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Mohamed Abdel-Hameed and Frank Proschan

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Department of Statistics
The Florida State University
Tallahassee, Florida 32306

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Abstract.

This paper extends results of Esary, Marshall, and Proschan (1973), Ann. of Probability, and Abdel-Hameed and Proschan (1972), FSU Statistics Report M243. We consider the life distribution of a device subject to a sequence of shocks occurring randomly in time according to a nonstationary pure birth process; given that k shocks have occurred in [0, t], the probability of a shock occurring in [t, t+Δ] is \( \lambda_k \lambda(t) \Delta + o(\Delta) \). We show that various fundamental classes of life distributions (such as those with increasing failure rate, or those with the "new better than used" property, etc.) are obtained under appropriate assumptions on \( \lambda_k, \lambda(t) \), and on the probability of surviving a given number of shocks.
1. Introduction and Summary.

Esary, Marshall, and Proschan (1973) consider the life distribution of a device subject to a sequence of shocks occurring randomly in time according to a **homogeneous Poisson process**. Abdel-Hameed and Proschan (1972) extend their results by assuming that shocks occur according to a **nonhomogeneous Poisson process**. In the present paper, we treat the more general case in which shocks occur according to a **nonstationary pure birth process** of the following sort: Shocks occur according to a Markov process; given that $k$ shocks have occurred in $[0, t]$, the probability of a shock occurring in $(t, t+\Delta]$ is $\lambda_k \lambda(t) \Delta + o(\Delta)$, while the probability of more than one shock occurring in $(t, t+\Delta] = o(\Delta)$. We shall refer to this as The Shock Model of Section I.

**Remark 1.1.** Note that in the stationary pure birth process, given that $k$ shocks have occurred in $[0, \Lambda(t)]$, the probability of a shock occurring in $[\Lambda(t), \Lambda(t) + \lambda(t) \Delta]$ (where $\Lambda(t) = \int_0^t \lambda(x) \, dx$) is of the same form: $\lambda_k \lambda(t) \Delta + o(\Delta)$. It follows immediately that the Shock Model of Section I may be obtained from the stationary pure birth process by the transformation $t \rightarrow \Lambda(t)$.

Let $\overline{P}_k$ denote the probability that the device survives $k$ shocks, and let $p_k = \overline{P}_{k-1} - \overline{P}_k$ denote the probability that the device fails on the $k^{th}$ shock. We assume throughout that the domain of $\overline{P}_k$ is $\{0, 1, \ldots\}$ and that $1 = \overline{P}_0 \geq \overline{P}_1 \geq \ldots$. Let $s_k(t)$ be the probability that exactly $k$ shocks occur in $[0, t]$. Then the probability $\overline{H}(t)$ that the device survives until time $t$ may be expressed as
\( \bar{H}(t) = \sum_{k=0}^{\infty} s_k(t) \bar{P}_k \), \hspace{1cm} (1.1)

with density \( h(t) \) given by
\[ h(t) = \sum_{k=0}^{\infty} s_k(t) \lambda_k \lambda(t) \rho_{k+1} \]. \hspace{1cm} (1.2)

Throughout, the domain of the time argument \( t \) of \( s_k, \bar{H}, \) and \( h \) is, of course, \([0, \infty)\).

Our basic interest in this paper is to show that each of the fundamental classes of life distributions treated in Abdel-Hameed and Proschan (1972) is "preserved" under the transformation (1.1) under appropriate assumptions on \( \lambda_k \) and \( \lambda(t) \). We thus extend the results of Abdel-Hameed and Proschan (1972) for a nonhomogeneous Poisson process to the present Shock Model of Section I.

The fundamental classes of life distributions are defined and discussed in Esary, Marshall, and Proschan (1973) and Abdel-Hameed and Proschan (1972). For ease of reference we give succinct definitions:

A life distribution \( F \) or survival function \( \bar{F} \) is said to be, or to have:

(i) a PF2 density if \( F \) has a density \( f \) which is log concave.

(ii) increasing failure rate (IFR) if \( \bar{F} \) is log concave.

