CONTINUOUS NONLINEAR PROGRAMMING WITH TWO SPATIAL DIMENSIONS: A RANDOM SEARCH PROBLEM

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SUMMARY

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A search problem is presented. A nonlinear programming problem with two spatial dimensions is proposed as a solution. Necessary and sufficient conditions are given for the existence of solutions. Duality, complementary slackness, and Kuhn-Tucker theorems are proven.

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1. Introduction: The Problem.

Consider the problem of searching a specified area of the ocean floor by means of a drogue equipped with scanning devices and a computer that can be programmed to control two aspects of the drogue's movement, namely the time spent stationary in any position and its depth. Let \( z(x,y) = (z_1(x,y), z_2(x,y)) \) denote the control function, where \( z_1(x,y) \) and \( z_2(x,y) \) represent the time spent and the depth at coordinates \( (x,y) \). It is assumed that, a priori, some regions are of greater interest than others, which will be reflected in the values to be taken by these controls. The drogue is towed successively to various starting points and, apart from control of \( z_1 \) and \( z_2 \), is allowed to drift with the current. We consider one such sweep starting at \( (x,y) = (0,0) \). It is known that the current in the vicinity of \( (0,0) \) will carry an object from \( (0,0) \) to point \( (X,Y) \) along some fixed, though deterministically unknown, path.
The probability density for the passage of the current through point \((x,y)\) is given by \(h(x,y)\). The value of the information gathered is measured by an increasing differentiable function \(\phi\) of \(z\). Restrictions are provided on \(z\) by the contour of the ocean bottom, denoted by \(c_2(x,y)\) and the maximum time necessary to obtain all information available at \((x,y)\) denoted by \(c_1(x,y)\). The fuel used to control \(z_1\) and \(z_2\) is limited by a maximum capacity \(F\). We may then formulate the problem as follows:

\[
\begin{align*}
\text{Maximize } & \int_0^x \int_0^y \phi(z_1(x,y), z_2(x,y)) \, dx \, dy, \\
\text{subject to constraints } & z(x,y) \geq 0, \\
& z_1(x,y) \leq c_1(x,y), \\
& z_2(x,y) \leq c_2(x,y), \\
& 0 \leq F - \int_0^x \int_0^y g(z(u,v), (x,y), (u,v)) h(u,v) \, du \, dv.
\end{align*}
\]

In Section 2 we introduce a class of continuous nonlinear programming problems which includes the above described problem. With mild restrictions on \(\phi, h,\) and \(g\), such as concavity and \(g(0,0,0,0) = 0\), we prove in Theorem 3.1 that an optimal control policy exists.
Duality theorems and a form of the Kuhn-Tucker Theorem are presented in later sections. Our approach to the problem is an extension of the work of Larson and Hanson (1973). Some of the proofs follow straightforwardly and hence are omitted.

Additional comments and remarks are given in Section 7.


Consider the following problem:

**Primal Problem I.**

Maximize

\[
\ell(z) = \int \int f(z(x,y)) \, dx \, dy,
\]

subject to

\[
f(z(x,y), (x,y)) \leq c(x,y) + h(Y(z,(x,y)), (x,y))
\]

for

\[
(0,0) \leq (x,y) \leq (\bar{x}, \bar{y})
\]

\[
z(x,y) \geq 0
\]

and

\[
z(x,y) = 0 \text{ if } \min \{x,y\} < 0
\]

Here \(z(x,y)\) is bounded measurable \(n\)-dimensional function (i.e. \(z(x,y) \in L^\infty_n(\mathbb{R})\)), \(Y(z,(x,y))\) is a mapping from \([L^\infty_n(\mathbb{R})] \times \mathbb{R}\) into \(E^p\) given by
\[ Y(z,(x,y)) = \int_{0}^{x} \int_{0}^{y} g(z(u,v),(x,y),(u,v)) \, du \, dv, \quad (2.5) \]

\( f(z(x,y),(x,y)), c(x,y), \) and \( h(Y(z,(x,y))) \in \mathbb{E}^{m}, \) and \( g(z,u,v),(x,y),(u,v)) \in \mathbb{E}^{p}. \)

The closed rectangle \([0,X] \times [0,Y]\) is represented by \( R\) and \( f \) is a real-valued, concave, differentiable function. Throughout this paper no notational distinction between row and column vector is made. The transpose of a matrix \( A \) will be denoted by \( A^{T} \).

