \]

However since \( \sum_{j=0}^{r} g_j(\tilde{z}(s), t, s) \geq 0, \)

\[ \sum_{j=0}^{r} \int_{0}^{t-a_j} g_j(\tilde{z}(s), t, s) ds \leq \sum_{j=0}^{r} \int_{0}^{t} g_j(\tilde{z}(s), t, s) ds. \]

Thus (21) holds.

It can be shown, using the same procedure as in the proof of Theorem 2 [1], that \( \tilde{z}(t) \) is optimal for this problem. The requirement that \( 0 \) is feasible for Primal Problem I is needed to ensure that \( \tilde{z}(t) \), as defined in [1], is feasible for Primal Problem I also. The reader is referred to [1] for details.

The dual of Primal Problem II is the following:

**Dual Problem II**

Minimize

\[ G(\underline{w}_{m+n+1}) = \max \left( \begin{array}{c} w_{mx1} \\ v_{nx1} \end{array} \right) = \int_{0}^{T} \left( f(\tilde{z}(t)) - \phi'(\tilde{z}(t))v(\tilde{z}(t)) \right) \]

\[ - (w'(t), v'(t)) \left[ f(\tilde{z}(t), t) \right] + (w'(t), v'(t)) \left[ [\phi'(\tilde{z}(t), t)]'\tilde{z}(t) \right] - (w'(t), v'(t)) \left[ \tilde{z}(t) \right] \]
\[
\begin{align*}
\mathcal{L} & = \int_0^T \left( \phi(\bar{z}(t)) - \bar{z}'(t)\nabla\phi(\bar{z}(t)) - w'(t)f(\bar{z}(t), t) + w'(t)[\nabla f'(\bar{z}(t), t)]' \right) dt \\
& - \sum_{j=0}^{r} \int_0^T [v_{g_j}'(\bar{z}(s), t, s)]'\bar{z}(s) ds dt,
\end{align*}
\]

subject to
\[
\begin{align*}
\mathcal{W}(t) & = \begin{bmatrix} w(t)_{mx1} \\ v(t)_{nx1} \end{bmatrix} \geq 0 \quad 0 \leq t \leq T, \\
[v_{f'}(\bar{z}(t), t), 1]_{v(t)} & \geq \nabla \phi(\bar{z}(t)) + \sum_{j=0}^{r} \int_0^T [v_{g_j}'(\bar{z}(t), s, t)]'_{s} ds dt, \\
\int_0^T [w(s)_{m1} \quad v(s)_{n1}] ds & \geq 0 \leq t \leq T.
\end{align*}
\]

Using the hypothesis of the theorem and establishing the appropriate relationships between the formulation of Primal Problem II and the primal problem considered in [1], we can conclude from Theorem 2 [1] that there
exists an optimal solution \( \tilde{w}(t) = \frac{\tilde{v}(t)}{\tilde{v}(t)} \) for Dual Problem II. In expression (20) the only term containing \( v(t) \) is \( v'(t)\tilde{z}(t) \) and since \( \tilde{z}(t) \geq 0 \) to minimize this expression implies \( \tilde{v}(t) = 0 \). From Theorem 2 the objective functions of the two problems are equal, that is,

\[
\int_0^T \{ \tilde{z}'(t)v(t) + \tilde{v}'(t)f(\tilde{z}(t), t) - \tilde{v}'(t)[v(t)'(\tilde{z}(t), t)]'\tilde{z}(t)
- \tilde{v}'(t)c(t) - \sum_{j=0}^{r} \int_t^T \tilde{v}'(s)g_j(\tilde{z}(t), s, t)ds + \sum_{j=0}^{r} \int_t^T \tilde{v}'(s) [v(t)g_j'(\tilde{z}(t), s, t)]'\tilde{z}(t)ds \} dt = 0.
\]

Now let

\[
\tilde{w}(t) = \begin{cases} \tilde{w}(t) & 0 \leq t \leq T \\ 0 & t > T \end{cases}.
\]