(iii) decreasing mean residual life (DMRL) if \( \int_0^\infty \frac{\bar{F}(t+x)}{\bar{F}(t)} \, dx \downarrow \text{ in } t. \)

(iv) increasing failure rate average (IFRA) if \( -\log \bar{F}(t) \) is starshaped. (A nonnegative function \( u(x) \) defined on \([0, \infty)\) is said to be starshaped if \( \frac{1}{x} u(x) \uparrow \), or equivalently, \( u(ax) \leq \alpha u(x) \) for \( 0 \leq \alpha \leq 1, x \geq 0 \).)
(v) new better than used (NBU) if \( \frac{\overline{F}(x)}{\overline{F}(t+x)} \geq \overline{F}(t) \) for all \( x, t \geq 0 \).

(vi) new better than used in expectation (NBUE) if
\[
\int_0^\infty \frac{F(x)}{\overline{F}(t)} \, dx \geq \int_0^\infty \frac{\overline{F}(t+x)}{\overline{F}(t)} \, dx \quad \text{for all } t \geq 0.
\]

By reversing the direction of monotonicity or inequality we may obtain dual classes of life distributions. Thus \( \overline{F} \) has decreasing failure rate (DFR) if \( \overline{F} \) is log convex, etc. Also we may obtain discrete versions of (i) through (vi). Thus a survival probability \( \overline{F}_k \) with support on \( \{0, 1, 2, \ldots \} \) has increasing failure rate if \( \log \overline{F}_k \) is concave, etc. More details may be found in Esary, Marshall, and Proschan (1973) and in Abdel-Hameed and Proschan (1972).

As key tools in deriving the results below, we use the methods of total positivity, and in particular the variation diminishing property of totally positive (TP) functions. See Karlin (1968), p. 21.
2. **Preservation of Life Distribution Classes.**

In this section we show that the life distribution classes defined in Section 1 are preserved under the transformation (1.1) in the sense that if the shock survival probability \( \{ P_k \} \) (appropriately modified, in some cases) belongs to a discrete version of one of the life distribution classes (i)-(vi), or the classes obtained by reversing the directions of inequality, then the continuous time survival probability \( R(t) \) belongs to the class. Of course, appropriate assumptions on \( \lambda_k \) and \( \lambda(t) \) are needed.

We shall use the following two lemmas concerning the composition of functions.

2.1. **Lemma.** Let \( u(t) = u_1(u_2(t)) \) and let \( u_1 \) be \( t \). Then:

(a) \( u_i \) convex (concave), \( i = 1,2 \) \( \Rightarrow \) \( u \) convex (concave).

(b1) \( u_i \) starshaped, \( i = 1,2 \) \( \Rightarrow \) \( u \) starshaped.

(b2) \( u_i(\alpha x) \geq \alpha u_i(x) \) for all \( 0 \leq \alpha \leq 1 \) and \( x \geq 0 \), \( i = 1,2 \) \( \Rightarrow \) \( u(\alpha x) \geq \alpha u(x) \) for all \( 0 \leq \alpha \leq 1 \) and \( x \geq 0 \).

(c) \( u_i \) superadditive (subadditive), \( i = 1,2 \) \( \Rightarrow \) \( u \) superadditive (subadditive).

The proof is straightforward, and hence is omitted. For the proof of the next lemma, we use the conventions: (1) \( \text{sgn } A = -1, 0, \) or \(+1\) depending on whether \( A < 0, = 0, \) or \( > 0 \). (2) \( A \overset{\text{sgn}}{=} B \) means \( \text{sgn } A = \text{sgn } B \). (3) \( A \overset{\text{sgn}}{<} B \) means \( \text{sgn } A < \text{sgn } B \).
2.2. Lemma. \((a)\) Let \(F\) be DMRL(IMRL) and \(g\) be a convex
(concave) \(\dagger\) function. Then \(K(t) = F(g(t))\) is DMRL(IMRL).

\((b)\) Let \(F\) be NBUE(NWUE) and \(g\) be a starshaped \(\dagger\) function
\((g \text{ be an increasing function such that } g(ax) \geq \alpha g(x), 0 \leq \alpha \leq 1)\).
Then \(K(t) = F(g(t))\) is NBUE(NWUE).