It will be assumed that all functions are bounded and measurable. Further, \( f = (f_{1}, \ldots, f_{m}) \), where \( f_{i} \) is a real-valued convex function, differentiable in its first argument throughout \( R \) and satisfies

\[ V_{k} f_{i} = \frac{\partial f_{i}}{\partial z_{k}} \geq 0, \quad k = 1, \ldots, n, \quad i = 1, \ldots, m, \text{ on } R, \quad (2.6) \]

\[ V_{k} f_{i} > 0 \text{ for some } i, \quad (2.7) \]

\[ f_{i}(0,(x,y)) = 0. \quad (2.8) \]

The function \( g = (g_{1}, \ldots, g_{p}) \) is such that \( g_{j} \) is a real-valued concave function, differentiable in its first argument throughout \( R \) with the property

\[ g_{j}(z(u,v),(x,y),(u,v)) = 0 \text{ if } \min \{u,v,x,y\} < 0. \quad (2.9) \]

The function \( h = (h_{1}, \ldots, h_{m}) \) is such that \( h_{i} \) is a real-valued concave function, differentiable in its first argument throughout \( R \), satisfying

\[ V_{k} h_{i} = \frac{\partial h_{i}}{\partial y_{k}} \geq 0 \quad k = 1, \ldots, p, \text{ on } R. \quad (2.10) \]
3. **Existence Theorems.**

The set of all \( z(\cdot, \cdot) \) which satisfy constraints (2.2) \(- (2.4) \) will be denoted by \( F \). An element of \( F \) is said to be a feasible solution to Primal Problem I. If there exists \( z^* \in F \) such that for all \( z \in F \) \( \ell(z^*) \geq \ell(z) \) then we say \( z^* \) is an optimal solution for Primal Problem I. In this section we show that the existence of a feasible solution is sufficient for the existence of an optimal solution.

**THEOREM 3.1.** If \( F \) is nonempty then there exists an optimal solution to Primal Problem I.

The proof of the theorem uses the following three lemmas. The first is a slight generalization of Levinson’s (1966) Lemma 2.1.

**LEMMA 3.2.** Let the uniformly bounded real-valued measurable functions

\[
\{f_j(x,y)\}_{j=1}^{\infty}
\]

converge weakly to \( f_0(x,y) \) on \( R \). Let

\[
\lim_{j \to \infty} \sup_j f_j(x,y) = f_u(x,y).
\]

Then, except on sets of measure zero

\[
f_0(x,y) \leq f_u(x,y) \quad \text{on } R.
\]

**LEMMA 3.3.** Let the real-valued measurable function \( g(x,y) \) satisfy

\[
g(x,y) \leq c \int \int g(u,v) \, du \, dv \quad c \geq 0
\]

(3.1)

for all \( (x,y) \in R \). Then \( g(\cdot, \cdot) \leq 0 \) on \( R \).
PROOF. If \( xy = 0 \) then the result is immediate. Suppose then that \( xy > 0 \).

Let \( N = \int_{0}^{X} \int_{0}^{Y} g(u,v) \, du \, dv \), then clearly \( g(\cdot, \cdot) \leq N \) on \( R \). We will show that for all natural numbers \( n \) and for all \( (x,y) \in R \)

\[
g(x,y) \leq N(cx)^{n}/n!
\]

(3.2)

For \( n = 0 \) (3.2) holds trivially. Assume that (3.2) holds for \( n = k \). It follows from (3.1) that for all \( (x,y) \in R \)

\[
x \quad y
\]

\[
g(x,y) \leq c \int_{0}^{X} \int_{0}^{Y} \frac{[(Ncxv)^{k}/k!]}{du \, dv}
\]

\[
= (N/(k+1))[(cx)^{k+1}/k!]
\]

\[
\leq [N(cx)^{k+1}]/(k+1)!
\]

Hence by induction (3.2) holds for all \( n \). Then \( g(x,y) \leq N(cx)^{n}/n! \) for all \( n \) and thus \( g(\cdot, \cdot) \leq 0 \) on \( R \).

