ON THE UNIVERSAL OPTIMALITY OF EXPERIMENTAL DESIGNS
WITH UNEQUAL BLOCK SIZES AND UNEQUAL REPLICATIONS.

by

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ON THE UNIVERSAL OPTIMALITY OF EXPERIMENTAL DESIGNS
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Abstract

It is shown that in a family of competing experimental designs with unequal
block sizes and unequal replications, a design which is completely symmetric
is universally optimal. To show this we have utilized some optimality
techniques recently developed by Kiefer. Two methods for the construction
of the optimal designs are pointed out. As an application of these results
it is shown that a resistant BIB design is two stage optimal.
1. **Introduction and Summary.** Suppose we want to run an experiment for the purpose of comparing \( v \) treatments using \( n \) experimental units which can be grouped with respect to their influence on the response into \( b \) blocks. Moreover, these \( b \) blocks can be classified with respect to their size into \( c \) classes of equal size blocks. Let \( b_m \) and \( k_m \) denote the number of blocks and the size of each block in the \( m \)-th class respectively, \( m = 1, 2, \ldots, c \geq 1 \). Suppose the model of response is the usual homoscedastic linear additive fixed effects model. Thus the \( n \) observations will be assumed uncorrelated with common variance \( \sigma^2 \) with the expectation of an observation on treatment \( i \) in an experimental unit in the \( j \)-th block of the \( m \)-th class is \( \mu + \tau_i + \beta_{j(m)} \) with the usual meaning of each component (assuming such an observation is made).

The problem of concern here is to design an efficient experiment based on these \( v \) treatments and \( n \) experimental units having the following desirable properties:

(i) the selected design can provide a set of \( v-1 \) unbiased estimators for a specified set of \( v-1 \) independent contrasts among \( \tau_i \)'s.

(ii) the selected design is the best with respect to an optimality criterion which operates on the covariance matrix of the estimators in (i).

For many reasonable optimality criteria the known results in the literature are summarized below:

(a) If \( c = 1 \) and \( k_1 \geq v \), then Kiefer (1958) has shown that a balanced block design (if it exists) is the best design. Note that if \( k_m = v \) a balanced block design is the usual randomized complete blocks design which always can be constructed.
(b) If \( c = 1 \) and \( k_1 < v \), then Kiefer (1958), Kshirsagar (1958) and Mote (1958) have shown that a BIB design (if it exists) is the best design. Takeuchi (1961) has proved that if a BIB design does not exist, then a PBIB design with two associate classes (if it exists) is the second best design. If the class of competing designs does not contain a PBIB design with two associate classes either, then Hering (1973) has shown that a \( t \)-associate group divisible design with the smallest \( t \) (if it exists) is the next best design.

The main purpose of this note is to utilize some optimality tools of Kiefer (1958, 1973) and characterize in a usable form a family of optimal designs for the case \( c \geq 1 \) and \( k_m < v \). We also mention two different methods which are helpful for the construction of these designs. Finally, as an application of these results we show that a resistant BIB design is two stage optimal.

2. **Characterization of A Universally Optimal Design.**

Let \( D \) denote the class of competing designs which satisfies the requirement (i). This means that for each \( d \in D \), the matrix \( C_d \), the information matrix of the design \( d \), for the \( v \) unknown parameters \( \tau_i \)'s derived from "normal equations" is of maximal rank \( v-1 \). Note that \( C_d \) has always zero row and column sums. In other words \( d \) is connected. Thus the variance of the best linear unbiased estimator of the contrast \( \sum a_i \tau_i \) is \( \sigma^2 a' C_d^{-1} a \) where \( a = (a_1, a_2, \ldots, a_v) \) and \( C_d^{-1} \) is a generalized inverse of \( C_d \). Or in general the covariance matrix for a given set of \( v-1 \) independent contrasts specified by the \((v-1)\times v\) matrix \( A' \) is \( \sigma^2 A' C_d^{-1} A = \sigma^2 V_d \) (Say). Therefore, if the optimality function is \( \phi \) then the problem is to characterize a \( d^* \) in \( D \) which minimizes \( \phi(V_d) \). A design which has this property is said to be \( \phi \)-optimal. Some popular optimality criteria are:
D - Optimality: \( \phi(V_d) = \det V_d \);

2.1 A - Optimality: \( \phi(V_d) = \text{Tr} V_d \);

E - Optimality: Maximum eigen value of \( V_d \).

