A TEST FOR SUPERADDITIVITY OF THE
MEAN VALUE FUNCTION OF A NON-HOMOGENEOUS
POISSON PROCESS

by

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Abstract

Let $N(t)$ be a non-homogeneous Poisson process with mean value function $\Lambda(t)$ and rate of occurrence $\lambda(t)$. We propose a conditional test of the hypothesis that the process is homogeneous, versus alternatives for which the mean value function is superadditive. Specific models leading to superadditivity are presented, and the superadditive test is compared, on the basis of consistency and asymptotic relative efficiency, with the Cox-Lewis test; the latter being directed to alternatives where $\lambda(t)$ is increasing.
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1. Introduction. In many applications of the Poisson process, it is important to determine whether the associated mean value function is linear (corresponding to a homogeneous Poisson process) or not. The usual alternative is that the mean value function is convex; i.e., that the event rate of the process is increasing.

In this paper, we consider an alternative hypothesis which arises in a number of applications, but which, as far as we know, has not been treated before. The alternative states, roughly speaking, that the expected number of events in any initial interval (i.e., of the form [0, t]) is no greater than the expected number of events in any interval of the same length occurring later (i.e., of the form [x, x + t]).

Specifically, let \{N(t), t \geq 0\} be a non-homogeneous Poisson process with mean value function \( \Lambda(t) = F'\{t\} \). Assume \( \Lambda(t) \) is differentiable with rate of occurrence \( \lambda(t) = (d/dt)\Lambda(t) \).

We observe the process on \([0, t^*]\) and consider a testing problem.

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where the null hypothesis $H_0$ asserts that the process is in fact a homogeneous Poisson process. We propose a test of

(1) $H_0: \lambda(t) \equiv \lambda, \ 0 \leq t \leq t^*, \ (\lambda \ unknown)$,

versus

(2) $H_1: \Lambda(t_1) + \Lambda(t_2) \leq \Lambda(t_1 + t_2), \ 0 \leq t_1 + t_2 \leq t^*$,

where in (2) the inequality is presumed strict for at least one $(t_1, t_2)$. The $H_1$ alternatives specify that, in the interval $[0, t^*]$, $\Lambda(t)$ is a superadditive function. When the inequality in (2) is reversed, $\Lambda(t)$ is said to be a subadditive function.

Section 2 contains two models which illustrate the importance of the $H_1$ alternatives and serve as partial motivation for our testing problem and proposed test. (Additional superadditive alternatives are introduced in Section 4.) The test itself, which is conditional on $\mathbb{N}(t^*) = n$, is defined in Section 3. Here we also provide a null distribution table and large sample normal approximation for ease in implementing the test. Section 4 presents comparisons of our test with the Cox-Lewis [Cox (1955), Cox and Lewis (1966, Chapter 3)] conditional likelihood ratio test of $H_0$ derived for the alternatives $\lambda(t) = \exp(a + \beta t)$, where $H_0$ corresponds to $\beta = 0$. The comparisons are on the basis of consistency and asymptotic relative efficiency. We provide one class of superadditive alternatives, corresponding to rates of occurrence $\lambda(t)$ which vary and in particular are not non-decreasing, for which the proposed test of superadditivity is consistent while the
Cox-Lewis test is not consistent. Proofs, of various underscored assertions that appear in the body of the text, are given in the Appendix.

2. Models that lead to superadditivity.

(i) A device subject to overhaul

Consider a device which is overhauled at fixed times \( t_0, 2t_0, \ldots \). After overhaul at time \( it_0 \), the device has failure rate \( \lambda(t) = \lambda_i(x) \) at time \( t = it_0 + x \), \( 0 \leq x < t_0 \). Since the device deteriorates with increasing age and increasing number of overhauls, we assume

(i) \( \lambda_i(x) \) is increasing in \( x \), \( 0 \leq x < t_0 \), for fixed \( i \),

(ii) \( \lambda_i(x) \) is increasing in \( i \), \( i = 0,1,2,\ldots \), for fixed \( x \), and

(iii) \( \lambda_{i+1}(0) < \lambda_i(t_0^-) \), i.e., overhaul reduces the failure rate.

If a part in the device fails at any time \( t \), the failed part is immediately fixed or replaced. The failure rate \( \lambda(t) \) of the device is then that of a functioning device at time \( t \). That is, failure and repair of a part have negligible effects on the overall failure rate of the device (since it consists of a great many parts).