**Lemma 3.4.** Let the real-valued measurable function \( g \) satisfy for \( c_{1} \geq 0 \) and \( c_{2} > 0 \)

\[
x \quad y
\]

\[
g(x,y) \leq c_{1} + c_{2} \int_{0}^{X} \int_{0}^{Y} g(u,v) \, du \, dv
\]

(3.3)

for all \( (x,y) \in R \). Then there exists a real-valued measurable function \( G \) satisfying (3.3), such that \( G(x,y) \geq g(x,y) \) for \( (x,y) \in R \).

**Proof.** The conditions on \( c_{1} \) and \( c_{2} \) ensure the existence of a real valued measurable function \( G(x,y) \), which satisfies the equality in (3.3). Let \( \varphi(x,y) = g(x,y) - G(x,y) \), then \( \varphi \) satisfies the conditions of Lemma 3.3 and hence \( g(x,y) \leq G(x,y) \) for all \( (x,y) \in R \).
Lemmas 3.3 and 3.4 are similar to Lemma 1 and Theorem 2 of Rasmussen (1973), but his condition of the continuity of $g$ is relaxed since $c_1$ and $c_2$ are constants. It should be noted that if $g$ is bounded below by zero then so is $G$ and furthermore $g(x,y) \leq G(x,y)$.

**Proof of Theorem 3.1.** By arguments similar to those used in Larson and Hanson (1973) it can be established that there exist constants $c_1 \geq 0$ and $c_2 > 0$ such that

$$\sum_{k=1}^{n} z_k(x,y) \leq c_1 + c_2 \int_{0}^{x} \int_{0}^{y} \sum_{k=1}^{n} z_k(u,v) \, du \, dv.$$  

By Lemma 3.4, or rather by the remark following it, there exists a real-valued measurable function $G(x,y)$ such that

$$\sum_{k=1}^{n} z_k(x,y) \leq G(x,y) \leq G(x,y)$$  

(3.4)

for all $(x,y) \in \mathbb{R}$. Thus elements of $F$ are uniformly bounded on $\mathbb{R}$.

From here the proof follows from techniques similar to those used in Larson and Hanson (1973) and is omitted.

4. **Weak Duality.**

Define

$$
\Psi(z, (x,y)) = c(x,y) + h(Y(z, (x,y)), (x,y)) - f(z, (x,y), (x,y)).
$$

It will be shown in this section that the dual of the primal problem I is Dual Problem I

Minimize

$$k(u,w) = \int_{0}^{Y} \int_{0}^{X} \{ \phi(u(x,y)) - u(x,y)[\nabla \phi(u(x,y))] \}$$
\[ + w(x,y)\psi(u(x,y)) + u(x,y)[Vf(u(x,y),(x,y))]w(x,y) \]
\[ - \int \int u(x,y)H^T(u(t,s),(x,y))w(t,s)dt\,ds\,dx \,dy \]
subject to
\[ u(x,y), w(x,y) \geq 0 \quad (4.1) \]
and
\[ [Vf(u(x,y),(x,y))]w(x,y) \geq \phi(u(x,y)) + \int \int H^T(u(t,s),(x,y))w(t,s)dt\,ds \quad (4.2) \]
for all \((x,y) \in \mathbb{R}\), and
\[ w(x,y) = 0 \quad (4.3) \]
for all \((x,y) > (X,Y)\). Here
\[ H(u(t,s),(x,y)) = I_R(x,y)[\psi(Y(u(t,s),(t,s)))]Vg(u(x,y),(t,s),(x,y))], \]
where \(\psi(Y(u(t,s),(t,s))\) and \(Vg(u(x,y),(t,s),(x,y))\) denote the \(m \times p\) and \(p \times n\) gradient matrices of \(h\) and \(g\) evaluated at the points \(Y(u(t,s))\) and \(u(x,y)\) respectively, and
\[ I_R(x,y) = 1 \text{ if } (x,y) \in \mathbb{R} \]
\[ = 0 \text{ otherwise.} \]

