Duels With Continuous Firing*

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Abstract

A game-theoretic model is proposed for the generalization of a discrete-fire silent duel to a silent duel with continuous firing. This zero-sum two-person game is solved in the symmetric case. It is shown that pure optimal strategies exist and hence also solve a noisy duel with continuous firing. A solution for the general non-symmetric duel is conjectured.
1. Introduction

In the classical duels, each player has a fixed number of "bullets", each of which may be fired at any time in [0, 1]. For $i = 1, 2$, associated with Player $i$ is a nondecreasing function $P_i$, from [0, 1] onto [0, 1], called his accuracy function. If Player $i$ shoots one bullet at time $t$, the probability that he hits his opponent, given that both players are still alive at time $t$, is $P_i(t)$. The first player to hit his opponent ends the game and receives payoff +1 from his opponent. As in a physical duel in which the opponents advance towards each other as time progresses, each player must choose his firing times weighing the advantages of shooting early (before he himself is hit) against those of shooting later when his accuracy is greater.

A bullet is noisy if a player can detect the shooting by his opponent of such a bullet; otherwise a bullet is silent. In a silent duel all bullets are silent; in a noisy duel all bullets are noisy. Silent duels were solved in their greatest generality by Restrepo [1957] and noisy duels by Fox and Kimeldorf [1969].

As an example of the formulation of a duel as a two-person zero-sum game, consider a silent duel with Player $i$ possessing $n_i$ bullets and accuracy function $P_i$ $(i = 1, 2)$. A pure strategy for Player $i$ is a set $(t_1, \ldots, t_{n_i})$ of times in [0, 1] at each of which he plans to shoot one bullet. A randomized strategy for Player $i$ is an $n_i$-dimensional distribution function $F_{T_1, \ldots, T_{n_i}}$ which is interpreted in the obvious manner. In the case $n_1 = n_2 = 1$, the payoff-kernel, $K$, is

$$K(s, t) = \begin{cases} P_1(s) - P_2(t)(1 - P_1(s)) & \text{if } s < t \\ P_1(s) - P_2(s) & \text{if } s = t \\ -P_2(t) + P_1(s)(1 - P_2(t)) & \text{if } s > t \end{cases}$$
For larger numbers of bullets, \( K \) can be defined recursively (see Karlin [1959, p. 154]).

We consider a generalization of the classical silent duels. Rather than being restricted to discrete firing, the players are now allowed to fire continuously with varying intensities of fire. This game provides a natural model for a battle between two adversaries equipped with rapid-fire artillery such as machine-guns, or, perhaps in the future, ray guns. However, its applicability is not limited solely to physical warfare. For example, the above model has long been used to describe a common situation in advertising in which two companies are competing for a contract or sale. A fuller treatment of this example can be found in Gillman [1953].

2. **Inadequacies of Previous Models for Silent Duels with Continuous Firing.**

Silent duels with continuous firing have appeared in the literature (in various forms) throughout the last 24 years. Early formulations of these duels (with only partial solutions) were given by Gillman [1950] and by Danskin and Gillman [1953], but the first rigorous solution of such a duel was due to Karlin [1959]. Yanovskaya [1969] studied a more general version of the game considered by Karlin.

In Karlin's formulation of the game Player \( i \) has a continuously differentiable, strictly decreasing accuracy function \( P_i \) with \( P_i(0) = 1 \) and \( P_i(\infty) = 0 \). Notice that Karlin, apparently for mathematical convenience, casts the game on \([0,\infty)\) (distance between the players) instead of on \([0,1]\) (elapsed time). \( P_i(t) \) is interpreted as the probability that Player \( i \) hits his opponent if Player \( i \) expends one unit of firepower when both players are alive at distance \( t \). Player \( i \) is assumed to have a fixed amount \( \alpha_i \) of firepower. A pure strategy for Player \( i \) is a nonnegative function on \([0,\infty)\) that is bounded by 1 and integrates to \( \alpha_i \).
Such a function is called an intensity function and represents the intensity of Player i's fire.