For this overhaul model, \( \Lambda(t) = \int_0^t \lambda(u)\,du \) is superadditive.

A proof of this assertion is to be found in part Al of the Appendix.

A graph of a typical function \( \Lambda(t) \) for this case is given in Figure 1.
FIG. 1. A superadditive function corresponding to the overhaul model
Note that by (iii), $\lambda(t)$ is not non-decreasing. Hence a test of $H_0$ versus $H^*_1$ "$\lambda(t)$ non-decreasing", such as the one proposed by Boswell (1966) (see also Boswell and Brunk, 1969), is not particularly appropriate for this model.

The testing problem $H_0$ versus $H^*_1$ can be viewed as a Poisson process analogue of the IFR testing problem considered by many authors; cf. Barlow, Bartholomew, Bremner, and Brunk (1972, Chapter 6) for procedures and references. In the next subsection (Sampling with replacement) we show how the testing problem $H_0$ versus $H_1$ can be viewed as a Poisson process analogue of the NBU testing problem considered by Hollander and Proschan (1972) (also see Hollander and Wolfe, 1973, Section 10.4).

(ii) Sampling with replacement

The following life testing program is particularly useful when the underlying distribution is exponential. At time 0, $N$ items are placed on test. Failed items are immediately replaced by new items. The successive failure times observed are $\tau(1) < \ldots < \tau(n)$, where $n$ may be $<$, $=$, or $>$ than $N$. The experiment ends at fixed time $t^*$. The starting time for the item that failed at time $\tau(i)$ is not known, $i = 1, \ldots, n$.

Let $F$ denote the (assumed common) underlying life distribution of each item. $F$ is said to be new better than used (NBU) if
(3) \[ \overline{F}(t_1 + t_2) \leq \overline{F}(t_1)\overline{F}(t_2), \text{ for all } t_1, t_2 \geq 0, \]
where \( \overline{F}(t) = 1 - F(t) \). Inequality (3) states that the probability \( \overline{F}(t_1) \) that a new item will survive to age \( t_1 \) is greater than the conditional probability \( \overline{F}(t_1 + t_2)/\overline{F}(t_2) \) that an unfailed item of age \( t_2 \) will survive an additional time \( t_1 \), for all \( t_1, t_2 \geq 0 \).
That is, a new item has stochastically greater life than a used item of any age. The NBU class was systematically studied by Marshall and Proschan (1972) in their analysis of replacement policies, and also plays an important role in shock models (Esary, Marshall, and Proschan, 1973). Hollander and Proschan (1972) devised a test of \( H_a \): equality in (3), i.e. \( F \) is exponential, versus \( H_b : F \) is NBU (and not exponential). Their test utilizes uncensored life lengths as the basic data.

Suppose that we wish to test \( H_a \) versus \( H_b \) on the basis of the failure times observed in this sampling with replacement situation. Note that \( F(t) = 1 - e^{-\lambda t} \), for some \( \lambda > 0 \), implies that \( \{N(t), t \geq 0\} \) is a homogeneous Poisson process with rate of occurrence \( \lambda(t) = N \lambda \), where \( N(t) = \) the number of failures during \( [0, t] \). On the other hand, \( F \) _NBU_ implies that

\[
(4) \quad \text{s.t. } N(t_1) \leq N(t_1 + t_2) - N(t_2), \text{ for all } t_1, t_2 \geq 0.
\]

A proof of this assertion can be found in part A2 of the Appendix.

Note that when (4) holds, \( H_1 \) (2) is true. If more complete information were available, that is, the starting time \( s_i \) (say) of an item that failed at time \( \tau_{(i)}, i = 1, \ldots, n \), we could test \( H_a \).
versus $H_b$ with the NBU test statistic $J_n$ proposed in equation (1.5) of Hollander and Proschan (1972). The statistic $J_n$ is based on the life lengths $X(i) = \tau(i) - s_i$. Since these data are not available in the situation described, we could instead use the $Q_n$ test, proposed in Section 3, that is based on the times of failure.