Let us show that \( (\tilde{z}(t), \tilde{w}(t)) \) is optimal for Dual Problem I. Clearly, constraints (9) and (10) are satisfied. Now for each \( t \in [0, T] \) from Dual Problem II

\[
[vf'(\tilde{z}(t), t)]\tilde{w}(t) \geq \phi(\tilde{z}(t)) + \sum_{j=0}^{r} \int_t^T [v(t)g_j'(\tilde{z}(t), s, t)]\tilde{w}(s)ds,
\]

that is,

\[
[vf'(\tilde{z}(t), t)]\tilde{w}(t) \geq \phi(\tilde{z}(t)) + \sum_{j=0}^{r} \int_t^T [v(t)g_j'(\tilde{z}(t), s, t)]\tilde{w}(s)ds.
\]

Since \( \sum_{j=0}^{r} [v(t)g_j'(\tilde{z}(t), s, t)] \geq 0 \),
\[
\begin{align*}
\sum_{j=0}^{r} \int_{t}^{T} [v_{j}(\bar{z}(t),s,t)]\bar{w}(s)ds & \geq \sum_{j=0}^{r} \int_{t+\alpha_j}^{T} [v_{j}(\bar{z}(t),s,t)]\bar{w}(s)ds \\
& = \sum_{j=0}^{r} \int_{t}^{T} [v_{j}(\bar{z}(t),s+\alpha_j,t)]\bar{w}(s+\alpha_j)ds.
\end{align*}
\]

Hence
\[
[v\phi'(\bar{z}(t),t)]\bar{w}(t) \geq \phi(\bar{z}(t)) + \sum_{j=0}^{r} \int_{t}^{T} [v_{j}(\bar{z}(t),s+\alpha_j,t)]\bar{w}(s+\alpha_j)ds.
\]

Thus \((\bar{z}(t), \bar{w}(t))\) is feasible for the dual problem. From (23),
\[
\begin{align*}
\int_{0}^{T} \{ & \bar{z}'(t)v\phi(\bar{z}(t)) + \bar{w}'(t)f(\bar{z}(t),t) - \bar{w}'(t)[v\phi'(\bar{z}(t),t)]\bar{z}(t) \\
& + \bar{w}'(t)c(t) - \sum_{j=0}^{r} \int_{t}^{T} \bar{w}'(s)g_{j}(\bar{z}(t),s,t)ds + \sum_{j=0}^{r} \int_{t}^{T} \bar{w}'(s)[v_{j}(\bar{z}(t),s,t)]d\bar{z}(s, t) \} d\bar{z}(t)dt = 0.
\end{align*}
\]

Hence
\[
(24) \quad \int_{0}^{T} \{ \bar{z}'(t)v\phi(\bar{z}(t)) + \bar{w}'(t)f(\bar{z}(t),t) - \bar{w}'(t)[v\phi'(\bar{z}(t),t)]\bar{z}(t)-\bar{w}'(t)c(t) \\
- \sum_{j=0}^{r} \int_{t+\alpha_j}^{T} \bar{w}'(s)g_{j}(\bar{z}(t),s,t)ds + \sum_{j=0}^{r} \int_{t+\alpha_j}^{T} \bar{w}'(s)[v_{j}(\bar{z}(t),s,t)]d\bar{z}(s, t) \} d\bar{z}(t)dt = 0
\]

since \(\bar{w}'(s) \geq 0\) and
\[
\left[ \sum_{j=0}^{r} v g'_j(\bar{z}(t), s, t) \right] \bar{z}(t) \leq \sum_{j=0}^{r} \bar{g}_j(\bar{z}(t), s, t).
\]

The last inequality follows from concavity,

\[
0 \leq \sum_{j=0}^{r} \bar{g}_j(0, s, t) \leq \sum_{j=0}^{r} \bar{g}_j(\bar{z}(t), s, t) - \left[ \sum_{j=0}^{r} v g'_j(\bar{z}(t), s, t) \right] \bar{z}(t).
\]

