ADMISSIBILITY OF ELLIPTICALLY CONTOURED ESTIMATORS OF THE MEAN OF A MULTIVARIATE NORMAL DISTRIBUTION WITH GENERALIZED SQUARED ERROR LOSS

by

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SUMMARY

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Let $X$ be a $p$-variate ($p \geq 3$) random vector normally distributed with
mean vector $\theta$ and unknown positive definite covariance matrix $\Sigma$. Let $A$
be a $p \times p$ Wishart matrix with parameters $(n, \Sigma)$, $n \geq p$, independent of $X$.
This paper considers the problem of estimating $\theta$ with respect to a
generalized squared error. A class of elliptically contoured estimators,
i.e., estimators of the form $h(X'A^{-1}X)$ for some real-valued measurable
function $h$, is characterized. Necessary and sufficient conditions for
such estimators to be admissible in the class of all nonrandomized
estimators are established. These conditions depend only on the
structure of the estimators. By modifying the Lin and Tsai estimators
(Ann. Statist. 1 (1973), 142-45) a class of admissible and minimax
estimators is obtained.

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1. Introduction. 

Let $X$ be a $p$-variate ($p \geq 3$) random vector normally distributed with 
mean $\theta$, and unknown positive definite covariance matrix $\Sigma$. Let $A$ be a 
$p \times p$ Wishart matrix with parameters $(n, \Sigma)$, $n \geq p$, independent of $X$. 
The problem under consideration is that of estimating $\theta$ by $\hat{\theta}$ with 
the generalized squared error loss function 

\begin{equation} 
L(\theta, \Sigma; \hat{\theta}) = (\hat{\theta} - \theta)' \Sigma^{-1} (\hat{\theta} - \theta). 
\end{equation} 

In this paper we obtain necessary and sufficient conditions for the 
admissibility of bounded risk elliptically contoured estimators, that is, 
estimators of the form $h(X' A^{-1} X) X$, where $h$ is a real-valued measurable 
function. 

For the case $\Sigma = I$ numerous results are available. Stein (1955) 
proved the inadmissibility of the usual estimator $X$. Strawderman (1971) 
obtained a class of proper Bayes minimax estimators for $p \geq 5$. Lin (1974) 
showed that Strawderman's estimators are admissible and minimax for $p \geq 3$. 
Strawderman and Cohen (1971) established necessary and sufficient conditions 

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for the admissibility of bounded risk generalized Bayes spherically symmetric estimators based on a theorem of Brown (1971).

For the case $E = \sigma^2 I$, Baranchik (1970) has obtained a class of minimax estimators, and Lin and Tsai (1973) a class of generalized Bayes minimax estimators.

The present problem, when $E$ is completely unknown, has been studied by James and Stein (1961), who obtained a minimax estimator. Their result is generalized to a class of minimax estimators by Lin and Tsai (1973). In the latter paper, a subclass of generalized Bayes minimax estimators is also obtained.

In Section 2 a class of elliptically contoured estimators is characterized, and necessary and sufficient conditions for the admissibility of such estimators is established. Section 3 gives a class of admissible minimax estimators which may be expressed in terms of an incomplete gamma function. Some remarks are presented in Section 4.

Since the loss function (1.1) is convex we will restrict our attention to the class of nonrandomized estimators $\mathcal{D}$. Throughout this paper we will also denote the classes of essentially equivariant and equivariant estimators by $\mathcal{D}^*$ and $\mathcal{D}^{**}$ respectively. For the definition of $\mathcal{D}^*$ and $\mathcal{D}^{**}$ consult, e.g., Zacks (1971), p.319. We further introduce the notation $\preceq$ to mean that the inequality holds for all parameters of an appropriate parameter space and it is strict for at least one parameter point.