Let $Q[f_1, P_1, t, h]$ be the conditional probability that Player i, using intensity function $f_1$ and with accuracy function $P_1$, hits his opponent in the interval $(t, t + h)$, given that both players are alive at distance $t + h$. Karlin assumed the model

$$(2.1) \quad Q[f_1, P_1, t, h] = P_1(t)f_1(t)h + o(h).$$

The payoff to Player 1 is assumed to be +1 if he hits Player 2 first and 0 otherwise. The expected payoff to Player 1 if the pure strategies $f_1$ and $f_2$ are used by Players 1 and 2, respectively, is derived from (2.1) to be

$$(2.2) \quad K[f_1, f_2] = \int_0^t f_1(t)P_1(t)\exp\left[\int_0^t f_1(s)P_1(s)ds + \int_0^t f_2(s)P_2(s)ds\right] dt.$$ 

Notice that while the functions $f_1$ and $f_2$ resemble probability density functions, there is no randomness involved. They are pure strategies representing the players' intensities of fire.

In solving this game, Karlin appealed to a general minimax theorem by Fleming [1954] which implies the existence of optimal pure strategies for the players. Yanovskaya [1969] attempted to solve the duel with Player 1 receiving payoff A if he wins, B if he loses, and 0 in the case of a tie. This results in a new payoff-kernel:

$$(2.3) \quad K[f_1, f_2] = \int_0^t (Af_1(t)P_1(t) - Bf_2(t)P_2(t))\exp\left[\int_0^t f_1(s)P_1(s)ds + \int_0^t f_2(s)P_2(s)ds\right] dt.$$ 

It is, however, not clear how she established the existence of optimal pure strategies since the payoff kernel (2.3) is no longer of the form required by Fleming's theorem.
In this section we argue heuristically that the model considered by Karlin and Yanovskaya is not appropriate. In the next section we rigorously present a mathematically well-defined game that is suggested by these heuristics to be a superior model for a silent duel with continuous firing.

A minor criticism of Karlin's model is his use of distance instead of time. It seems more natural and more consistent with other duels to cast the duel on [0,1]. In the future, we shall assume all duels are played on [0,1].

A major difficulty with his model lies in his requirement that the intensity functions $f_1$ and $f_2$ be bounded. Since Karlin has both $f_i$ and $p_i$ bounded by 1, he does not allow a player to concentrate his fire at a particular time (by firing with extreme intensity). Thus, for example, even if a player knows or suspects that his opponent is out of ammunition, he cannot wait until time 1 and concentrate his fire there as can be done in discrete-fire duels.

In fact, Karlin's duel is not really a generalization of a discrete-fire duel. While a player is "allowed" to fire continuously, he is no longer able to fire "discretely" (by shooting with high intensity). The model we shall present is a true generalization; the players may fire continuously but do not lose the ability to concentrate their fire.

To motivate our continuous-fire duel consider a silent discrete-fire duel with $n$ bullets each and assume for the moment that Player 2 is not shooting, so that Player 1 is essentially shooting at a target. If Player 1 chooses $n$ times $t_k$, where $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq 1$, at each of which he shoots one bullet, we define $S(t_1, t_2, \ldots, t_n)$ to be the probability that he hits his target (at least once).

In a duel with continuous firing, a pure strategy for Player 1 is an absolutely continuous, nondecreasing function $F$ on [0,1] satisfying $F(0) = 0$ and $F(1) \leq 1$. We call $F$ a **cumulative intensity function** and its derivative $f$ an **intensity function**.
There is again assumed to be a function $S(f)$ representing the probability that Player 1 hits his opponent if he chooses to fire in $[0,1]$ according to $f$ and Player 2 is not firing. Our problem is to determine what $S$ should be in order to make this a natural generalization of a discrete-fire duel.

For any intensity function $f$, let $S_I(f)$ denote the probability that Player 1 hits the target in the interval $I$ using intensity function $f$. For fixed $t \in [0,1)$, consider the intensity functions

$$f_{t,n}(s) = \begin{cases} n & \text{if } t < s < t + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

which, as $n$ gets large, concentrate more and more firepower at time $t$. Then, for $0 \leq t < 1$, we assume $\lim_{n \to \infty} S_{[0,1]}(f_{t,n})$ exists. We call this limit $P(t)$ and assume further that $P(0) = 0$, $\lim_{t \to 1} P(t) = 1$, and that $P$ exists and is continuous and positive on $(0,1)$.