From Lemma 3, we have that the left hand side of (24) is less than or equal to zero. Thus

\[
(25) \quad \int_{0}^{T} \{ \bar{z}'(t) v f(\bar{z}(t)) + \bar{w}'(t)f(\bar{z}(t), t) - \bar{w}'(t)[\bar{v} f'(\bar{z}(t), t)]' \bar{z}(t) - \bar{w}'(t)c(t)
\]

\[- \sum_{j=0}^{r} \int_{0}^{T} \bar{w}'(s+\alpha_j) g_j(\bar{z}(t), s+\alpha_j, t) ds + \sum_{j=0}^{r} \int_{0}^{T} \bar{w}'(s+\alpha_j) [v g'_j(\bar{z}(t), s+\alpha_j, t)]' \bar{z}(t) ds \} dt = 0.
\]

From our remark at the end of Lemma 3, we conclude that \((\bar{z}(t), \bar{w}(t))\) is optimal for Dual Problem I. Expression (25) shows that the values of the objective functions for the primal and dual problems are equal.

Q.E.D.

From the hypothesis of the theorem, it is seen that if \(r = 0\), \(a_0 = 0\), \(g_0(z(s), t, s) = k(t, s)z(s)\), and \(f(z(t), t) = B(t)z(t)\), which is the formulation considered by Levinson [9], Tyndall [10], [11], and Grinold [2], [3], then the requirement that
\[ (c(t) - f(\bar{z}(t), t) + \sum_{j=0}^{t} g_j(\bar{z}(s), t, s))' \bar{z}(t) + \int_{0}^{t} g_0(\bar{z}(s), t, s) ds \]

\[ - \int_{0}^{t} \left[ \sum_{j=0}^{r} g_j'(\bar{z}(s), t, s) \right]' \bar{z}(s) ds \geq 0 \]

becomes \( c(t) \geq 0 \). This is one of Levinson's conditions for the duality theorem to hold. The other conditions are given by (5), (6), and (19).

**Corollary 2.1:**

Under assumptions (18) and (19), if \( c(t) \geq 0 \) and \( f(0, t) = 0 \), then Theorem 2 holds.

**Proof:** Since \( f(0, t) = 0 \) and \( c(t) \geq 0 \), 0 is clearly a feasible solution to Primal Problem I. Moreover since \( f \) is convex and \( \sum_{j=0}^{r} g_j \) is concave

\[ 0 = f(0, t) \geq f(\bar{z}(t), t) - \left[ \sum_{j=0}^{r} g_j'(\bar{z}(t), t) \right]' \bar{z}(t) \]

and

\[ 0 \leq \sum_{j=0}^{r} g_j(0, t, s) \leq \sum_{j=0}^{r} g_j(\bar{z}(s), t, s) - \left[ \sum_{j=0}^{r} g_j'(\bar{z}(s), t, s) \right]' \bar{z}(s). \]

Thus the conditions of Theorem 2 are satisfied.

Q.E.D.

**Theorem 3** (Complementary Slackness Principle):

If \( \bar{z}(t) \) and \( \bar{w}(t) \) are extremal solutions for Primal and Dual Problems I then
\begin{equation}
(26) \quad \int_0^T \tilde{\nu}'(t)\{f(\tilde{z}(t), t) - c(t)\} - \sum_{j=0}^r \int_0^t g_j(\tilde{z}(s-\alpha_j), t, s-\alpha_j) ds dt = 0
\end{equation}

and

\begin{equation}
(27) \quad \int_0^T \tilde{z}'(t)\{[\nu f'([\tilde{z}(t), t])]\tilde{\nu}(t) - \nu \phi([\tilde{z}(t)]) - \sum_{j=0}^r \int_0^T [\nu g_j'([\tilde{z}(t), s+\alpha_j], t)] \tilde{\nu}(s+\alpha_j) ds dt = 0.
\end{equation}