The relationship among these optimality criteria is well known and can be found in Keifer's papers on optimal designs. One sees that the major difficulty with this approach is the computation of \( C_d^- \) for each \( d \in D \). Also this approach is somewhat unattractive because the \( \phi \)-optimal design will in general depend on the choice of \( A' \). Recently, Keifer (1973) has introduced an optimality criterion which operates on \( C_d^- \) with many interesting implications on \( V_d \). This new optimality criterion is given below.

**Definition 2.1.** A design \( d^* \) in \( D \) is said to be universally optimal if \( d^* \) minimizes \( \phi(C_d) \) for any real valued \( \phi \) having the following properties:

(i) \( \phi \) is convex;

(ii) \( \phi(bC_d) \) is nonincreasing in the scalar \( b \geq 0 \);

(iii) \( \phi \) is invariant under each permutation of rows and (the same on) columns.

This means that the optimal design does not depend on the way one labels the treatments.

A useful characterization of functions with properties (i), (ii) and (iii) in Definition 2.1 is unknown to this author. The following statistically useful family of functions with those properties is due to Keifer (1973).
\( \phi_p(C_d) = \left( \frac{1}{v-1} \sum_{i=1}^{v-1} \lambda_{d_i}^{-p} \right)^{1/p}, \quad 0 < p < \infty, \)

where \( \{\lambda_{d_i}, i = 1, 2, \ldots, v\} \) denotes the set of non zero eigen values of \( C_d \).

Note that the optimality criteria defined in (2.1) are special cases of (2.2) for \( p = 1 \) and its limiting values.

Before proceeding further we need the following definition.

**Definition 2.2.** A design \( d \) in \( D \) is said to be completely symmetric if \( C_d \) is of the form \( \alpha I_v + \beta J_v \), where \( \alpha, \beta, I_v \) and \( J_v \) are scalars; \( v \times v \) identity matrix and \( v \times v \) matrix with all entries equal one respectively.

Following Kiefer (1973) we have for our problem:

**Proposition 2.1.** If there is a \( d^* \) in \( D \) with the properties that

(a) \( d^* \) is completely symmetric,

(b) \( \text{Tr} C_{d^*} = \max_{d \in D} \text{Tr} C_d \),

then \( d^* \) is universally optimal in \( D \).

Condition (b) in Proposition 2.1 is usually difficult to establish. In the following lemma we show that there will be no difficulty if we avoid multiple assignment of a treatment to a block.

**Lemma 2.1.**

\[ \text{Tr} C_d = n - \sum_{m=1}^{c} b_m, \text{ for each } d \in D. \]
Proof. Upon appropriate rewriting we can express $C_d$ as

$$C_d = R_d - \sum_{m=1}^{c} \frac{1}{k_m} N_{dm} N'_{dm},$$

Here $R_d$ is a $v \times v$ diagonal matrix with $r_{di}$ in its $(i,i)$ entry, where $r_{di}$ is the number of experimental units assigned to treatment $i$ by design $d$. $N_{dm}$ is the treatment-block incidence matrix generated by design $d$ for the $m$-th class of blocks.

Thus we have

$$\text{Tr } C_d = \sum_{i=1}^{V} r_{di} - \sum_{m=1}^{c} b_m$$

$$= n - \sum_{m=1}^{c} b_m.$$

This leads to the following conclusion.

Theorem 2.1. Any completely symmetric design in $D$ is universally optimal.

We now outline two procedures for the construction of some completely symmetric designs.

Procedure One. Try to construct, if possible, a sub BIB design based on $v$ and each class of blocks. Having done this the whole design which is the union of $c$ BIB designs will be of course completely symmetric.

Example 2.1. Suppose $v = 4$, $c = 2$, $b_1 = 4$, $k_1 = 3$, $b_2 = 6$ and $k_2 = 2$. In this case two sub BIB designs are possible and the whole symmetric design is given below.

$d^*$:  
\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\},
\{1,2\}, \{1,4\}, \{2,4\}, \{1,3\},
\{2,3\}, \{3,4\},

with $C_{d^*} = \frac{26}{6} I_4 - \frac{7}{6} J_4.$
Procedure two. It has been shown by John (1964) and Hedayat and Federer (1972) that by proper unionization and augmentation one may construct a variance balanced incomplete block design. Since any variance balanced incomplete block design is completely symmetric, thus one should see whether the integers $v, b_m, k_m; m = 1, 2, \ldots, c$ fall in the domain of the results obtained by these authors. We now give two examples which have been constructed using this procedure.