3. The superadditive test. Let

$$\theta(\Lambda) = \int \int \{G(t_1 + G(t_2) - G(t_1 + t_2)) \ dG(t_1)dG(t_2),$$

where

$$t_1 + t_2 \leq t^*$$

(5)

where

$$G(t) = \begin{cases} 0, & t < 0 \\ \Lambda(t)/\Lambda(t^*), & 0 \leq t \leq t^* \\ 1, & t > t^*. \end{cases}$$

(6)

Consider the following U-statistic corresponding to $\theta(\Lambda)$. Define

$$Q_n = (n(n-1)(n-2))^{-1} \sum [2\phi(\tau_{a_3}, \tau_{a_1}) - \phi(\tau_{a_3}, \tau_{a_1} + \tau_{a_2}) \phi(\tau_{a_1} + \tau_{a_2}, t^*)],$$

where $\tau_1, \ldots, \tau_n$ are independent and identically distributed according to $G$, the $\sum$ is over all $n(n-1)(n-2)$ permutations $(a_1, a_2, a_3)$ of integers $1, 2, \ldots, n$ with $a_1 \neq a_2, a_1 \neq a_3, a_2 \neq a_3$, and

$$\phi(a, b) = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise.} \end{cases}$$

(7)
Straightforward algebra shows that $Q_n$ can be written in a form more convenient for computation and tabulation, namely,

$$Q_n = 2K_n / \{n(n-1)(n-2)\},$$

where

$$K_n = \sum [\phi(\tau(a_3) + \tau(a_2), t^*) - \{\phi(\tau(a_3), \tau(a_1) + \tau(a_2))\phi(\tau(a_1) + \tau(a_2), t^*)\}],$$

$\tau(1) < \tau(2) < \ldots < \tau(n)$ are the ordered $\tau$'s (we associate the $\tau(i)$'s with the times at which events occur in $[0, t^*)$, and the $\sum$ is over all $n(n-1)(n-2)/6$ choices of subscripts such that $1 \leq a_1 < a_2 < a_3 \leq n$.

Our test statistic $Q_n$ is motivated as follows. Given $N(t^*) = n$, $\tau(1), \ldots, \tau(n)$ have the same distribution as the order statistics in a random sample of size $n$ from the distribution $G$ defined by (6) [cf. Parzen (1962, p. 143)], and it follows that $E(Q_n) = \beta(\Lambda)$. Thus $Q_n$ can be viewed as an estimator of $\beta(\Lambda)$.

When $H_0$ is true, $G(t)$ is the uniform distribution on $[0, t^*)$ and $\beta(\Lambda) = 0$. When $H_1$ obtains, $\beta(\Lambda)$ will be negative and thus large negative values of $Q_n$ yield an appropriate critical region.

Consider the test which rejects $H_0$ if $Q_n \leq q_n, \alpha$, where $q_n, \alpha$ satisfies $P_0 \{Q_n \leq q_n, \alpha\} = \alpha$. In the sequel this test is called the superadditive test. (Here $P_0$ indicates that the probability is conditional on $N(t^*) = n$ and $H_0$ being true. Note that the value of $Q_n$ is invariant under the transformation $\tau'(i) = \tau(i)/t^*$, $i = 1, \ldots, n, t^* = 1$. Hence, without loss of generality, we can take
t* = 1 and $P_o$ calculations can be made by taking $G$ to be uniform on $[0, 1]$.

Table 1, based on Monte Carlo sampling (10,000 replications), gives upper and lower critical points of $K_n$ in the $\alpha = .01, .025, .075,$ and $.10$ regions for $n = 4(1)20(5)35$. The lower tail is to be used for tests of $H_0$ versus the $H_1$ superadditive alternatives; the upper tail is to be used for tests of $H_0$ versus subadditive alternatives. Upper tail entries $C_\alpha$ are such that $P_o \{K_n \geq C_\alpha \} \leq \alpha$ and lower tail entries are such that $P_o \{K_n \leq C_\alpha \} \leq \alpha$. 
### TABLE 1

Critical values of the \( K_n \) statistic \( n = 4(1)20(5)35 \)