Proof. \((a)\) First assume \(F\) is DMRL with density \(f\). Then
\[
\frac{d}{dt} \left[ \int_t^\infty \frac{R}{R(t)} \right] \text{sgn} = -\frac{R^2(t)}{R(t)} + \frac{dk}{dt} \int_t^\infty \frac{R}{R(t)}
\]
\[= -\left[ F(g(t)) \right]^2 + f(g(t)) \left( g'(t) \int_t^\infty F(g(x))dx \right)
\]
\[\leq -\left[ F(g(t)) \right]^2 + f(g(t)) \left( \int_t^\infty F(g(x)) dx \right) \cdot g'(x)dx, \text{ since } g' \dagger.
\]
Let \(y = g(x)\) in the integral. Then
\[
\frac{d}{dt} \left[ \int_t^\infty \frac{R}{R(t)} \right] \text{sgn} \leq -\left[ F(g(t)) \right]^2 + f(g(t)) \int_t^\infty F(y)dy \leq 0,
\]
since \(F\) is DMRL. It follows that \(K\) is DMRL.

In case \(F\) is DMRL without a density, we obtain the same
conclusion by limiting arguments.

For \(F\) IMRL, the desired result can be obtained by similar
arguments, simply reversing inequalities.

\((b)\). Write
\[
K(t) \int_0^\infty \frac{R}{R(t)} - \int_0^\infty \frac{R}{R(t)} = K(t) \int_0^\infty \frac{R}{R(t)} - K(t) \int_0^\infty \frac{R}{R(t)}
\]
\[= F(g(t)) \int_0^\infty \frac{R(x)}{g(x)}dx - F(g(t)) \int_0^\infty \frac{R(x)}{g(t)}dx
\]
\[\geq F(g(t)) \int_0^\infty \frac{g(t)}{g(t)} \int_0^\infty F(g(t)) \int_0^\infty \frac{g(t)}{g(t)} \int_0^\infty \frac{R(x)}{g(t)}dx,
\]
since \(g\) is starshaped. Let \(y = \frac{g(t)}{g'(t)} x\). The last expression becomes
\[ \mathbb{E}(g(t)) \text{sgn}(g(t)) \int_0^\infty f(y) \, dy - \mathbb{E}(g(t)) \int_0^\infty \frac{f(y) \, dy}{g(t)} \]

where \( F \geq 0 \), since \( F \) is NBUE. It follows that \( K \) is NBUE.

For \( F \) NWUE, the desired result can be obtained by similar arguments, simply reversing inequalities. \( \| \)

We may now give sufficient conditions for the preservation of the PF_2 property.

2.3 Theorem. Let \( \lambda_k^\uparrow, \lambda_{k+1}^\downarrow \) and PF_2, and \( \lambda(t)^\uparrow \).

(a) Then \( h(t) / \lambda(t) \) is PF_2. (b) If, in addition, \( \lambda(t) \) is log concave, then \( h(t) \) is PF_2.

Proof. (a) By Remark 1.1, we may write

\[ h(t) = \sum_{k=0}^\infty z_k(\Lambda(t)) \lambda_k \lambda(t) p_{k+1} , \]

where \( z_k(t) \) denotes the probability of \( k \) shocks in \([0,t] \) when \( \lambda(t) \equiv 1 \). Thus \( u(t) \) def. \( \frac{h(\Lambda^{-1}(t))}{\lambda(\Lambda^{-1}(t))} = \sum_{k=0}^\infty z_k(t) \lambda_k p_{k+1} \).

By Karlin (1966), p. 178,

\[ z'_k(t) = - \lambda_k z_k(t) + \lambda_{k-1} z_{k-1}(t), k \geq 1, \text{and } z'_o(t) = -\lambda_o z_o(t). \]

