RECENT RESEARCH ON CLASSES OF LIFE DISTRIBUTIONS
USEFUL IN MAINTENANCE MODELLING

by

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Abstract

This is an expository survey paper prepared for presentation at the Logistics Research Conference, held at The George Washington University, Washington, D.C., May 8-10, 1974, and sponsored by the Office of Naval Research and the George Washington University in cooperation with the Air Force Office of Scientific Research and the Army Research Office.

Several physically motivated broad classes of life distributions have been introduced recently and shown to be natural and useful in the study of maintenance policies, such as age replacement, block replacement, and replacement at failure only. These classes include the "new better than used" class, the "new better than used in expectation" class, and their duals. Recent papers studying these classes are surveyed, and results are summarized concerning the following: (a) replacement policy comparisons, (b) models giving rise to these classes, (c) preservation of classes under standard reliability operations, (d) bounds on reliability and mean life for both individual components and systems, and (e) statistical inference for these classes. Additional classes of life distributions such as the "decreasing mean residual life" class, the "decreasing percentile residual life" class, and their duals are briefly discussed. Inclusion relationships among all of these classes and reliability classes studied earlier are exhibited.
1. Introduction.

In this paper we shall survey some classes of life distributions recently introduced for use in the study of maintenance policies.

We consider mainly two types of planned maintenance policies:

1.1. Definition. Under an age replacement policy, a unit is replaced upon failure or at age T, whichever comes first.

1.2. Definition. Under a block replacement policy, the unit in operation is replaced upon failure and at times, T, 2T, 3T, ... .

An age replacement policy is more difficult and costly to administer since it requires keeping track of the age of the device. However, it does have the advantage that fewer relatively new items are replaced.

Block replacement is generally used in the maintenance of a "block" of similar items, such as the set of neon tubes used on a given floor of an office building. Minimal record keeping is involved, since all tubes are replaced at regular intervals, and in addition, failed tubes are replaced at failure. Intuitively, it seems clear that block replacement will lead to replacement of a greater number of unfailed items.

In addition to these two planned replacement policies, we shall consider the policy defined in:

1.3. Definition. Under a replace at failure only policy, the unit is replaced only when it fails.

Note that the sequence of failure intervals corresponds to a renewal process.

In Section 2, we compare stochastically the three replacement policies as to number of failures during operation, number of planned replacements, total number of removals, etc.

We find that certain classes of life distributions arise naturally and play a crucial role in the comparison and study of these maintenance policies:
1.4. **Definition.** A life distribution $F$ (or survival function $\overline{F}$) is said to be **New Better than Used (NBU)** if

$$\overline{F}(x + y) \leq \overline{F}(x)\overline{F}(y) \quad \text{for all } x, y \geq 0.$$  \hspace{1cm} (1.1)

This means that for each $x > 0$, the probability $\overline{F}(x)$ that a new item survives a period of length $x$ is greater than the corresponding probability that an unfailed item of age $y$ survives an additional period of length $x$. Another way of stating this is that a used item has stochastically smaller remaining life length than does a new item. Mathematically, (1.1) states that $-\log \overline{F}(t)$ is superadditive.

1.5. **Definition.** A life distribution $F$ (or survival function $\overline{F}$) is said to be **New Better than Used in Expectation (NBUE)** if the mean $\mu$ of $F$ is finite and

$$\int_0^\infty [\overline{F}(t + x)/\overline{F}(t)]dx \leq \mu \hspace{1cm} (1.2)$$

for all $t \geq 0$ such that $\overline{F}(t) > 0$. Note that $\int_0^\infty [F(t + x)/F(t)]dx$ represents the conditional mean remaining life of a unit of age $t$. Hence (1.2) states that a used unit of age $t$ has smaller mean remaining life than a new unit if $F$ is NBUE.

By reversing the direction of inequality in (1.1) and (1.2) respectively, we obtain dual classes, New Worse than Used (NWU) and New Worse than Used in Expectation (NWUE). The results stated below for NBU and NBUE have obvious duals for NWU and NWUE; these dual results will not be explicitly stated.