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For large \( n \) values not covered by Table 1, we can turn to a normal approximation. Note that as \( t^* \to \infty \), the number \( n \) of events in \( [0, t^*] \) converges a.s. to \( \infty \). The asymptotic normality of \( Q_n \), suitably normed, follows directly from Hoeffding's (1948) U-statistic theory. Direct calculations yield

\[
\begin{align*}
(11) & \quad E_o(Q_n) = 0, \\
(12) & \quad \text{Var}_o(Q_n) = \binom{n}{3}^{-1} \sum_{c=1}^{\frac{n}{3}} \binom{3}{c} (\binom{n}{3-c}) \xi_c,
\end{align*}
\]

where \( \xi_1 = 1/270, \xi_2 = 1/108, \xi_3 = 1/27 \), and

\[
(13) \quad \lim_{n} n \text{Var}_o(Q_n) = 1/30.
\]

Thus the normal approximation to the superadditive test rejects when \((30n)^{1/2} Q_n \leq -z_\alpha\), where \( z_\alpha \) is the upper \( \alpha \) - percentile point of a standard normal distribution.

To this point, the observation period of the process was assumed to be \([0, t^*]\). If instead, observation ends when a pre-assigned number of events have occurred, the superadditive test can still be applied with \( t^* \) equal to the period of observation, and \( n \) replaced by \( n^* = n - 1 \), the number of events preceding the one at \( t^* \).

4. Comparisons with the Cox-Lewis test. Cox (1955) and Cox and Lewis (1966, p. 45) derived and investigated a conditional test of \( H_o \) designed for the alternatives \( \lambda(t) = \exp \{a + \beta t\} \), with \( \beta > 0 \). The test is conditional on \( N(t^*) = n \), and utilizes the statistic
\( S_n = \frac{1}{i=1} \tau_i \).

If \( \beta > 0 \), the rate \( \lambda(t) \) of occurrence is increasing, \( S_n \) would tend to be large, and thus large values of \( S_n \) form the critical region for the Cox-Lewis test. Critical values and normal approximations are readily available since the distribution of \( S_n \) under \( H_0 \) is that of a sum of \( n \) independent random variables, each being uniform on \([0, t^*]\). The appropriate normal deviate under \( H_0 \) is \( U_n = \{S_n - (nt^* / 2)\} / [t^*(n/12)^{1/2}] \).

The Cox-Lewis test is primarily designed to detect alternatives where \( \lambda(t) \) is increasing. Taking, without loss of generality, \( t^* = 1 \), it is easy to show that the test, which rejects for large values of \( S_n \), will be consistent against those alternatives for which \( E(S_n / n) \) exceeds its null expected value 1/2. Equivalently, the test will be consistent when \( \int_0^1 \lambda(t) dt > 1/2 \), where \( \lambda(t) = 1 - \Lambda(t) \).

Similarly, it can be shown that the superadditive test is consistent against those alternatives for which \( \beta(\Lambda) \) is less than 0. In particular, superadditivity of \( \Lambda \) (with the inequality in (2) strict for at least one \((t_1, t_2)\)) and the continuity of \( \Lambda \) insure that \( \beta(\Lambda) < 0 \) and hence the superadditive test is consistent against the \( H_1 \) alternatives.

We now consider a class \( A \) of mean value functions for which the superadditive test is consistent whereas the Cox-Lewis test is not consistent. Let
A = \{\Lambda: \Lambda(t) \text{ is superadditive on } [0, 1/2]; \; \Lambda(1/2) = 1/2; \}

(15)

\Lambda(t) \leq t \text{ on } [0, 1/2]; \; \Lambda(t) = 1 - \Lambda(1-t) \text{ for } 1/2 \leq t \leq 1).

For \( \Lambda \in A \), \( \Lambda(t) \) is superadditive on \([0, 1]\). A proof of this assertion can be found in part A3 of the Appendix. Thus the superadditive test is consistent against \( \Lambda \)'s that are members of \( A \). Note, however, that

\[
\int_0^{1/2} \Lambda(t) dt = \int_0^{1/2} \Lambda(t) dt + \int_0^{1/2} \Lambda(t) dt = \int_0^{1/2} \Lambda(t) dt + \int_0^{1/2} \Lambda(t) dt = 1/2,
\]

and hence the Cox-Lewis test will not be consistent against members of \( A \). A graph of a typical function \( \Lambda \in A \) is given in Figure 2.
FIG. 2. A superadditive function in the class $A$
Of course, we could also produce classes of mean value functions for which the Cox-Lewis test is consistent and the superadditive test is not consistent. The point of the class $\Lambda$ is to emphasize that the superadditive test is designed for different alternatives than is the Cox-Lewis test. Consider, for example, the mean value function 

$$\Lambda(t) = \begin{cases} 2t^2 & \text{for } 0 \leq t < 1/2, \\ -2t^2 + 4t - 1 & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