\[
\begin{vmatrix}
& \frac{z_{k_1}(t_1)}{z_{k_2}(t_1)} & \frac{z_{k_1}(t_2)}{z_{k_2}(t_2)} \\
\frac{u'(t_1)}{u'(t_2)} & \frac{z_{k_1}(t_1)}{z_{k_2}(t_1)} & \frac{z_{k_1}(t_2)}{z_{k_2}(t_2)} \\
\frac{u(t_1)}{u(t_2)} & \frac{z_{k_1}(t_1)}{z_{k_2}(t_1)} & \frac{z_{k_1}(t_2)}{z_{k_2}(t_2)} \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
\lambda_{k_1} (\lambda_{k_1+1} p_{k_1+2} - \lambda_{k_1} p_{k_1+1}) & \lambda_{k_2} p_{k_1+1} \\
\lambda_{k_2} (\lambda_{k_2+1} p_{k_2+2} - \lambda_{k_2} p_{k_2+1}) & \lambda_{k_2} p_{k_2+1} \\
\end{vmatrix}
\]
Since the intervals between successive shocks in a stationary pure birth process are independent (nonidentical) exponential random variables, it follows that \( z_k(t) \) is TP (Karlin and Proschan, 1960, Theorem 3). Thus, for \( t_1 < t_2 \), the first determinant in the summand is nonnegative.

Since \( \lambda_k p_{k+1} \downarrow \) and \( \lambda_k \uparrow \), the second determinant in
the summand is

\[
\begin{vmatrix}
\lambda_{k_1+1} p_{k_1+2} & \lambda_{k_1} p_{k_1+1} \\
\lambda_{k_2+1} p_{k_2+2} & \lambda_{k_2} p_{k_2+1}
\end{vmatrix}
= \begin{vmatrix}
\lambda_{k_1+1} p_{k_1+2} & \lambda_{k_1} p_{k_1+1} \\
\lambda_{k_2+1} p_{k_2+2} & \lambda_{k_2} p_{k_2+1}
\end{vmatrix} \geq 0,
\]

since \( \lambda_k p_{k+1} \) is PF2.

Thus \( u(t) \) is log concave. By differentiating \( u(t) \) and using the hypothesis \( \lambda_k p_{k+1} \downarrow \), we may show \( u(t) \downarrow \). Since \( \Lambda \) is convex and \( u \) is decreasing, it follows by Lemma 2.1(a) that

\[
\frac{h(t)}{\lambda(t)} \quad \text{is log concave.}
\]

(b) follows immediately from the fact that the product of PF2 functions is PF2. \( \|$\)

Remark. Note that if \( \lambda_k \) and \( p_k \) are each PF2, the hypothesis \( \lambda_k p_{k+1} \) PF2 holds.

The following theorem provides sufficient conditions for preservation of the IFR property.
2.4 Theorem. Let \( \lambda_k^1 \), \( \lambda(t) \), \( \lambda_k^2 \), and \( \overline{P}_k \) be IFR.

Then \( \overline{H}(t) \) is IFR.

Proof. Let \( z_k(t) \) be defined as in Theorem 2.3, and let

\[
\overline{H}_1(t) = \sum_{k=0}^{\infty} z_k(t) \overline{P}_k.
\]  

(2.1)

Then by Remark 1.1, we may write

\[
\overline{H}(t) = \overline{H}_1(\Lambda(t)).
\]  

(2.2)

By Lemma 2.1(a), it suffices to show that \( \overline{H}_1 \) is IFR.

To this end, write

\[
D = \begin{vmatrix} h_1(t_1) & \overline{H}_1(t_1) \\ h_1(t_2) & \overline{H}_1(t_2) \end{vmatrix} = \begin{vmatrix} \sum_{k=0}^{\infty} z_k(t_1) \lambda_{k+1} \overline{P}_{k+1} & \sum_{k=0}^{\infty} z_k(t_1) \overline{P}_k \\ \sum_{k=0}^{\infty} z_k(t_2) \lambda_{k+1} \overline{P}_{k+1} & \sum_{k=0}^{\infty} z_k(t_2) \overline{P}_k \end{vmatrix}
\]

\[
= \sum_{k_1 < k_2} \begin{vmatrix} z_{k_1}(t_1) & z_{k_2}(t_1) \\ z_{k_1}(t_2) & z_{k_2}(t_2) \end{vmatrix} \begin{vmatrix} \lambda_{k_1} \overline{P}_{k_1+1} & \overline{P}_{k_1} \\ \lambda_{k_2} \overline{P}_{k_2+1} & \overline{P}_{k_2} \end{vmatrix},
\]

by the basic composition theorem (Karlin, 1968, p.17). For \( t_1 < t_2 \), the first determinant is nonnegative by Theorem 3 of Karlin and Proschan (1960). The second determinant is nonpositive since \( \lambda_k^1 \leq \lambda_k^2 \) and \( \overline{P}_k \) is IFR. It follows that \( \overline{H}_1 \) is IFR. ||