2. **Necessary and Sufficient Conditions for Admissibility.**

In this section a group of transformations, which leaves the problem invariant, is defined. It is shown that the class $\mathcal{D}^{**}$ of equivariant estimators under this group is identical to the class of all elliptically
contoured estimators. Necessary and sufficient conditions are established
in Theorem 2.2 for the admissibility of bounded risk elliptically contoured
estimators. These conditions depend only on the structure of the estimators.
Specifically, let $G$ be the group of nonsingular transformations defined by

\[(2.1) \quad X \rightarrow CX, \quad A \rightarrow CAC',\]

where $C$ is any $p \times p$ nonsingular matrix. The induced groups of trans-
formations on the parameter space and the space $D$ are, respectively, given
by

\[(2.2) \quad \theta \rightarrow C\theta, \quad E \rightarrow CEC',\]

and

\[(2.3) \quad \delta(X, A) \rightarrow C\delta(X, A)\]

Under the group $G$ the class $D^{**}$ of equivariant estimators is character-
ized by the following lemma.

**Lemma 2.1.** The estimator $\delta(X, A)$ is in $D^{**}$ if and only if

\[(2.4) \quad \delta(X, A) = h(X'A^{-1}X)X\]

for some real-valued measurable function $h$.

**Proof.** The sufficiency is trivial. We will prove the necessary condition.
Suppose that $\delta(X, A)$ is equivariant, i.e.,

\[(2.5) \quad \delta(CX, CAC') = C\delta(X, A)\]

for any $p \times p$ nonsingular matrix $C$. Since $A$ is positive definite with
probability one (see Dykstra (1970)), we may let $C = A^{-1/2}$, where $A^{1/2}$ is a
nonsingular square root of \( A \), i.e., \( A = (A^2)^{1/2}(A^2)^{1/2} \). Then (2.5) becomes

\[(2.6) \quad A^{-k} \delta(X, A) = \delta(A^{-k}X, I) = g(A^{-k}X)\]

for some \( p \times 1 \) vector-valued measurable function \( g \) which is a function only of \( A^{-k}X \). From (2.6) it follows that \( \delta(X, A) \) is of the form

\[(2.7) \quad \delta(X, A) = h(A^{-k}X)X\]

for some real-valued measurable function \( h \). Moreover, the equivariance of \( \delta(X, A) \) implies the invariance of \( h(A^{-k}X) \). Hence \( h \) is a function of the maximal invariant statistic \( X'A^{-1}X = (A^{-k}X)'(A^{-k}X) \). \( \square \)

We now state the necessary and sufficient conditions for the admissibility of bounded risk elliptically contoured estimators.

**THEOREM 2.2.** A bounded risk elliptically contoured estimator \( h(X'A^{-1}X)X \) of \( \theta \) is admissible relative to the loss function (1.1) if and only if the following conditions hold:

i) \( g(z'z) = \exp\left[\frac{1}{2} \int_0^\infty (h(y) - 1)dy\right] = \exp\left[-\frac{1}{2}(z - \mu)'(z - \mu)\right]dF(\mu) \)

for some measure \( F(\mu) \) and for all \( p \)-vector \( z \), and

ii) \( \int_1^\infty [r^{p-1}g(r^2)]^{-1}dr = \infty \).

The proof of this theorem depends on Theorem 4.4.1 of Strawderman and Cohen (1971) and the following two lemmas. Lemma 2.4 may be of independent interest and is presented in a general context.

**LEMMA 2.3.** A bounded risk elliptically contoured estimator \( h(X'A^{-1}X)X \) is admissible in \( \mathcal{D}^{**} \) if and only if the conditions of Theorem 2.2 are satisfied.
PROOF. The risk of \( h(X'A^{-1}X)X \) is given by

\[
(2.8) \quad E\{[h(X'A^{-1}X)X - \theta]'A^{-1}[h(X'A^{-1}X)X - \theta]\} = E\{E\{[h(X'A^{-1}X)X - \gamma]'A^{-1}[h(X'A^{-1}X)X - \gamma]\}|A\} = E\{E\{[h(Y'Y) - \mu]'[h(Y'Y) - \mu]\}|A\}.
\]

The second equality of (2.8) is obtained since the risk of an equivariant estimator is constant on each orbit, the risk may be evaluated at the point \((\theta, A) = (\gamma, A)\), where \(\gamma = A^{-1}\theta\). Here \(X\) given \(A\) is now regarded as \(N(\gamma, A)\) and \(Y\) given \(A\) as \(N(\mu, I)\), where \(\mu = A^{-1}\theta\). Note that the conditional expectation in the last expression of (2.8) does not depend on \(A\); it may be interpreted as the risk of \(h(Y'Y)Y\), a spherically symmetric estimator of \(\mu\), relative to the loss function \(L(\mu; \hat{\mu}) = (\mu - \hat{\mu})'(\mu - \hat{\mu})\).