Theorem 4.1 is presented as a weak form of duality, stating that for feasible solutions the value of the objective function for the dual problem is greater than or equal to that of the primal problem. Denote by \(\mathcal{G}\) the class of all feasible solutions to Dual Problem I. That is, \(\mathcal{G}\) is the set of all \((u,w)\) which satisfy (4.1) - (4.3).
THEOREM 4.1. Let \( z \in F \) and \((u,w) \in G\),
then \( \ell(z) \leq k(u,w) \).

Lemma 4.2 is an easy consequence of Fubini's Theorem. It is needed in the proof of Theorem 4.1.

**Lemma 4.2.**

\[
\int \int \int w(x,y)h(z(x,y),(t,s))z(t,s)dtdsdxdy
\]

\[
\int \int \int z(x,y)h^T(z(t,s),(x,y))w(t,s)dtdsdxdy.
\]

For clarity we make the further identification:

\[
\Gamma(u,w,(x,y)) = V\phi(u(x,y)) - [Vf(u(x,y),(x,y))]w(x,y)
\]

\[
+ \int \int H^T(u,(t,s),(x,y))w(t,s)dtds.
\]

Theorem 4.3 establishes a sufficient condition for the optimality of \( z \) and \((u,w)\).

THEOREM 4.3. Let \( z \in F \) and \((u,w) \in G\). Suppose that \( \ell(z) = k(u,w) \). Then \( z \) is optimal for Primal Problem I and \((u,w)\) is optimal for Dual Problem I.

5. Constraint Qualification.

To obtain strong duality, a constraint qualification similar to that of Kuhn and Tucker (1951) are needed. We proceed as in Larson and Hanson (1973).

**Definition 5.1.** For each \( z \in F \) define \( D(z) = \{ \gamma : z + \tau \gamma \in F \text{ if } \tau \in [0,0] \text{ for some } \sigma > 0 ; \gamma \in L^\infty_n(\mathbb{R}) \} \).
DEFINITION 5.2. Define $\bar{D}(z)$ to be the closure of $D(z)$ under the $L_\infty^n$ norm. That is, $\gamma \in \bar{D}(z)$ if there exist $\{\gamma(j)\} \in D(z)$ such that $\max_{kk} \|\gamma^{(j)}_k - \gamma_k\| \to 0$ as $j \to \infty$.

Let $\delta\Psi(z;\gamma)(x,y)$ denote the Frechet differential of $\Psi(z;\gamma)$ at the point $z$. Since $f$, $h$, and $g$ are differentiable $\delta\Psi(z;\gamma)(x,y)$ exists and is given by

$$\delta\Psi(z;\gamma)(x,y) = \int\int h(z,(x,y),(t,s))dt\,ds$$

$$- [\nabla f(z(x,y),(x,y))]\gamma(x,y).$$

DEFINITION 5.3. Let $z \in F$, then define $D(z)$ to be the set of all $\gamma$ satisfying

i) $\gamma \in L_\infty^n(\mathbb{R})$

ii) $\gamma(x,y) = 0$ if $\min \{x,y\} < 0$

iii) $\gamma_k(x,y) \geq 0$ a.e. on $\{(x,y) \in \mathbb{R} : Z_k(x,y) = 0\}$,

$$k = 1,\ldots,n,$$ and

iv) $\delta\Psi_i(z;\gamma)(x,y) \geq 0$ a.e. on $\{(x,y) \in \mathbb{R} : \gamma_i(x,y) = 0\}, \quad i = 1,\ldots,m$.

