A CLASS OF CONTINUOUS NONLINEAR
COMPLEMENTARITY PROBLEMS

by

Erwin P. Bodo¹ and Morgan A. Hanson

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The Florida State University
Department of Statistics
Tallahassee, Florida

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SUMMARY

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The concept of continuous nonlinear complementarity is defined. Basic properties and existence theorems are proven. Applications to continuous linear and nonlinear programming are presented. Kuhn-Tucker type conditions are established.

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1. Introduction.

In recent years many authors, including Cottle (1966), Cottle and Dantzig
(1968), and Dorn (1961) have been investigating complementarity; that is, the
question of existence of an n-vector $x$ which satisfies the system of inequalities

$$x \geq 0, \quad Mx + b \geq 0, \quad x(Mx + b) = 0,$$

(1.1)

where $M$ is an $n \times n$ matrix and $b$ is an $n$-vector of real numbers.

Complementarity problems are elegant generalizations of certain linear pro-
gramming, quadratic programming, and game theory problems. This unifying
concept can be expanded to include a broader class of problems such as non-
linear programming by appropriately modifying (1.1); namely, consider the
problems of finding an $n$-vector which satisfies the system of inequalities

$$x \geq 0, \quad f(x) \geq 0, \quad xf(x) = 0,$$

(1.2)

where $f$ is a mapping of $E^n$ into itself. A system of the type (1.2) has
been studied among others by Cottle (1966), who gave sufficient conditions for
the existence of \( x \), and by Karamardian (1969), who established sufficient conditions for the existence of a unique solution.

We extend the results of Karamardian (1969) to the case where \( x \) is a bounded measurable function which maps some finite interval into \( E^n \). This is also a generalization of the work of Farr and Hanson (1974a, 1974b) and Larson and Hanson (1974). The development of this paper parallels that of Karamardian's (1969). Some of the proofs are similar in nature.

In Section 2 a class of continuous complementarity problems is defined and basic properties for a general \( f \) are presented. Sufficient conditions for the existence of a unique solution are given in Section 3. In Section 4 the case of a differentiable \( f \) is discussed. Finally, Section 5 contains applications to continuous linear and nonlinear programming. Kuhn-Tucker type conditions are also given therein.

Throughout this paper all functions considered are assumed to be bounded and measurable. \( L^\infty_n \) is viewed as a topological metric space with the usual norm metric \( || \cdot ||_\infty \). Hence all functions that are equal everywhere, except perhaps on a set of measure zero are considered to be identical. No notational distinction is made between row and column vectors. The transpose of a matrix \( A \) will be denoted by \( A^T \).

2. The Continuous Complementarity Problem.

We consider the question of the existence of a function \( x : [0, T] \rightarrow E^n \) which satisfies the following system of inequalities:

\[
x(\cdot) \geq 0, \quad f(x, \cdot) \geq 0, \quad x(\cdot)f(x, \cdot) = 0.
\] (2.1)
Here \( f : L_0^n[0,T] \times [0,T] \to E^n \) and is continuous in its first argument.

Let \( P \) denote the class of functions with domain \([0,T]\) and range \( E_+ \), the set of positive reals. For each \( p \) in \( P \) define two sets

\[
S_p = \{ x | x \in L_0^n[0,T], x^{(*)} \geq 0, \text{ex}^{(*)} = p^{(*)} \} \tag{2.2}
\]

and

\[
S_p^* = \{ x | x^{(*)} = \sum_{i=1}^n \alpha_i e_i, \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0 \} \tag{2.3}
\]

where \( e = (1, \ldots, 1) \) is the summation vector and \( e_i \) is the \( i \)-th unit vector in \( E^n \).

**Definition 2.1.** A set \( A \in L_0^n[0,T] \) is closed if for each sequence \( \{ a^{(j)} \}_{j=1}^\infty \) in \( A \) for which there exists an \( a^0 \in L_0^n[0,T] \) such that \( \max_k ||a^{(j)}_k - a^0_k||_\infty \to 0 \) as \( j \to \infty \), \( a^0 \in A \).