To obtain sufficient conditions for the preservation of the IFRA property, we shall find it helpful to use the following lemma.
2.5. Lemma. Let \( f(0) = 0, f(t) \uparrow \). For each \( 0 \leq \beta < \infty \), let there exist a starshaped function \( g_{\beta}(t) \) satisfying:

\[
    g_{\beta}(t) - f(t) \geq 0 \quad \text{for} \quad 0 \leq t \leq \beta \\
    \leq 0 \quad \text{for} \quad \beta \leq t < \infty.
\]

(2.3)

Then \( f \) is starshaped.

Proof. Since \( g_{\beta}(t) \) is starshaped, then for \( 0 \leq t \leq \beta \),

\[
    \frac{t g_{\beta}(\beta)}{\beta} \geq g_{\beta}(t) \geq f(t),
\]

whereas for \( \beta \leq t < \infty \),

\[
    \frac{t g_{\beta}(\beta)}{\beta} \leq g_{\beta}(t) \leq f(t).
\]

Thus for each \( \beta > 0 \), \( f(t) \) lies below the line \( \frac{g_{\beta}(\beta)}{\beta} t \) for \( 0 \leq t \leq \beta \) and above this line for \( \beta \leq t < \infty \).

It follows that \( f(\alpha t) \leq \alpha f(t) \) for \( 0 \leq \alpha < 1, \ t > 0 \). Thus \( f \) is starshaped. \( \square \)

We may now state and prove the result for preservation of the IFRA property.

2.6. Theorem. Let \( \lambda_k \uparrow, \Lambda(t) \) be starshaped, and \( \mathbf{P}_k \) be IFRA. Then \( \mathbf{H}(t) \) is IFRA.

Proof. By Lemma 2.1 (bl), it suffices to show that \( \mathbf{H}_1 \) defined in (2.1) is IFRA. To this end, let

\[
    \mathbf{H}_p(t) = \sum_{k=0}^{\infty} z_{k}(t) p^k \quad \text{for} \quad 0 \leq p \leq 1,
\]

(2.4)

an IFR survival probability by Theorem 2.4. Then \( \mathbf{P}_k \) IFRA implies \( \mathbf{P}_k - p^k \) changes sign at most once, and if once, from + to - for each fixed \( p, 0 \leq p \leq 1 \). Using the variation diminishing property of the totally positive kernel \( z_{k}(t) \), it follows that
\[
\overline{H}_1(t) - \overline{H}_p(t) = \sum_{k=0}^\infty z_k(t) (\overline{P}_k - p_k)
\]

changes sign at most once (in \(t\)), and if once, in the order \(+, -\). Since the log function is monotone, we see that

\(- \log \overline{H}_p(t) - \big[ - \log \overline{H}_1(t) \big]\) changes sign at most once, and if once, in the order \(+, -\). Moreover, given \(0 \leq \alpha < \infty\), by varying \(p\) between \(0\) and \(1\), \(- \log \overline{H}_p(\alpha) - \big[ - \log \overline{H}_1(\alpha) \big]\) varies continuously between \(\lambda_\alpha \) (a positive quantity) and \(- \log \overline{H}_1(\alpha)\) (a negative quantity). Thus there exists \(p_\alpha\) such that \(- \log \overline{H}_{p_\alpha}(\alpha) - \big[ - \log \overline{H}_1(\alpha) \big] = 0\).

Applying Lemma 2.5, we conclude that \(\overline{H}_1\) is IFRA. Thus \(\overline{H}\) is IFRA.