It is easily seen that $\Lambda \in \Lambda$. Also $\lambda(t) = \frac{4t}{t}$ for $0 \leq t \leq 1/2, \frac{4}{4} - 4t$ for $1/2 \leq t \leq 1$. Thus the rate of occurrence $\lambda$ is increasing on $[0, 1/2]$ and decreasing on $[1/2, 1]$. Tests such as the Cox-Lewis test and Boswell's (1966) test were designed primarily to detect situations where the rate of occurrence is increasing, whereas our superadditive test is intended to detect, in addition, situations where $H_1$ holds, but where the rate of occurrence may not be increasing due, for example, to seasonal variation. Note that the superadditive test will also be consistent against those $\Lambda$ alternatives for which $\lambda(t)$ is increasing, since then $\Lambda(t)$ will be convex, and thus superadditive. (This implication follows from the definitions of convexity, superadditivity, and the fact that $\Lambda(0) = 0$.)

Another class of superadditive functions, which could arise as a result of seasonal variation, is given by

$$\Lambda(t) = \begin{cases} 0, & 0 \leq t \leq (a+b)/\lambda \\ -a + \lambda t + b \sin ct, & t > (a+b)/\lambda, \end{cases}$$

where $\lambda, a, b, c > 0$, $a \geq 3b$, $\lambda \geq bc$. (These functions are motivated
by, but different from, the mean value functions of a particular class of non-homogeneous Poisson processes considered by Willis (1964).)

From (16), \( \Lambda''(t) = \lambda(t) = \lambda + bc \cos ct \geq 0 \), for \( t > (a+b)/\lambda \), and \( \lambda(t) = 0 \) for \( 0 \leq t \leq (a+b)/\lambda \). Furthermore, for \( t > (a+b)/\lambda \), \( \Lambda''(t) = -bc^2 \sin ct \) which is positive for \( (2n+1)\pi/c < t < (2n+2)\pi/c \), and negative for \( 2n\pi/c < t < (2n+1)\pi/c \), for \( n = 0,1,2,\ldots \). Thus \( \Lambda \) is alternating convex and concave in successive intervals of length \( \pi/c \).

A graph of \( \Lambda(t) \) is given in Figure 3.
FIG. 3 A superadditive function in the class defined by (16)
We next consider the asymptotic relative efficiency of the superadditive test with respect to the Cox-Lewis test for two situations where \( \lambda(t) \) is increasing; situations which could a priori be viewed as favorable to the Cox-Lewis S test. Let \( \Lambda_\sigma \) be a sequence of alternatives with \( \sigma_n \to \sigma_o \), where \( \Lambda_\sigma(t) = \lambda t \), corresponding to \( H_o(1) \). The Pitman asymptotic relative efficiency of the superadditive test with respect to the S test is

\[
e(Q, S) = \lim_{n} \left[ \frac{\text{Var}_o(S_n)}{\text{Var}_o(Q_n)} \right] [\beta'(\sigma_o)/\mu'(\sigma_o)]^2,
\]

where we take, without loss of generality, \( t^* = 1 \), and where

\[
\beta(\sigma) = \frac{1}{(\Lambda_\sigma(1))^{3/2}} \int (\Lambda_\sigma(t_1) + \Lambda_\sigma(t_2) - \Lambda_\sigma(t_1 + t_2)) \lambda_\sigma(t_1)\lambda_\sigma(t_2) dt_1 dt_2, \quad t_1 + t_2 \leq 1
\]

\[
\mu(\sigma) = n(\Lambda_\sigma(1))^{-1/2} \int_0^1 t \lambda_\sigma(t) dt,
\]

are the means of \( Q_n \) and \( S_n \) respectively for the alternative \( \Lambda_\sigma \), where \( \text{Var}_o(S_n) \) is given by (12), \( \text{Var}_o(S_n) = n/12 \), and \( \beta'(\sigma_o)/\mu'(\sigma_o) \) is the derivative of \( \beta(\sigma)/\mu(\sigma) \) with respect to \( \sigma \), evaluated at \( \sigma = \sigma_o \).