We call $P$ Player 1's accuracy function. Loosely speaking, $P(t)$ represents the probability that he hits his target at time $t$ if he shoots all of his ammunition at this time.

Let $P_T(f,P,t,h)$ be the probability that Player 1, using intensity function $f$ with accuracy function $P$, hits his target in the interval $(t,t+h)$. As in previous models, we assume that there exists a function $H$, depending on $P$, such that

$$P_T(f,P,t,h) = f(t)H(t)h + o(h). \tag{2.4}$$

Karlin's assumption (2.1) is essentially (2.4) with $H(t) = P(t)$. We shall show that the correct function to take is $H(t) = -\log[1 - P(t)]$. 

It follows easily from (2.4) that

\[(2.5) \quad S_{1}(\gamma) = 1 - \exp[-\int_{I} H(s)f(s)ds] \]

for any subinterval \(I\) of \((0,1)\). In particular, for a given \(t \in [0,1)\),

\[(2.6) \quad S_{(0,1)}(f_{t,n}) = 1 - \exp[-n \int_{t}^{1} H(s)ds]. \]

Since by assumption the left hand side of (2.6) has a limit \(P(t)\),
\[
\lim_{n \to \infty} [1 - \exp(-n \int_{t}^{1} H(s)ds)] = 1 - \exp[-H(t)] = P(t),
\]
and hence

\[(2.7) \quad H(t) = -\log [1-P(t)]. \]

Thus, we have heuristically motivated adopting a new model for a silent duel with continuous firing. That is, Karlin's assumption (2.1) is replaced by

\[(2.8) \quad P_{T}(f_{1}, P_{1}, t, h) = f_{1}(t)A_{1}(t)h + o(h), \]

where \(A_{1}(t) = -\log [1-P_{1}(t)]\). From (2.8) a new payoff-kernel

\[(2.9) \quad K[f_{1}, f_{2}] = \int_{0}^{t} (f_{1}(t)A_{1}(t) - f_{2}(t)A_{2}(t)) \exp[-\int_{0}^{t} f_{1}(s)A_{1}(s)ds - \int_{0}^{t} f_{2}(s)A_{2}(s)ds]dt \]

can be derived in a straightforward manner.

3. The Proposed Model

The heuristics of the previous section motivate the introduction of the following zero-sum two-person game, which we can interpret as an appropriate model for a silent duel with continuous firing.
For $i = 1, 2$, associated with Player $i$ is a nonnegative, strictly increasing, absolutely continuous function $P_i$ defined on $[0,1]$ with $P_i(0) = 0$ and $P_i(1) = 1$. The function $P_i$, Player $i$'s accuracy function, is known beforehand to both players. There are fixed nonnegative numbers $M_1$ and $M_2$ (also known beforehand to both players), representing the amounts of ammunition available to Players 1 and 2, respectively. A pure strategy for Player $i$ is a nondecreasing, absolutely continuous function $F_i$ (with derivative $f_i$) on $[0,1]$ satisfying $F_i(0) = 0$ and $F_i(1) \leq M_i$. The functions $f_i$ and $F_i$ are called intensity functions and cumulative intensity functions, respectively. The payoff-kernel is given by (2.9).

Our model differs from the ones treated originally by Karlin and Yanovskaya in several ways:

1) The game is now played on $[0,1]$ instead of on $[0,\infty)$.

2) It is no longer assumed that the intensity functions $F_1$ and $F_2$ are bounded.

3) The payoff-kernel, given by (2.9), differs from any previously considered.

As a consequence of these differences, the methods of Karlin and Yanovskaya cannot be used to solve our game. Section 5 discusses some of the problems encountered in attempting to apply their methods.

4. Solution in the Symmetric Case

In this section, the game defined in Section 3 is solved in its simplest form. It is assumed that both players have the same accuracy function $P$, and, without loss of generality, that the time scale has been transformed so that $P(t) = t$. It is further assumed that $M_1 = M_2$, which without loss of generality, can be taken to be 1, so that any candidate $F_i$ for an optimal pure strategy satisfies $F_i(1) \leq 1$. We shall discuss the game under less restrictive assumptions in the next section.
Because $K[f_1, f_2] = K[f_2, f_1]$, the value of the game (if it exists) is zero and both players have the same set of optimal strategies. Hence, it is only necessary to find an optimal strategy for one player.