**Example 2.2.** Suppose $v = 5, c = 2, b_1 = 8, k_1 = 4, b_2 = 6$ and $k_2 = 2$. A completely symmetric design based on procedure two is

$$
\{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\},
$$

$$
\{2,3,4,5\}, \{1,2,3,5\}, \{1,2,4,5\},
$$

$$
\{1,3,4,5\}, \{2,3,4,5\}, \{1,2\},
$$

$$
\{1,4\}, \{2,4\}, \{1,3\}, \{2,3\}, \{3,4\}.
$$

With

$$
C_{d^*} = \frac{15}{2} I_5 - \frac{6}{4} J_5.
$$

Note that in this design $r_{d^*1} = 9, i = 1, 2, 3, 4$ and $r_{d^*5} = 8$. Also note that this design is obtained by augmenting an experimental unit, containing a new treatment viz., to each block of size 3 in Example 2.1 and then duplicating it. It is interesting to observe that neither of the sub designs in this example is a BIB design based on the entire set of treatments.

**Example 2.3.** Suppose $v = 5, c = 2, b_1 = 2, k_1 = 4, b_2 = 4$ and $k_2 = 2$. Based on these parameters a completely symmetric design using procedure two can be constructed.
\{1,2,3,4\}; \{1,2,3,4\}, \{1,5\},

\text{d}^*:

\{3,5\}, \{2,5\}, \{4,5\},

with \(C_{d^*} = \frac{5}{2} I_5 - \frac{1}{2} J_5\).

Here \(r_{d^*i} = 3\), \(i = 1,2,3,4\) and \(r_{d^*5} = 4\).

Remark. From the above examples it is clear that equal replications and/or equal block sizes are not necessary conditions for the existence of a completely symmetric design.

3. Two Stage Optimality of Resistant BIB Designs. The basic theory and application of resistant BIB designs can be found in Hedayat and John (1974). Some additional results are recently found by Most (1973). For the completeness sake, the definition of a resistant BIB design is given below.

**Definition 3.1.** If a set of \(t\) treatments is removed from a BIB design in \(v\) treatments, \(v > t\), and if the remaining incomplete block design retains the variance balanced property (or equivalently it is completely symmetric), then we say that the BIB design is resistant to removal of the given set of \(t\) treatments.

Suppose we are interested in comparing \(v\) treatments using \(b\) blocks each of size \(k < v\). Further suppose that we are not sure that we are going to get data on a subset of \(t\) treatments for one reason or another (see Hedayat and John (1974) for more details). If universal optimality is the criterion of interest and if it is possible to construct a resistant BIB design, then this design would be the best choice. This is so because if nothing went wrong with the experiment then the chosen design is of course universally optimal. However, if we failed to collect observations on the designated \(t\) treatments then since
the remaining design associated with a resistant BIB design is completely symmetric, thus it will be optimal in the class of remaining designs associated with other competing designs. For some \( v, b \) and \( k \) it is possible to construct resistant and susceptible (non resistant) BIB designs. This being so we should be careful in our choice of a completely symmetric design in the first place.

For the record we may formally state this as a corollary. First we introduce the following definition.

**Definition 3.2.** We say a design \( d \) is two stage \( \phi \)-optimal with respect to a subset of \( t \) treatments, if \( d \) and \( d \) without these \( t \) treatments are \( \phi \)-optimal in their corresponding competing designs.

**Corollary 3.1.** A BIB design is two stage universally optimal if and only if it is a resistant BIB design.

The sufficiency part of this corollary is given above. The necessary part can be concluded from a result given in Hedayat and John (1974) and the preceding arguments.

4. **Closing Remarks** Recently Shah and Raghavarao (1973) have obtained the following result:

**Theorem 4.1.** A two non-interacting factor design is D-(A- or E-) optimal for the estimation of all main effects if and only if it is D-(A- or E-) optimal for the estimation of main effects of any one factor.

Therefore, in our case if \( \mathbf{d}^* \) is optimal for the estimation of treatment contrasts will be optimal for the estimation of block contrasts and will also be optimal for the contrasts among the treatments and block effects. The reader is referred to Shah and Raghavarao for more details.
REFERENCES


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