From (2.2) and (2.3) it is evident that \( S_p^* \) is a proper subset of \( S_p \) for each \( p \) in \( P \). Further basic properties of \( S_p^* \) are established in Lemma 2.2.

**Lemma 2.2.** For each \( p \in P \), \( S_p^* \) is a nonempty, compact, convex subset of \( L_0^n[0,T] \).
PROOF. For each \( i, \ i = 1, \ldots, n, \) \( p(\cdot)e_i \) is in \( S_p^* \) and hence \( S_p^* \) is nonempty. By virtue of its definition \( S_p^* \) is the convex hull of \( \{p(\cdot)e_i\}_{i=1}^n \) and is therefore convex, which, together with the Bolzano-Weierstrass Theorem ensures the compactness of \( S_p^* \).

The main result of this section and the foundation for the remaining sections is Theorem 2.5. Its proof uses a result of Himmelberg (1972) which we include for completeness as Theorem 2.3. The assumptions for Himmelberg's Theorem are established in Lemma 2.4.

**THEOREM 2.3.** (Himmelberg) Let \( A \) be a nonvoid convex subset of a separated locally convex space. Let \( g: A \to A \) be an upper semicontinuous (u.s.c.) point to set function such that \( g(a) \) is closed and convex for each \( a \in A \) and \( g(A) \) is contained in some compact subset of \( A \). Then there exists \( \bar{a} \in A \) such that \( \bar{a} \in g(\bar{a}) \).

For each \( p \in P \) define a mapping \( \Gamma_p(x) \) on \( S_p^* \) by

\[
\Gamma_p(x) = \{y \in S_p^*, \ y(\cdot)f(x, \cdot) = \min_{z \in S_p^*} z(\cdot)f(x, \cdot)\}. \tag{2.4}
\]

**LEMMA 2.4.** For each \( p \in P, \) \( \Gamma_p \) is an u.s.c. point to set mapping. Furthermore, for each \( x \in S_p^*, \) \( \Gamma_p(x) \) is a closed, convex subset of \( S_p^* \).

**PROOF.** From the continuity of \( f(x, \cdot) \) and the compactness of \( S_p^* \) it follows that \( \Gamma_p(x) \) is nonempty. Hence \( \Gamma_p \) is a point to set mapping.
To show upper semicontinuity we prove that both \( \Gamma_p(x) \), for each \( x \in S_p^* \), and \( D = \{ (x, y) \mid x \in S_p^*, \ y \in \Gamma_p(x) \} \) are closed subsets of \( S_p^* \) and \( S_p^* \times S_p^* \), respectively. Clearly \( S_p^* \) is closed. So let \( x \) be an element of \( S_p^* \) and let \( \{ y^{(j)} \} \_{j=1}^\infty \) be a sequence in \( \Gamma_p(x) \) converging to \( y^0 \). Since \( S_p^* \) is closed \( y^0 \in S_p^* \). For every \( j \), \( y^{(j)} \in \Gamma_p(x) \) and hence by definition,

\[
y^{(j)}(\cdot)f(x, \cdot) \leq z(\cdot)f(x, \cdot), \text{ for each } z \in S_p^*. \tag{2.5}
\]

Since the right side of (2.5) is independent of \( j \) we have, in the limit,

\[
y^0(\cdot)f(x, \cdot) \leq z(\cdot)f(x, \cdot), \text{ for each } z \in S_p^*. \tag{2.6}
\]

Hence \( \Gamma_p(x) \) is closed. Consider a sequence \( \{(x^{(j)}, y^{(j)})\}_{j=1}^\infty \) in \( D \) converging to \( (x^0, y^0) \). \( S_p^* \) is closed and thus \( (x^0, y^0) \in S_p^* \times S_p^* \). Furthermore, for every \( j \), \( y^{(j)} \in \Gamma_p(x^{(j)}) \), and

\[
y^{(j)}(\cdot)f(x^{(j)}, \cdot) \leq z(\cdot)f(x^{(j)}, \cdot), \text{ for each } z \in S_p^*. \tag{2.7}
\]