**Theorem 1.** An optimal pure strategy for either player is the cumulative intensity function $F$ defined by

$$
(4.1) \quad F(t) = \begin{cases} 
0 & \text{if } t < a \\
1 + \frac{1}{2} \log (1-t) & \text{if } a \leq t < 1 \\
1 & \text{if } t \geq 1,
\end{cases}
$$

where $a = 1 - \exp[-\frac{1}{2}]$.

**Proof:** Let $G$ be the set of all cumulative intensity functions bounded by 1. It is sufficient to show that for each $G \in G$,

$$
(4.2) \quad K[F, G] \geq 0.
$$

For the moment, assume $G$ (with derivative $g$) is any function in $G$ with $G(a) = 0$. Let $K_1[F, G] = \int_a^t f(t) \log(1-t) \exp[\int_a^t f(s) \log(1-s) ds + \int_a^t g(s) \log(1-s) ds] dt$

(where $f$ is the derivative of $F$) and let $K_2[F, G] = \int_a^t g(t) \log(1-t) \exp[\int_a^t f(s) \log(1-s) ds + \int_a^t g(s) \log(1-s) ds] dt$. Then $K[F, G] = K_1[F, G] - K_2[F, G]$ and it must be shown that

$$
(4.3) \quad K_1[F, G] \geq K_2[F, G].
$$

Let $Q(t) = \exp[\int_a^t g(s) \log(1-s) ds]$. Then $K_1[F, G] = 2 \frac{3}{2} \int_a^t Q(t) \log(1-t) \frac{3}{2} \log(1-t)^{-1} dt$.

An integration by parts yields $K_1[F, G] = 1 - 2 \frac{1}{2} \int_a^t Q(t) g(t) \log(1-t) \frac{1}{2} dt$. 
On the other hand, direct calculation yields
\[
K_2[F,G] = 2^{1/2} \frac{1}{a} \int_a^{\infty} q(t)g(t)\left[-\log(1-t)\right]^{1/2}dt.
\]

Thus, the problem reduces to showing that
\[
\frac{1}{a} \int_a^{\infty} q(t)g(t)\left[-\log(1-t)\right]^{1/2}dt \leq \frac{1}{\sqrt{2}}.
\]

But,
\[
\frac{1}{a} \int_a^{\infty} q(t)g(t)\left[-\log(1-t)\right]^{1/2}dt \leq \left[\int_a^{\infty} q(t)^2g(t)^2\left[-\log(1-t)\right]^{1/2}dt\right]^{1/2} \leq \left[\int_a^{\infty} g(t)^2\left[-\log(1-t)\right]^{1/2}dt\right]^{1/2}.
\]

Thus, (4.2) is satisfied for any \(G \in G\) with \(G(a) = 0\). A standard and elementary argument establishes that (4.2) is satisfied for all \(G \in G\) since it is "unwise" for any player to shoot in \([0,a)\) because his accuracy is improving with time while his opponent (using \(F\)) is not yet shooting.

It is interesting to note that in this silent duel with continuous firing there are optimal pure strategies, while in silent duels with discrete firing the only optimal strategies are randomized. A consequence of the existence of such optimal pure strategies in the continuous-fire duel is that our solution to a silent duel with continuous firing is also a solution to a duel with continuous firing in which some or all bullets are noisy. This observation follows from the well-known "spy-proof property" of optimal strategies.
Notice that the duel just solved can be considered a limit of discrete-
fir e duels. More specifically, suppose that each player must shoot exactly one
nth of his ammunition at each of n times. Then, essentially, they would be in
a silent duel with n "bullets" and accuracy functions \( P_n(t) = 1 - (1-t)^{1/n} \). The
solution to such a duel is well known (see Restrepo [1957]): there exist a
set of n well-defined constants \( a_{ni} \) (where \( 0 < a_{n1} < a_{n2} < \ldots < a_{nn} < 1 \)) and
each player should shoot his \( k^{th} \) bullet according to a certain known probability
distribution function on \( [a_{nk}, a_{n,k+1}] \) independently of the times he fires
his other bullets. The following theorem justifies our assertion that the
continuous-fire duel proposed here is a correct generalization of the classical
discrete-fire silent duel.