In our discussion of the application of the NBU, NBUE, NWU, and NWUE classes in the study of maintenance policies, we will find it helpful to recall the definitions of other classes of life distributions, some of which have already played a significant role in reliability and life testing.
1.6. Definitions. A distribution function $F$ or survival function $\bar{F}$ is said to be or to have:

(i) **Increasing Failure Rate (IFR)** if $\frac{\bar{F}(x + t)}{\bar{F}(t)}$ is decreasing in $t$ whenever $x > 0$.

When $F$ has a density, this is equivalent to the condition that for some version $f$ of the density, the hazard rate $r(t) \triangleq f(t)/\bar{F}(t)$ is increasing in $t$. Also, $F$ is IFR if and only if $\log \bar{F}$ is concave. To say that the life distribution $F$ of an item is IFR is to say that the residual life length of an unfailed item of age $t$ is stochastically decreasing in $t$.

(ii) **Decreasing Mean Residual Life (DMRL)** if $\int_0^\infty \frac{[\bar{F}(x + t)/\bar{F}(t)]}{dx}$ is decreasing in $t$.

To say that the life distribution $F$ of an item is DMRL is equivalent to saying that the residual life of an unfailed item of age $t$ has a mean that is decreasing in $t$.

(iii) **Increasing Failure Rate Average (IFRA)** if $[\bar{F}(t)]^{1/t}$ is decreasing in $t > 0$.

When a failure rate exists, this is equivalent to the condition that the failure rate average $t^{-1} \int_0^t r(u)du$ is increasing in $t$. Equivalently, this condition says that $-t^{-1} \log \bar{F}(t)$ is increasing in $t$; i.e., that $-\log \bar{F}(t)$ is a starshaped function.

Dual classes are obtained by replacing "decreasing" by "increasing" and "increasing" by "decreasing" in (i), (ii), and (iii); these classes are called respectively, **Decreasing Failure Rate (DFR)**, **Increasing Mean Residual Life (IMRL)**, and **Decreasing Failure Rate Average (DFRA)**.

The inclusion relations among the classes defined above may be graphically displayed as follows:
See, for example, Bryson and Siddiqui (1969).

An additional class of distributions has recently been introduced by Haines and Singpurwalla (1974); it is defined and discussed in Section 6.
2. Replacement Policy Comparisons.

A major purpose of planned replacement policies is to minimize the probability of failure during operation. Thus it is of importance to compare stochastically the numbers of failures observed under the three types of maintenance policies described in Definitions 1.1, 1.2, and 1.3. Recall that random variable \( X \) is stochastically larger than random variable \( Y \) if \( P[X > x] \geq P[Y > x] \) for each real \( x \).

Let

\[ N(t) = \text{the number of failures during } [0, t] \text{ under a "replace at failure only" policy, i.e., the number of renewals in an ordinary renewal process}, \]
\[ \hat{N}(t) = \text{the number of renewals during } [0, t] \text{ in a stationary renewal process, i.e., a renewal process that starts at time } -\infty. \]
\[ N_A(t, T) = \text{the number of failures during } [0, t] \text{ using an age replacement policy with planned replacement age } T, \]
\[ N_B(t, T) = \text{the number of failures during } [0, t] \text{ using a block replacement policy with planned replacement interval } T. \]

Marshall and Proschan (1972) prove the following results:

\[ \hat{N}(t)_{st} > N(t) \text{ for each } t \geq 0 \iff F \text{ is NBUE.} \quad (2.1) \]

Thus the NBUE class is the largest class of life distributions for which the number of failures observed in a "replace at failure only" policy is larger stochastically in the long run than it is when the process first starts.

\[ N(t)_{st} > N_A(t, T) \text{ for each } t \geq 0, T \geq 0 \iff F \text{ is NBU.} \quad (2.2) \]

Thus the NBU class is the largest class of life distributions for which age replacement stochastically decreases the number of failures. It follows that the NBU class is the natural class of life distributions in the study of age replacement policies.
A similar result holds for block replacement policies:

\[ N(t)^{st} > N_B(t, T) \] for each \( t \geq 0, \ T \geq 0 \iff \ F \text{ is NBU.} \] (2.3)

Next we compare block replacement policies calling for different intervals between planned replacements:

\[ N_B(t, kT)^{st} > N_B(t, T) \] for each \( t \geq 0, \ T \geq 0, \ k = 1, 2, \ldots \iff \ F \text{ is NBU.} \] (2.4)

Thus the NBU class is the largest class of life distributions for which a planned replacement interval length of an integer multiple of \( T \), as compared with one of length \( T \), results in more failures stochastically.