In this case Strawderman and Cohen (1971, Theorem 4.4.1.) established that the conditions of Theorem 2.2 are necessary as well sufficient for the admissibility of bounded risks spherically symmetric estimators of \(\mu\). Therefore these conditions are necessary and sufficient for the admissibility of \(h(X'A^{-1}X)X\) in the class of elliptically contoured estimators \(D^*\). □

**Lemma 2.4.** Let \(X\) be a random variable on a space \(X\), and let \(\tau = \{ P(\theta_1, \theta_2) : \theta_1 \in \theta_1, \theta_2 \in \theta_2 \}\) be a family of probability measures on \(X\). Let \(L((\theta_1, \theta_2); \hat{\theta}_1(X))\) represent the loss incurred when estimating \(\theta_1\) by \(\hat{\theta}_1(X)\). Assume that the elements of \(\tau\) are equivalent. Then \(\hat{\theta}_1(X)\) is admissible in \(D^*\) if and only if it is admissible in \(D^{**}\).

**Proof.** The necessary condition is obvious. We will prove the sufficient condition. Assume \(\hat{\theta}_1(X)\) is not admissible in \(D^*\). Then there exists an
estimator \( \hat{\theta}_1(X) \) in \( D^* \) such that

\[
(2.9) \quad \int_{X} \{L((\theta_1, \theta_2); \hat{\theta}_1^*(x)) - L((\theta_1, \theta_2); \hat{\theta}_1(x))\}dP(\theta_1, \theta_2)(x) \leq 0.
\]

By hypothesis \( X \) may be partitioned into \( X_1 \) and \( X_2 \) such that \( \hat{\theta}_1^*(x) \in D^{**} \) for \( x \in X_2 \) and \( \hat{\theta}_1^*(x) \notin D^{**} \) for \( x \in X_1 \), with \( P(\theta_1, \theta_2)(X_1) = 0 \) for all \( \theta_1 \) and \( \theta_2 \). Then (2.9) is equivalent to

\[
\int_{X_2} \{L((\theta_1, \theta_2); \hat{\theta}_1^*(x)) - L((\theta_1, \theta_2); \hat{\theta}_1(x))\}dP(\theta_1, \theta_2)(x) \leq 0.
\]

But \( \hat{\theta}_1(X) \) is admissible in \( D^{**} \). This contradiction proves the lemma. \( \Box \)

**PROOF OF THEOREM 2.2.** The admissibility of \( h(X'A^{-1}X)X \) in \( D^{**} \) is equivalent to its admissibility in \( D^* \) by Lemma 2.4. On the other hand the group \( G \) is locally compact, the generalized Hunt-Stein Theorem (Zacks (1971), p.346) may be applied to establish the essential completeness of \( D^* \). Therefore the admissibility of \( h(X'A^{-1}X)X \) is equivalent to its admissibility in \( D \). \( \Box \)

3. A Class of Admissible Minimax Estimators.

Lin and Tsai (1973) obtained a class of minimax estimators of the form

\[
(3.1) \quad h(X'A^{-1}X)X = [1 - r(X'A^{-1}X)/X'A^{-1}X]X
\]

where \( r(\cdot) \) is nonnegative nondecreasing and bounded by \( 2(p - 2)/(n - p + 3) \). The following lemma extends the upper bound of \( r(\cdot) \) to \( 2(p - 2) \). It can be proven by the equivariance argument employed in Lemma 2.3 and a result of Strawderman (1971). The proof is omitted.

**LEMMA 3.1.** An estimator given by (3.1) is minimax if \( r(\cdot) \) is nonnegative nondecreasing, and bounded by \( 2(p - 2) \).
The main result of this section is stated below.

**Theorem 3.2.** The class of estimators

\[
\delta_m(x, A) = \left[ 1 - \frac{1}{\int_0^\infty \lambda^{-p-a+1} \exp(-\lambda y) d\lambda} \right] x,
\]

where \( y = X' A^{-1} X \), is admissible and minimax for \( p \geq 3 \) with \( 3 - \frac{1}{p} \leq a \leq 2 \).