We first obtain sufficient conditions for preservation of the new better than used property in a model in which shocks occur according to a generalized renewal process:

2.7, Theorem. Let the interval between the \(k-1^{st}\) and \(k^{th}\) shock be governed by NBU distribution \(F_k\), with \(F_k(t) \uparrow\) in \(k\) for each \(t \geq 0\), and all intervals mutually independent. Let \(\overline{F}_k\) be NBU. Then survival probability

\[
\overline{H}_2(t) = \sum_{k=0}^\infty P[N(t) = k] \overline{P}_k
\]

is NBU, where \(N(t)\) denotes the number of shocks in \([0, t]\).
Proof. \[ \bar{H}(t_1+t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P[N(t_1) = k_1] \prod_{k_1+k_2} \]

\[ \cdot P[N(t_1+t_2) - N(t_1) = k_2 | N(t_1) = k_1] \prod_{k_1+k_2} \]

\[ \leq \sum_{k_1} \sum_{k_2} P[N(t_1) = k_1] P[N(t_1+t_2) - N(t_1) = k_2 | N(t_1) = k_1] \prod_{k_1} P_{k_1} \prod_{k_2} P_{k_2} \] [since \( P_k \) is NBU]

\[ = \sum_{k_1} \sum_{k_2} P[N(t_1) = k_1] P[N(t_1+t_2) - N(t_1) < k_2 | N(t_1) = k_1] \prod_{k_1} P_{k_1} \prod_{k_2} P_{k_2} \]

\[ \leq \sum_{k_1} \sum_{k_2} P[N(t_1) = k_1] \prod_{k_1} P_{k_1} \sum_{k_2} P[N(t_2) < k_2] P_{k_2} \]

[since \([k_1, k_2]\) is NBU]

\[ \leq \sum_{k_1} \{ \sum_{k_2} P[N(t_1) = k_1] \prod_{k_1} P_{k_1} \} P[N(t_2) < k_2] P_{k_2} \]

\[ = \sum_{k_1} \sum_{k_2} P[N(t_1) = k_1] \prod_{k_1} P_{k_1} \prod_{k_2} P_{k_2} \]

and last shock occurred at \( t_1 \) \( P_{k_2} \) [since intervals between shocks are governed by NBU distributions]

\[ \leq \sum_{k_2} \{ \sum_{k_1} P[N(t_1) = k_1] \prod_{k_1} P_{k_1} \} P[N(t_2) < k_2] P_{k_2} \]

[since \( P_{k_1} \) in \( k_1 \) = \( \bar{H}(t_1) \bar{H}(t_2) \).]}

As an immediate consequence of Theorem 2.7, we obtain NBU preservation results for The Shock Model of Section 1.

**2.8 Theorem.** Let \( \lambda_k \) \( \uparrow \), \( \Lambda(t) \) be superadditive, and \( P_k \) be NBU. Then \( \bar{H}(t) \) is NBU.

Proof. Simply use Lemma 2.1(c) and Theorem 2.7.}

To develop sufficient conditions for the preservation of the DMRL and NBUE properties, we shall need the following lemma:
2.9 Lemma. Let $z_k(t)$ be as defined in Theorem 2.3. Then

(a) $\int_0^t z_k(u) \, du = \frac{1}{\lambda_k} \sum_{j=k+1}^{\infty} z_j(t),$

(b) $\int_0^\infty z_k(u) \, du = \frac{1}{\lambda_k} \sum_{j=0}^{k} z_j(t).$

Proof. (a) We proceed by induction. First note that

\[ \int_0^t z_0(u) \, du = \int_0^t e^{-\lambda_0 u} \, du = \frac{1}{\lambda_0} (1 - e^{-\lambda_0 t}). \]

Thus the desired conclusion holds for $k = 0.$

Next assume the result holds for $k = k_0 - 1 \geq 1.$ Since

$z_{k_0}(t) = \lambda_{k_0-1} z_{k_0-1}(t) - \lambda_{k_0} z_{k_0}(t)$ [Karlin, 1966, p. 178],

then by integrating, we have

\[ z_{k_0}(t) = \lambda_{k_0-1} \int_0^t z_{k_0-1}(u) \, du - \lambda_{k_0} \int_0^t z_{k_0}(u) \, du. \]

Thus

\[ \int_0^t z_{k_0}(u) \, du = \frac{1}{\lambda_{k_0}} \left[ \lambda_{k_0-1} \int_0^t z_{k_0-1}(u) \, du - z_k(t) \right] \]

\[ = \frac{1}{\lambda_{k_0}} \left[ \sum_{j=k_0}^{\infty} z_j(t) - z_k(t) \right] \] [by inductive hypothesis]

\[ = \frac{1}{\lambda_{k_0}} \left[ \sum_{j=k_0+1}^{\infty} z_j(t) \right]. \]

Thus the desired conclusion holds for $k = k_0.$
(b) The result follows from (a) in conjunction with
the identity \( \int_0^\infty z_k(u)du = \frac{1}{\lambda_k} \).