A similar result holds for age replacement:

\[ N_A(t, kT)^{st} > N_A(t, T) \] for each \( t \geq 0, \ T \geq 0, \ k = 1, 2, \ldots \iff \ F \text{ is NBU.} \] (2.5)

A finer comparison is possible if we confine ourselves to the more highly structured class of IFR distributions:

\[ N_A(t, T_1)^{st} < N_A(t, T_2) \] for each \( T_1 < T_2, \ t \geq 0 \iff \ F \text{ is IFR.} \] (2.6)

Note that this stochastic comparison no longer requires an integer multiple of planned replacement interval \( T \).

Up till now, our comparisons have been confined within each class of replacement policies. Next we compare age versus block replacement policies. Let \( R_A(t, T) (R_B(t, T)) \) denote the number of removals during \([0, t]\), including both failed and unfailed items, using an age (block) replacement policy with planned replacement interval \( T \). Barlow and Proschan (1964) prove the following:

For each \( t > 0, \ T > 0, \)

For every life distribution, \( R_A(t, T)^{st} < R_B(t, T); \) \hspace{1cm} (2.7)

For an underlying IFR life distribution, \( N_A(t, T)^{st} > N_B(t, T). \) \hspace{1cm} (2.8)
3. Models for the NBU and NBUE Classes.

A number of physically motivated models have been proposed which yield the NBU and NBUE classes of life distributions.

Coherent systems of repairable components.

Ross (1974) considers a coherent system of components. (See Barlow and Proschan, 1974, Chapters 1 and 2, for a discussion of coherent systems.) Component $i$ has exponential lifelength with failure rate $\lambda_i$; upon failure, repair is initiated requiring exponential repair period with repair rate $v_i$. All lifelengths and repair periods are mutually independent. Ross proves that the time to the first system failure has an NBU distribution.

He further notes, by contrast, the interval of time between later successive system failures need not be NBU.

Shock models. Esary, Marshall, and Proschan (1973) consider a device subject to shocks occurring in time according to a Poisson process with shock rate $\lambda$. The probability that the device survives $k$ shocks is $\bar{F}_k$, where $1 = \bar{F}_0 \geq \bar{F}_1 \geq \bar{F}_2 \geq \cdots$. Then the survival probability $\bar{H}(t)$ over time is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \bar{F}_k \quad \text{for } t \geq 0.$$  \hspace{1cm} (3.1)

Esary, Marshall, and Proschan obtain results in which various classes of life distributions are "preserved" under the transformation (3.1). Specifically, they prove:

3.1. Theorem. (a) If $\bar{F}_k$ is discrete NBU (i.e., $\bar{F}_{k+\ell} \leq \bar{F}_k \bar{F}_\ell$ for each $k = 0, 1, 2, \ldots$; $\ell = 0, 1, 2, \ldots$), then $\bar{H}(t)$ is NBU.

(b) If $\bar{F}_k$ is discrete NBUE (i.e., $\sum_{j=0}^{\infty} \bar{F}_j \geq \sum_{j=0}^{\infty} (\bar{F}_{k+j}/\bar{F}_k)$ for $k = 0, 1, 2, \ldots$), then $\bar{H}(t)$ is NBUE.
A-Hameed and Proschan (1973, 1975) obtain similar preservation results in the more general models:

(1) Shocks occur according to a nonstationary Poisson process.
(2) Shocks occur according to a birth process.

Preservation of classes of life distributions. An important question in formulating classes of life distributions is the following:

For which standard reliability operations is the class of life distributions closed?

For example, is the convolution of NBU distributions itself NBU? Note that the convolution of distributions corresponds to the addition of independent lifelengths; such an operation arises routinely in the determination of spares kits.