**Proof.** We will first prove the minimaxity. Upon integration by parts of \( \int_0^\infty \lambda^{-p-a+1} \exp(-\lambda y) d\lambda \) and a little algebra, \( \delta_m(x, A) \) may be expressed as

\[
\delta_m(x, A) = \left[ 1 - \frac{p - 2a + 2}{y} + \frac{2\exp(-\lambda y)}{y \int_0^\infty \lambda^{-p-a} \exp(-\lambda y) d\lambda} \right] x.
\]

For this estimator, the function \( r(y) \) of Lemma 3.1 is now

\[
r(y) = p - 2a + 2 - \frac{2\exp(-\lambda y)}{\int_0^\infty \lambda^{-p-a} \exp(-\lambda y) d\lambda},
\]

which is less than or equal to \( 2(p - 2) \) since \( a \geq 3 - \frac{1}{p} \). In view of (3.2), it is also apparent that \( r(y) \) is nonnegative. The nondecreasingness of \( r(y) \) is implied by that of \( \int \lambda^{-p-a} \exp[\lambda(1 - \lambda)y] d\lambda \). Thus \( \delta_m(x, A) \) is minimax by Lemma 3.1.

It remains to show the admissibility. Lin (1974) proves that the function
\[ h(y) = 1 - \frac{1}{\int_0^{\lambda^{2p-a}} \exp(-\frac{1}{2}\lambda y) d\lambda} \]

satisfies the conditions of Theorem 2.2 for \( a \geq 2 \). Hence the estimators (3.2) are also admissible. \( \square \)

Note that the integral expression in (3.3) may be replaced by an incomplete gamma function, that is,

\[ \int_0^1 \lambda^{2p-a} \exp(-\frac{1}{2}\lambda y) d\lambda = (2/y)^{2p-a+1} \int_0^{1/2y} t^{2p-a} \exp(-t) dt \]

\[ = (2/y)^{2p-a+1} \left[ \Gamma(\lambda^{2p-a+1}) - \Gamma(\lambda^{2p-a+1}, 1/2y) \right], \]

where \( \Gamma(\cdot) \) and \( \Gamma(\cdot, \cdot) \) are the gamma and incomplete gamma functions, respectively.


The following question is posed: Under what conditions on \( a \) and \( g \) are the estimators

\[ \int_0^1 \lambda^{2p-a+1} g(\lambda x') \lambda x d\lambda \]

admissible? Note that the Lin and Tsai (1973) estimators are of the form (4.1) with \( g(\lambda y) = (1 + \lambda y)^{-\frac{3}{2}(n-u+p+2)} \), and that the modified estimators (3.2) are of the same form with \( g(\lambda y) = \exp(-\frac{1}{2}\lambda y) \).

In conclusion we observe that the case \( \Sigma = \sigma^2 I \) can be treated in a fashion similar to that of completely unknown \( \Sigma \) treated in this paper.
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Admissibility of Elliptically Contoured Estimators of the Mean of a Multivariate Normal Distribution With Generalized Squared Error Loss

4. **Descriptive Notes**  
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5. **Authors**  
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13. **Abstract**

Let \( X \) be a \( p \)-variate (\( p \geq 3 \)) random vector normally distributed with mean vector \( \theta \) and unknown positive definite covariance matrix \( \Sigma \). Let \( A \) be a \( p \times p \) Wishart matrix with parameters \((n, \Sigma)\), \( n \geq p \), independent of \( X \). This paper considers the problem of estimating \( \theta \) with respect to a generalized squared error. A class of elliptically contoured estimators, i.e., estimators of the form \( h(X^T A^{-1} X) \) for some real-valued measurable function \( h \), is characterized. Necessary and sufficient conditions for such estimators to be admissible in the class of all nonrandomized estimators are established. These conditions depend only on the structure of the estimators. By modifying the Lin and Tsai estimators (Ann. Statist. 1, 142-45) a class of admissible and minimax estimators is obtained.

14. **Key Words**

Multivariate normal mean, admissible, minimax, elliptically contoured estimators, equivariant, locally compact, essentially complete.