For the alternatives \( \Lambda(t) = \alpha t + (\alpha t^2)/2 \), \( \alpha > 0 \), \( \sigma > 0 \), with \( H_o \) achieved at \( \sigma = \sigma_o = 0 \), we find \( \beta'(0) = -1/(24\alpha) \), \( \mu'(0) = n/(12\alpha) \), and so (17) yields \( e(Q, S) = (5/8) = .625 \).

For the alternatives \( \Lambda(t) = t^\sigma \), \( \sigma \geq 1 \), with \( H_o \) achieved at \( \sigma = \sigma_o = 1 \), we find \( \beta'(1) = -1/6 \), \( \mu'(1) = n/4 \), and so (17) yields \( e(Q, S) = (10/9) = 1.11 \).
A comparison of a different nature is obtained by applying the Q and S statistics to the same set of data. Table 2 is based on data that appeared as Table 1 of Maguire, Pearson and Wynn (1952) and can also be found as Table 1.1 of Cox and Lewis (1966, p. 4). The data are the intervals $y_i$ (say) in days between successive coal-mining disasters (a disaster is said to occur if at least ten people die) in Great Britain for the period December 6, 1875, to May 29, 1951.
### TABLE 2

Days between successive coal-mining disasters

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*Source: B. A. Maguire, E. S. Pearson, and A. H. A. Wynn (1952).*
The $\tau(i)$'s (ordered event times) are obtained by cumulating the $y_i$'s, so that $\tau(1) = y_1$, $\tau(2) = y_1 + y_2$, etc. We applied the superadditive and S tests using $n = 108$ and $t = \sum_{i=1}^{109} y_i = 26.263$.

The normal deviate $\left(30 \cdot 108\right)^{1/2} Q_{108} = 4.97$ is significant at $P = .000003$, and this can be viewed as strong evidence of subadditivity.

This is in agreement with the visual indication of subadditivity one obtains from Figure 1.1 of Cox and Lewis (1966, p. 3). (Their figure is a plot of the total number of events that have occurred at or before time $t$, versus $t$.) For the Cox-Lewis test, the normal deviate is $U_{108} = -3.76$, which is also highly significant; $P = .000085$.

Additional inferential procedures for non-homogeneous Poisson processes can be found in Cox and Lewis (1966), Lewis (1972a), Lewis (1972b), Lewis (1974), and Watson (1974). Watson (1974) and Serfling (1974) provide concise summaries and additional references.

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REFERENCES


APPENDIX: PROOFS OF UNDERSCORED ASSERTIONS

A1: For the overhaul model of Section 2.1, \( \Lambda(t) = \int_0^t \Lambda(u) du \) is superadditive.

Proof. For \( 0 \leq t_1 \leq t_2 \), define \( \Lambda(t_1, t_2) = \Lambda(t_1 + t_2) - \Lambda(t_1) - \Lambda(t_2) \). We consider two cases.

Case 1. Let \([t_1/t_0] + [t_2/t_0] = [(t_1 + t_2)/t_0]\), where \([a]\) denotes the largest integer that is less than or equal to \(a\). Then by assumption (ii) of Section 2.1, we have \( \Lambda([t_1/t_0]t_0) + \Lambda([t_2/t_0]t_0) \leq \Lambda([(t_1 + t_2)/t_0]t_0) \).

Also, by assumption (ii), we have \( \Lambda(t_1) - \Lambda([t_1/t_0]t_0) \leq \Lambda([(t_1 + t_2)/t_0]t_0 + t_1 - [t_1/t_0]t_0) \)

\[-\Lambda([(t_1 + t_2)/t_0]t_0).\]

Similarly, by assumptions (i) and (ii) we find \( \Lambda(t_2) - \Lambda([t_2/t_0]t_0) \leq \Lambda(t_1 + t_2) - \Lambda([(t_1 + t_2)/t_0]t_0 + t_1 - [t_1/t_0]t_0) \).

Thus, adding the three inequalities, we conclude that \( \Lambda(t_1, t_2) \geq 0 \).

Case 2. Write \( t_1 = i_1t_0 + s_1, t_2 = i_2t_0 + s_2, t_3 = t_1 + t_2 = i_3t_0 + s_3 \), where \( 0 \leq s_i < t_0, i = 1, 2, 3 \). Suppose \( s_1 + s_2 \geq t_0 \).