We summarize the situation in Table 3.1:

**Table 3.1. Preservation of Life Distribution**

**Classes Under Reliability Operations.**

<table>
<thead>
<tr>
<th>Class</th>
<th>Formation of Coherent Systems</th>
<th>Convolutions</th>
<th>Arbitrary Mixtures</th>
<th>Mixtures of Distributions That Don't Cross</th>
</tr>
</thead>
<tbody>
<tr>
<td>NBU</td>
<td>Preserved</td>
<td>Preserved</td>
<td>Not Preserved</td>
<td>Not Preserved</td>
</tr>
<tr>
<td>NBUE</td>
<td>Not Preserved</td>
<td>Preserved</td>
<td>Not Preserved</td>
<td>Not Preserved</td>
</tr>
<tr>
<td>NWU</td>
<td>Not Preserved</td>
<td>Not Preserved</td>
<td>Not Preserved</td>
<td>Preserved</td>
</tr>
<tr>
<td>NWUE</td>
<td>Not Preserved</td>
<td>Not Preserved</td>
<td>?</td>
<td>Preserved</td>
</tr>
</tbody>
</table>

The operation "formation of coherent systems" refers to the situation in which a coherent system is formed of independent components, not subject to repair. An arbitrary mixture $F$ of distributions $F_1, \ldots, F_n$ is given by:

$$F = p_1 F_1 + \cdots + p_n F_n,$$

(3.2)
where each \( p_i \geq 0 \) and \( \sum_1^n p_i = 1 \). Finally, distributions \( F_1 \) and \( F_2 \) are said not to cross if there is no pair \( t_1, t_2 \) such that \( F_1(t_1) - F_2(t_1) > 0 \) and \( F_1(t_2) - F_2(t_2) < 0 \).

The preservation results summarized in Table 3.1 are proved in Marshall and Proschan (1972) and Esary, Marshall, and Proschan (1970).
4. Bounds for the NBU and NBUE Classes.

Bounds for individual components. Marshall and Proschan (1972) develop the following simple bound for the NBU distribution.

4.1. Theorem. Let \( F \) be NBUE with mean \( \mu \). Then

\[
F(t) \leq \frac{t}{\mu} \quad \text{for} \quad t \leq \mu. \tag{4.1}
\]

The bound is sharp.

It is interesting to note that the bound (4.1) cannot be improved even if \( F \) is restricted to the smaller NBU class. Haines and Singpurwalla (1974) do obtain a stronger bound for the NBUE class by assuming additional information is known, as stated in:

4.2. Theorem. Let \( F \) be NBUE with mean \( \mu \) and \( \overline{F}(t_o) = \alpha \) for some \( 0 \leq t_o \leq \mu \). Then

\[
\overline{F}(t) \geq \begin{cases} 
\max (\alpha, \frac{\mu - t}{\mu}) & \text{for} \quad 0 \leq t \leq t_o \\
\frac{1}{\mu} [\mu - t_o - \alpha (t - t_o)] & \text{for} \quad 0 \leq t_o \leq t \leq \beta,
\end{cases}
\]

where \( \beta = \frac{\mu + t_o \alpha - t}{\alpha} \).

For the NBU class, Marshall and Proschan (1972) present the following bound.

4.3. Theorem. Let \( F \) be NBU and \( \overline{F}(t_o) = \alpha \). Then

\[
\overline{F}(t) \geq \alpha^{1/k} \quad \text{for} \quad \frac{t_o}{k + 1} < t \leq \frac{t_o}{k} \quad \text{and} \quad k = 1, 2, \ldots ,
\]

while

\[
\overline{F}(t) \leq \alpha^k \quad \text{for} \quad kt_o \leq t < (k + 1)t \quad \text{and} \quad k = 0, 1, 2, \ldots .
\]

These bounds are sharp.
The upper bound is itself an NBU survival function; the lower bound is not.

**Bounds on system mean life.** The bounds given above have been for individual NBU and NBUE components. Marshall and Proschan (1970, 1972) give bounds for the mean life of series and parallel systems of NBUE components.

**4.4 Theorem.** Let \( \mu_s(\mu_p) \) be the mean life of a series (parallel) system of \( n \) independent NBUE components with mean lives respectively of \( \mu_1, \ldots, \mu_n \).

Then

\[
\mu_s \geq \left( \sum_{i=1}^{n} \mu_i^{-1} \right)^{-1},
\]

and

\[
\mu_p \leq \int_0^\infty [1 - \prod_{i=1}^{n} (1 - e^{-x/\mu_i})] dx.
\]

The bounds are sharp.