LEMMA 5.4. $\bar{D}(z) \subseteq D(z)$.

CONSTRAINT QUALIFICATION 5.5. If $z$ is an optimal solution to Primal Problem 1, then it is assumed that $\bar{D}(z) = D(z)$.

Theorems of strong duality, complementary slackness, and related results will be proved under the following two assumptions:

\[ H(\tilde{z}, (x,y), (t,s)) \geq 0, \ (t,s) \leq (x,y) \in R \]  

(6.1)

and

\[ \psi(\tilde{z}, (x,y)) - \delta \psi(\tilde{z}; z) (x,y) \geq 0, \ (x,y) \in R. \]  

(6.2)

Here \( \tilde{z} \) is an optimal solution for Primal Problem I. In Lemma 6.2 it will be proven that (6.2) is satisfied if \( z(x,y) \equiv 0 \) is feasible.

**Theorem 6.1.** Under Constraint Qualification 5.5 and assumptions (6.1) and (6.2), there exists an optimal solution \((\tilde{u}, \tilde{w})\) for Dual Problem I. Furthermore, \( \tilde{u} = \tilde{z} \), where \( z \) is the optimal solution for Primal Problem I, also \( \mathcal{L}(z) = k(\tilde{z}, \tilde{w}) \).

**Lemma 6.2.** Let \( z \equiv 0 \) be an element of \( F \). Then (6.2) holds.

Lemma 6.2 yields the following corollary of Theorem 6.1.

**Corollary 6.3.** Suppose \( 0 \in F \). Then under Constraint Qualification 5.5 and assumption (6.1) there exists an optimal solution \((\tilde{u}, \tilde{w})\) for Dual Problem I. Furthermore, \( \tilde{u} = \tilde{z} \), where \( \tilde{z} \) is the optimal solution for Primal Problem I, also \( \mathcal{L}(\tilde{z}) = k(\tilde{z}, \tilde{w}) \).

It should be noted that (6.1) holds if \( \sum_{k=1}^{n} g_i \geq 0, \ k = 1, \ldots, n, \ i = 1, \ldots, p \).

**Theorem 6.4.** Suppose \( \tilde{z} \) and \( \tilde{w} \) are optimal for Primal and Dual Problem I. Then

\[ \int_{0}^{X} \int_{0}^{Y} \psi(\tilde{z}, (x,y)) dx dy = 0 \]  

(6.3)

and
\[ Y^X \int \int z(x,y) \Gamma(\tilde{z},\tilde{v}, (x,y)) \, dx \, dy = 0. \]

0 0

Theorem 6.4 is the Complementary Slackness Principle.

**Theorem 6.5.** (Kuhn-Tucker Theorem) A necessary and sufficient condition for the optimality of \( \tilde{z} \in F \) is the existence of a function \( w_0(x,y) \) satisfying

i) \( \Gamma(\tilde{z}, w_0, (x,y)) \leq 0 \) \( (x,y) \in R \),

\[ Y^X \int \int \tilde{z}(x,y) \Gamma(\tilde{z}, w_0, (x,y)) \, dx \, dy = 0, \]

0 0

ii) \( \int \int \tilde{z}(x,y) w_0(x,y) \, dx \, dy = 0, \)

\[ Y^X \]

0 0

iii) \( \int \int w_0(x,y) \Psi(\tilde{z}, (x,y)) \, dx \, dy = 0, \)

0 0

iv) \( w_0(x,y) I_R(x,y) = 0. \)

7. **Concluding Remarks.**

It should be noted that the extension to two continuous variables of the works of Farr and Hanson (1974a), Farr and Hanson (1974b), Levinson (1966), and Grinold (1969) can be accomplished without difficulty in a fashion similar to ours. In fact, the extension of Grinold's Duality Theorem (1969) is assumed for the proof of Theorem 6.1.

Generalizations to more than two continuous variables can be achieved if a Gronwall type lemma is proven for the appropriate number of variables.
REFERENCES