**Theorem 2.** In the classical symmetric discrete-fire silent duel, let
\( \{t_{n1}, t_{n2}, \ldots, t_{nn} \} \) represent any pure strategy in the support of the optimal
randomized strategy. Let \( F_n \) be the discrete distribution function whose saltus
at points \( t_{nk} \) is \( 1/n \). Then \( F_n \) converges weakly to \( F \) given by (4.1). Hence the
solutions to the n-bullet symmetric silent duel with accuracy function \( P_n(t) = 1 - (1-t)^{1/n} \) converge to the solution (4.1) of the symmetric continuous-fire
duel with accuracy function \( P(t) = t \).

The proof of Theorem 2 involves lengthy but straightforward calculations
and is therefore omitted.

In a noisy duel in which each player has n bullets and accuracy function
\( P_n(t) = 1 - (1-t)^{1/n} \), it is well known (see Blackwell and Girshick [1954, Section
2.7] or Fox and Kimeldorf [1969]) that if each player plays optimally, he will
fire his \( k^{th} \) bullet at time
\[
t_{nk} = 1 - \left( 1 - \frac{1}{2(n+k-1)} \right)^n \quad \text{for } k = 1, 2, \ldots, n.
\]
Straightforward calculation yields the following result.

**Theorem 3.** If \( G_n \) is the discrete distribution function whose saltus at points \( t_n \) is \( 1/n \), then \( G_n \) converges weakly to \( F \). Hence the solutions to the \( n \)-bullet symmetric noisy duels with accuracy function \( P_n(t) = 1 - (1-t)^{1/n} \) converge to the solution (4.1) of the symmetric continuous-fire duel with accuracy function \( P(t) = t \).

5. **The General Case**

This section considers the continuous-fire duel defined in Section 3 under less restrictive assumptions concerning the accuracy functions and the amounts of ammunition available to the players.

The accuracy functions \( P_i \) are no longer assumed to be necessarily equal, and \( M_1 \) and \( M_2 \), the amounts of ammunition available to the players, are assumed to be any nonnegative numbers.

In this more general form, the duel is considerably more difficult to solve. The method of solution used in Section 4 seems to be inapplicable here. The methods of proof used by Karlin and Yanovsky also fail even in the equal accuracy case. For example, to apply their methods it would be necessary to assume that \( A_2'(t)/(A_1^A \lambda_2) \) is bounded on \([0,1]\), where \( A_1 = -\log[1-P_1(t)] \). However, even in the simplest case where \( P_1(t) = P_2(t) = t \), this assumption does not hold.

For \( \lambda > 0 \) and \( 0 < a < 1 \), let

\[
(5.1) \ f_{\lambda a}(t) = \begin{cases} \frac{A_2'(t)}{A_2(t)[A_1(t) + \frac{1}{\lambda} A_2(t)]} & \text{if } a < t < 1 \\ 0 & \text{otherwise} \end{cases}
\]
and

\[ \frac{A_1'(t)}{A_1(t)[\lambda A_1(t) + A_2(t)]} \text{ if } a < t < 1 \]

\[ g_{\lambda a}(t) = \begin{cases} 1 & \text{if } a < t \leq 1, \\ 0 & \text{otherwise} \end{cases} \]

where \( A_i = -\log[1 - P_i] \).

We conjecture that there exists a pair \((a, \lambda), 0 < a < 1, \lambda > 0\), such that \( \int_0^1 f_{\lambda}(t)dt = M_1, \int_0^1 g_{\lambda}(t)dt = M_2 \), that for this pair \((a, \lambda)\) the pure strategies given by (5.1) and (5.2) are optimal for the respective players, and that the value of the game is \( [\lambda A_1(a) - A_2(a)]/[\lambda A_1(a) + A_2(a)] \). We propose to furnish a proof of these conjectures in a forthcoming report.

An open problem in the study of discrete-fire silent and noisy duels is the asymptotic behavior of the optimal strategies as the numbers of bullets get large. Reasoning parallel to ours in the symmetric case suggests that our conjectured solution to the non-symmetric continuous-fire duels should provide new insights into the solution of this problem.
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