Note that the bounds represent respectively, the mean life of a series and parallel system of exponential components.
5. Statistical Inference.

Given a sample $X_1, \ldots, X_n$ from life distribution $F$, Hollander and Proschan (1972) propose a test of the hypothesis,

$H_0$: The distribution $F$ is exponential with unspecified scale parameter, vs. the alternative hypothesis,

$H_1$: The distribution $F$ is NBU (and not exponential).

The test statistic proposed is motivated by consideration of the parameter

$$\gamma(F) \overset{\text{def}}{=} \int \left[ (F(x)F(y) - F(x + y))dF(x)dF(y) \right]. \quad (5.1)$$

Note that the integrand is identically 0 when $F$ is exponential, and nonnegative when $F$ is NBU.

The test statistic is developed by first replacing in (5.1) the unknown distribution $F$ by the empirical distribution $F_n$. Next the $U$-statistic which is asymptotically equivalent is used, since $U$-statistics have many desirable properties and a fully developed theory.

The test statistic essentially counts the number of triples of ordered observations $X_{(i)} < X_{(j)} < X_{(k)}$ such that $X_{(i)} + X_{(j)} > X_{(k)}$, and rejects for large values. The statistic is unbiased, asymptotically normal, and is consistent.

The asymptotic relative efficiency of the test statistic is determined relative to statistics designed against IFR alternatives (since no other test statistics have yet been proposed against NBU alternatives). The NBU test statistic proposed shows reasonably good asymptotic relative efficiency, especially if one takes into account the fact that the class of IFR alternatives is more restricted than is the class of NBU alternatives.

To permit application of the test, small sample null tail probabilities are derived, and additional critical values are obtained by Monte Carlo sampling. Tables are provided for sample sizes up to 50. For larger sample sizes, the
asymptotic normality may be used, along with the calculated asymptotic mean and variance.

Other than the test for NBU, statistical inference for the NBU, NBUE, NWU, and NWUE has not been developed as yet. A useful contribution would be to develop an estimator with desirable properties for say, the NBU class of distributions.
6. Related Classes of Life Distributions.

Several other classes of life distributions have been proposed for use in the study of maintenance policies. In certain respects, they seem less appropriate for maintenance modelling than do the NBU and NBUE classes and their duals, discussed above. However, we summarize some of the recent work on these classes. We consider in particular the DMRL class, defined in 1.6, and the following new class introduced and studied by Haines and Singpurwalla (1974):

6.1. Definition. A distribution $F$ has Decreasing Percentile Residual Life (DPL) if for some $\alpha$, $0 < \alpha < 1$, the $100\alpha$th percentile of the residual life of an item of age $t$ decreases in $t \geq 0$.

A DPL distribution is somewhat similar to a DMRL distribution in that in both cases, a parameter measuring residual life is a decreasing function of age. Haines and Singpurwalla (1974) develop a number of properties of the DPL class, and relate it to classes of life distributions developed earlier. A typical result states:

6.2. Theorem. An IFR distribution is DPL for each value of $\alpha$ in $(0, 1)$.

One seeming disadvantage of the DPL class is that except for the inclusion stated in Theorem 6.2, the DPL class neither contains nor is contained in any of the other classes of life distributions. Another somewhat discouraging aspect of the DPL class is that none of the preservation properties shown in Table 3.1 can be claimed for either the DPL or its dual class.

Haines and Singpurwalla obtain bounds on the survival function of both the DPL and the DMRL distribution functions having a known mean and a percentile. They present a series of graphs portraying various bounds for the various classes of distributions; the graphs show how the bound improves as additional information is used.

Finally, we mention an interesting display of the empirical mean residual life for a group of cancer patients, shown in Bryson and Siddiqui (1969). The
graph shows quite dramatically that the mean residual lifelength is decreasing in time measured from diagnosis of cancer; i.e., the population is DMRL. It would be desirable to have an estimator for the DMRL class having optimal properties.
REFERENCES


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14. Key Words

- Maintenance policies
- Life distributions
- Survey paper
- Age replacement
- Block replacement
- New better than used
- New better than used in expectation
- Decreasing mean residual life
- Decreasing percentile residual life