Then \( \Lambda(i_1t_0) + \Lambda(i_2t_0) \leq \Lambda((i_1+i_2)t_0) \).
since $\Lambda((n+1)t_0) - \Lambda(nt_0) \uparrow$ in $n$ by assumption (ii). Also, by (ii),

$$\Lambda(t_1) - \Lambda(i_1t_0) \leq \Lambda((i_1+i_2)t_0 + t_1 - i_1t_0) - \Lambda((i_1+i_2)t_0),$$

and, by (i) and (ii),

$$\Lambda(t_2) - \Lambda(i_2t_0) \leq \Lambda(t_3) - \Lambda(t_3-(t_2-i_2t_0)).$$

Thus, adding inequalities, we conclude that $\Delta(t_1, t_2) \geq 0$, and the proof is complete.

Note that assumption (iii) of Section 2.1 is not needed, and was not used, to obtain superadditivity. This assumption does however insure that $\Lambda$ is not convex and thus provides a class of alternatives against which, for example, Roswell's (1966) test is not particularly appropriate but against which the superadditive test is appropriate.

A2: For the sampling with replacement model of Section 2.2, $F_{NBU}$ implies that $N(t_1)^{st} N(t_1+t_2) - N(t_2)$, for all $t_1, t_2 \geq 0$.

Proof. In socket $j$, the item on test is new at time $0$ and has age $\geq 0$ at time $t_2$. Note that $P(X_1+\ldots+X_k \leq t_1) = P(N_j(t_1) \geq k)$ and $P(X_1^*+X_2^*+\ldots+X_k^* \leq t_1) = P(N_j^*(t_2+t_1) - N_j(t_2) \geq k)$, where $N_j(t) =$ number of failures in socket $j$ during $[0, t]$, $X_1^*$ is the remaining life of the item at time $t_2$, and $X_1, \ldots, X_k, X_1^*, \ldots, X_k^*$ are independent and identically distributed according to $F$. Since $F$ is $NBU$, $X_1^{st} \leq X_1$. Thus
P(N_j(t_1) \geq k) \leq P(N_j(t_2 + t_1) - N_j(t_2) \geq k),

i.e., N_j(t_1) \leq N_j(t_2 + t_1) - N_j(t_2). Since N(t) = \sum_{j=1}^{N} N_j(t), we have N(t_1) \leq N(t_2 + t_1) - N(t_2). This completes the proof.

A3: For \( A \in A \), where \( A \) is defined by (15), \( A(t) \) is superadditive on \([0, 1]\).

Proof. Let \( 0 \leq t_1, t_2 \), and \( t_1 + t_2 \leq 1/2 \). Then \( A(t_1) + A(t_2) \leq A(t_1 + t_2) \) since \( A \) is superadditive on \([0, 1/2]\). Next let \( 0 \leq t_1, t_2 \leq 1/2 \) with \( t_1 + t_2 > 1/2 \). Then \( A(t_1) + A(t_2) \leq t_1 + t_2 \leq A(t_1 + t_2) \). Finally, let \( 0 \leq t_1 < 1/2 < t_2 \) with \( t_1 + t_2 \leq 1 \). Then \( A(t_1) + A(t_2) = A(t_1) + 1 - A(1-t_2), A(t_1 + t_2) = 1 - A(1-t_1-t_2) \), and therefore

\[ A(t_1 + t_2) - A(t_1) - A(t_2) = A(1-t_2) - A(t_2) - A(1-t_1-t_2) \geq 0, \]

since \( t_1, 1 - t_1 - t_2 \in [0, 1/2] \) and \( A \) is superadditive on \([0, 1/2]\). This completes the proof.
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13. Abstract

Let $N(t)$ be a non-homogeneous Poisson process with mean value function $\Lambda(t)$ and rate of occurrence $\lambda(t)$. We propose a conditional test of the hypothesis that the process is homogeneous, versus alternatives for which the mean value function is superadditive. Specific models leading to superadditivity are presented, and the superadditive test is compared, on the basis of consistency and asymptotic relative efficiency, with the Cox-Lewis test; the latter being directed to alternatives where $\lambda(t)$ is increasing.
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