Proof of Theorem 3:

From (3) and (11),

\[ \int_0^T \tilde{z}'(t)[\nu f'(\tilde{z}(t), t)]\tilde{\nu}(t) dt \geq \int_0^T \tilde{z}'(t)\nu \phi(\tilde{z}(t)) dt 
\]

\[ + \int_0^T \tilde{z}'(t)\{\sum_{j=0}^r \int_0^T [\nu g_j'(\tilde{z}(t), s+\alpha_j, t)] \tilde{\nu}(s+\alpha_j) ds dt \}
\]

\[ = \int_0^T \tilde{z}'(t)\nu \phi(\tilde{z}(t)) dt + \sum_{j=0}^r \int_0^T \int_0^t \tilde{z}'(s-\alpha_j)[\nu g_j'(\tilde{z}(s-\alpha_j), t, s-\alpha_j)] \tilde{\nu}(t) ds dt, \]

by Lemma 2,

\[ \int_0^T \{\tilde{\nu}'(t) c(t) + \tilde{z}'(t)\nu f'(\tilde{z}(t), t)\} \tilde{z}(t) 
\]

\[ - \tilde{\nu}'(t)f(\tilde{z}(t), t) + \sum_{j=0}^r \int_0^t \tilde{\nu}'(t) g_j(\tilde{z}(s-\alpha_j), t, s-\alpha_j) ds 
\]

\[ - \sum_{j=0}^r \int_0^t \tilde{\nu}'(t)[\nu g_j'(\tilde{z}(s-\alpha_j), t, s-\alpha_j)] \tilde{\nu}(s-\alpha_j) ds + \sum_{j=0}^r \int_0^t \tilde{z}'(s-\alpha_j) 
\]

\[ [\nu g_j'(\tilde{z}(s-\alpha_j), t, s-\alpha_j)] \tilde{\nu}(t) ds dt, \]
using Lemma 2, and (25).

Hence

\[ \int_0^T \bar{w}'(t)\{f(\bar{z}(t),t) - c(t) - \sum_{j=0}^{r} \int_0^t g_j(\bar{z}(s-a_j),t,s-a_j)ds\}dt \geq 0. \]

From (9) and (2) it follows that

\[ \int_0^T \bar{w}'(t)\{f(\bar{z}(t),t) - c(t) - \sum_{j=0}^{r} \int_0^t g_j(\bar{z}(s-a_j),t,s-a_j)ds\}dt \geq 0. \]  

Thus result (26) follows.

From (28) and Lemma 2 it follows that

\[ \int_0^T w'(t)f(\bar{z}(t),t)dt \leq \int_0^T \bar{w}'(t)c(t)dt + \sum_{j=0}^{r} \int_0^T \bar{w}'(s+a_j) g_j(\bar{z}(t),s+a_j,t)ds dt \]

\[ = \int_0^T \{\bar{z}'(t)\nu(\bar{z}(t)) + \bar{w}'(t)f(\bar{z}(t),t) \]

\[ - \bar{w}'(t)[\nu f'(\bar{z}(t),t)]'\bar{z}(t) - \sum_{j=0}^{r} \int_0^t \bar{w}'(s+a_j) g_j(\bar{z}(t),s+a_j,t)ds + \sum_{j=0}^{r} \int_0^t \bar{w}'(s+a_j) g_j(\bar{z}(t),s+a_j,t) \]

\[ s+a_j, t)ds dt, \text{ from (25).} \]

Thus

\[ \int_0^T \bar{z}'(t)\{[\nu f'(\bar{z}(t),t)]\bar{w}(t) - \nu(\bar{z}(t)) - \sum_{j=0}^{r} \int_0^t [\nu g_j'(\bar{z}(t),s+a_j,t)] \]

\[ \bar{w}(s+a_j)ds \}dt \leq 0. \]
Using (ii) and (3), expression (27) is established. Q.E.D.