We may now obtain preservation results in the DMRL and
NBUE cases.

2.10 Theorem. Let \( \lambda(t) \uparrow \) and \( \frac{1}{\bar{P}_k} \sum_{j=k}^\infty (\bar{P}_j/\lambda_j) \uparrow \).
Then \( \bar{H}(t) \) is DMRL.

Proof. By Lemma 2.2(a), it suffices to prove that \( \bar{H} \),
is DMRL. By Lemma 2.9(b), for any real value \( c \), we have

\[
\int_0^\infty \bar{H}_1(u)du - c\bar{H}_1(t) = \sum_{k=0}^\infty \frac{1}{\lambda_k} \bar{P}_k \sum_{j=0}^k z_j(t) - c \sum_{j=0}^\infty z_j(t) \bar{P}_j
\]

\[
= \sum_{j=0}^\infty z_j(t) \left[ \sum_{k=j}^\infty \frac{\bar{P}_k}{\lambda_k} - c\bar{P}_j \right].
\]

By hypothesis, \( \sum_{k=j}^\infty \frac{\bar{P}_k}{\lambda_k} - c\bar{P}_j \) has at most one change
of sign, and if one, from + to -. By the variation diminishing
property, it follows that the same statement may be made for

\[
\int_0^\infty \bar{H}_1(u)du - c\bar{H}_1(t). \text{ Thus } \int_0^\infty \bar{H}_1/\bar{H}_1(t) \uparrow.\]

2.11. Theorem. Let \( \Lambda(t) \) be starshaped and

\[
\sum_{j=0}^\infty (\bar{P}_j/\lambda_j) \geq \frac{1}{\bar{P}_k} \sum_{j=k}^\infty (\bar{P}_j/\lambda_j) \text{ for } k = 0, 1, 2, \ldots .
\]

Then \( \bar{H} \) is NBUE.
Proof. By Lemma 2.2(b), it suffices to show that $\bar{H}_1$ is NBUE. We may write

$$
\bar{H}_1(t) \int_0^\infty \bar{H}_1(u) du = \sum_{k=0}^\infty z_k(t) \prod_{j=0}^k \left( \frac{\bar{P}_j}{\lambda_j} \right)
$$

$$
\geq \sum_{k=0}^\infty z_k(t) \prod_{j=k}^\infty \left( \frac{\bar{P}_j}{\lambda_j} \right) \text{ [by hypothesis]}
$$

$$
= \sum_{j=0}^\infty \frac{\bar{P}_j}{\lambda_j} \sum_{k=0}^j z_k(t) = \int_0^\infty \bar{H}_1(u) du \text{ [by Lemma 2.9b].}
$$

Thus $\bar{H}_1$ is NBUE. ||

For each of the theorems of this section concerning the preservation of life distribution classes, a dual theorem exists for the preservation of the dual life distribution class. For example, the dual of Theorem 2.4 states:

2.4'. Theorem. Let $\lambda_k \uparrow$, $\lambda(t) \uparrow$, and $\bar{P}_k$ be DFR. Then $\bar{H}(t)$ is DFR.

In each case, the dual theorem is obtained from the theorem by reversing the direction of monotonicity, and/or inequality.
REFERENCES.


Shock Models with Underlying Birth Process

Technical Report, May 1973

Mohamed Abdel-Hameed
Frank Proschan

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This paper extends results of Esary, Marshall, and Proschan (1973), Ann. of Probability, and Abdel-Hameed and Proschan (1972), FSU Statistics Report M243. We consider the life distribution of a device subject to a sequence of shocks occurring randomly in time according to a nonstationary pure birth process; given that $k$ shocks have occurred in $[0,t]$, the probability of a shock occurring in $[t,t+\Delta]$ is $\lambda_k \lambda(t)\Delta + o(\Delta)$. We show that various fundamental classes of life distributions (such as those with increasing failure rate, or those with the "new better than used" property, etc.) are obtained under appropriate assumptions on $\lambda_k$, $\lambda(t)$, and on the probability of surviving a given number of shocks.