**Theorem 4. (Kuhn-Tucker Theorem):**

For \( \tilde{z}(t) \) to be extremal for Primal Problem I, it is necessary and sufficient that there exists an m×1 vector \( w_0(t) \) such that

\[
(i) \quad \left[ \nabla f(\tilde{z}(t),t) \right] w_0(t) - \nabla \phi(\tilde{z}(t)) - \sum_{j=0}^{r} \int_{t}^{T} \left[ \nabla g_j(\tilde{z}(t),s+a_j,t) \right] w_0(s+a_j) ds \geq 0
\]

\[
(ii) \quad \int_{0}^{T} \tilde{z}'(t) \left[ \nabla f(\tilde{z}(t),t) \right] w_0(t) - \nabla \phi(\tilde{z}(t)) - \sum_{j=0}^{r} \int_{t}^{T} \left[ \nabla g_j(\tilde{z}(t),s+a_j,t) \right] w_0(s+a_j) ds
\]

\[ \quad dt = 0 \]

\[
(iii) \quad \int_{0}^{T} w_0(t) \left[ f(\tilde{z}(t),t) - c(t) \right] - \sum_{j=0}^{r} \int_{0}^{t} g_j(\tilde{z}(s-a_j),t,s-a_j) ds \right] dt = 0
\]

\[
(iv) \quad w_0(t) \geq 0 \quad \text{for} \quad 0 \leq t \leq T \]

\[
\quad w_0(t) = 0 \quad \text{for} \quad t > T.
\]

Proof of Theorem 4. The existence of an extremal solution is given by Theorem 1. Sufficiency -

Let \( z(t) \) be any feasible solution of Primal Problem I. Thus
\[
\int_0^T \phi'\left(\tilde{z}(t)\right) dt - \int_0^T \phi'\left(\bar{z}(t)\right) dt \leq \int_0^T (z(t) - \bar{z}(t))' \nu \phi(\tilde{z}(t)) dt,
\]

since \( \phi \) is concave,

\[
\leq \int_0^T (z(t) - \bar{z}(t))' \left[ \nu f'\left(\tilde{z}(t), t\right) \right] w_0(t)
\]

\[
- \sum_{j=0}^r \int_0^T \left[ \nu g_j'\left(\tilde{z}(t), s+\alpha_j, t\right) \right] w_0(s+\alpha_j) ds dt, \quad \text{by (3)}
\]

and conditions (i) and (ii),

\[
- \int_0^T w_0'(t) [\nu f'\left(\tilde{z}(t), t\right)]' (z(t) - \bar{z}(t)) dt
\]

\[
- \sum_{j=0}^r \int_0^T \int_0^T w_0'(s+\alpha_j) [\nu g_j'\left(\tilde{z}(t), s+\alpha_j, t\right)]' (z(t) - \bar{z}(t)) dsdt
\]

\[
\leq \int_0^T w_0'(t) [f(z(t), t) - f(\bar{z}(t), t)] dt
\]

\[
+ \sum_{j=0}^r \int_0^T \int_0^T w_0'(s+\alpha_j) [g_j(\tilde{z}(t), s+\alpha_j, t) - g_j(z(t), s+\alpha_j, t)] dsdt,
\]

since \( f \) is convex and \( g_j \) is concave,

\[
= \int_0^T w_0'(t) [f(z(t), t) - c(t) - \sum_{j=0}^r \int_0^t g_j(z(s-\alpha_j), t, s-\alpha_j) ds] dt
\]
by Lemma 2,

\[ \leq 0 \quad \text{by conditions (iii), (iv) and constraint (2)}. \]

Thus \( \bar{z}(t) \) is optimal for Primal Problem I.

Necessity:

The necessity of the Kuhn-Tucker conditions follows from Theorems 2 and 3 as the solution \( \bar{w}(t) \) of the Dual Problem I has the required properties.

We note that restrictions (18) and (19) can be relaxed for the sufficiency part of the theorem.

Q.E.D.

3. Example of a Continuous Time Model with Time Delayed Constraints

We consider a generalization of a water storage model given by Koopmans[7]. Suppose there is a confluent system of rivers supplying water to a major dam and hydroelectric plant on the main stream. There are also dams and hydroelectric plants on n of the tributaries. Let \( \Omega_i, \ i=1, \ldots, n \), be the initial store of water in the reservoir on tributary i and \( \Omega_0 \) the initial store in the main reservoir. Let \( \theta_i, \ i=0, \ldots, n \), be the reservoir capacities, \( u_i(t) \) the spillage of water through dam i at time t, and \( v_i(t) \) the rate of discharge of water through the turbines at dam i where i goes from 0 to n. \( \xi_i(t), \ i=1, \ldots, n \), will denote the rate of inflow of water into reservoir i at time t. The rate of inflow of water into the main reservoir from its undammed tributaries will be denoted as \( \xi_0(t) \).
Suppose that it takes $q_i, i=1, \ldots, n,$ units of time for water released from dam $i$ to reach the main reservoir. These will represent the time lags of our model.

The determination of the optimum water storage policy for the entire system of dams is for the planning period $[0, T]$ and it is subject to two sets of constraints, the reservoir constraints and flow constraints. The reservoir constraints can be formulated as:

\begin{equation}
0 \leq \Omega_i + \int_0^t (\xi_i(t) - u_i(t) - v_i(t)) dt \quad 0 \leq t \leq T, \quad i=1, \ldots, n,
\end{equation}

\begin{equation}
\Omega_i + \int_0^t (\xi_i(t) - u_i(t) - v_i(t)) dt \leq \Theta_i \quad 0 \leq t \leq T,
\end{equation}

and

\begin{equation}
0 \leq \Omega_0 + \int_0^t \left\{ \xi_0(t) + \sum_{j=1}^n (u_j(t-a_j) + v_j(t-a_j)) - u_0(t) - v_0(t) \right\} dt \quad 0 \leq t \leq T,
\end{equation}

\begin{equation}
\Omega_0 + \int_0^t \left\{ \xi_0(t) + \sum_{j=1}^n (u_j(t-a_j) + v_j(t-a_j)) - u_0(t) - v_0(t) \right\} dt \leq \Theta_0, \quad 0 \leq t \leq T.
\end{equation}
The quantities $\xi_i(t) - u_i(t) - v_i(t)$, $i = 1, \ldots, n$, and

$$\xi_0(t) + \sum_{j=1}^{n} (u_j(t-\alpha_j) + v_j(t-\alpha_j)) - u_0(t) - v_0(t)$$

designate that the net excess of inflow over outflow through turbines $v_i(t)$ or spillways $u_i(t)$, $i = 0, \ldots, n$, becomes a net increase to the store of water at dam $i$.

The flow constraints are

\begin{align}
(31) & \quad 0 \leq u_i(t) \leq \beta_i(t) \\
(32) & \quad 0 \leq v_i(t) \leq \phi_i \quad \text{for } 0 \leq t \leq T,
\end{align}

$i = 0, \ldots, n$. $\beta_i(t)$ is the maximum allowable spillage at time $t$ through dam $i$ and $\phi_i$ is the turbine capacity for that dam.

The output of electricity from the turbine is assumed to be proportional to $v_i$. This factor of proportionality, $\lambda_i$, is a function of the store of water, $W_i$, in reservoir $i$.

Therefore the power output of a discharge $v_i$ at dam $i$ is $v_i \lambda_i(W_i)$. The units of water flow and electric power are chosen in such a way that $\lambda_i(W_i)$ satisfies the following,

$$0 \leq \lambda_i(0) \leq \lambda_i(W_i) \leq \lambda_i(W_i') \leq \lambda_i(\Omega_i) = 1 \text{ for}$$
\[ 0 \leq \dot{w}_i \leq \dot{w}_1 \leq \dot{w}_2 \leq \cdots \leq \dot{w}_n, \quad i=0, \ldots, n. \]

The final flow constraint is then

\[ (33) \quad \sum_{i=0}^{n} v_i(t)\lambda_i(\dot{w}_i(t)) \leq g(t) \quad 0 \leq t \leq T, \]

where \( g(t) \) is the demand for power at time \( t \).

Koopmans points out that in some cases where the range of reservoir surface levels is small in comparison with the drop in elevation from the intake gates to the water below the reservoir, the conversion factor can be taken to be 1.

We make this assumption for each of the reservoirs so that constraint (33) becomes

\[ (34) \quad \sum_{i=0}^{n} v_i(t) \leq g(t) \quad 0 \leq t \leq T. \]

If power demand exceeds the hydroelectric supply, additional power is available from thermal generating stations. Let \( \psi(s) \) be the cost of the thermal generation at rate \( s \) where it is assumed that \( \psi(s) \) is a twice differentiable increasing convex function of \( s \). We also assume it has a positive slope at \( s=0 \) and \( \psi(0)=0 \). The total operating cost of thermal generation for the period \([0,T]\) is

\[ \int_{0}^{T} \psi(g(t)) - \sum_{i=0}^{n} v_i(t)) dt. \]
The problem is to minimize this integral by proper choice of the functions
\( u_i(t), v_i(t), i=0, \ldots, n \), subject to constraints (27), (28), (29), (30), (31), (32), and (34).

This model can be shown to satisfy the assumptions concerning the functions of our time delayed problem and thus by Theorem 1 of this paper, if the water storage problem is feasible, it has an optimal solution; and necessary and sufficient conditions for a solution are given by the Kuhn-Tucker conditions in Section 2.
REFERENCES


**UNCLASSIFIED**

Security Classification

**DOCUMENT CONTROL DATA - R&D**

1. Originating Activity
   The Florida State University
   Department of Statistics
   Tallahassee, Florida

2a. Report Security Classification
    Unclassified

2b. Group

3. Report Title
   Continuous Time Programming with Nonlinear Time Delayed Constraints

4. Descriptive Notes
   Technical Report, December, 1972

5. Author(s)
   William H. Farr, The Florida State University
   Morgan A. Hanson, The Florida State University

6. Report Date
   December 1972

7a. Total No. of Pages
    32

7b. No. of References
    11

8a. Contract or Grant No.
    AFOSR-72-2345

9a. Originator's Report Number(s)
    AFOSR-H Technical Report No. 5

8b. Project No.
    9769

9b. Other Report No(s).
    FSU Statistics Report M 248

10. Availability/Limitation Notices
    Releasable without limitations on dissemination

11. Supplementary Notes

12. Sponsoring Military Activity
    U.S. Air Force
    Air Force Office of Scientific Research
    1400 Wilson Boulevard
    Arlington, Virginia 22209

13. Abstract

   A class of continuous time nonlinear programming problems relating to problems of acquisition, stockpiling, and distribution of materials is given. Nonlinearity appears in both the objective function and the constraints, and provision is made for time lags in the constraints. Necessary and sufficient conditions for the existence of solutions are established and optimal solutions are characterized in terms of a duality theorem. This Report is an extension of the authors' previous results presented in Report AFOSR-H No. 3.

<table>
<thead>
<tr>
<th>Key Words</th>
<th>Link A</th>
<th>Link B</th>
<th>Link C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical Programming</td>
<td>Role</td>
<td>Role</td>
<td>Role</td>
</tr>
<tr>
<td>Nonlinear Programming</td>
<td>Wt.</td>
<td>Wt.</td>
<td>Wt.</td>
</tr>
<tr>
<td>Continuous Time Programming</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Production Control</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inventory Control</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time Delayed Control</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>