It follows from the continuity of \( f \) and (2.7) that

\[
y^0(\cdot)f(x^0, \cdot) = \lim_{j \to \infty} y^{(j)}(\cdot)f(x^{(j)}, \cdot) \leq \lim_{j \to \infty} z(\cdot)f(x^{(j)}, \cdot) = z(\cdot)f(x^0, \cdot)
\]

for all \( z \in S_p^* \) and hence \( D \) is closed. Thus \( \Gamma_p \) is u.s.c.

The simple proof of convexity is omitted.
THEOREM 2.5. For each $p \in P$ there exists an $x^p \in L_{\infty}^n[0,T]$ such that

$$x^p(\cdot) \geq 0, \quad \text{ex}^p(\cdot) = p(\cdot), \quad f(x^p, \cdot) \geq \frac{x^p(\cdot)f(x^p, \cdot)}{p(\cdot)} \quad e. \quad (2.8)$$

PROOF. For each $p$ in $P$, $S^*_p$ and $\Gamma_p$ meet the conditions of Theorem 2.3. Hence there exists a fixed point $x^p$ of $\Gamma_p$, that is, $x^p \in \Gamma_p(x^p)$. Then it follows that

$$x^p(\cdot) \geq 0, \quad \text{ex}^p(\cdot) = p(\cdot), \quad x^p(\cdot)f(x^p, \cdot) \leq z(\cdot)f(x^p, \cdot) \quad (2.9)$$

for all $z \in S^*_p$. In particular (2.9) must hold for $z(\cdot) = e^p_i(\cdot), \quad i = 1, \ldots, n$. Substituting $e^p_i(\cdot)$ for $z(\cdot)$ in (2.9) we have

$$x^p(\cdot) \geq 0, \quad \text{ex}^p(\cdot) = p(\cdot), \quad x^p(\cdot)f(x^p, \cdot) \leq p(\cdot)f(x^p_i, \cdot)$$

$i = 1, \ldots, n$, or equivalently

$$x^p(\cdot) \geq 0, \quad \text{ex}^p(\cdot) = p(\cdot), \quad f(x^p, \cdot) \geq \frac{x^p(\cdot)f(x^p, \cdot)}{p(\cdot)} \quad e.$$

This proves the theorem.

COROLLARY 2.6. Let $\Lambda$ be a mapping of $P$ into $L_{\infty}^n[0,T]$ defined by

$$\Lambda(p) = \{x^p | x^p \in \Gamma_p(x^p)\}. \quad (2.10)$$
Then $\Lambda$ is such that;

i) for each $p \in P$, $\Lambda(p)$ is nonempty

and

ii) every $x^p \in \Lambda(p)$ satisfies (2.8).

The following lemma characterizes solutions to (2.1) as elements of
$\Lambda(p)$ for some $p \in P$.

**Lemma 2.7.** If there exists $\bar{p} \in P$ such that $\bar{x}(\cdot)f(\bar{x}, \cdot) = 0$ for
some $\bar{x} \in \Lambda(\bar{p})$, then $\bar{x}$ is a solution to the problem (2.1). Conversely, if
there exists $\bar{p} \in P$ such that $\bar{x} \in S^*_p$ is a solution to the problem, then
$\bar{x} \in \Lambda(\bar{p})$.

**Proof.** Suppose $\bar{x}(\cdot)f(\bar{x}, \cdot) = 0$ for some $\bar{x} \in \Lambda(\bar{p})$. Then

$$0 = \bar{x}(\cdot)f(\bar{x}, \cdot) \leq z(\cdot)f(\bar{x}, \cdot)$$

(2.11)

for all $z \in S^*_p$ and therefore $f(\bar{x}, \cdot) \geq 0$. Hence $\bar{x}$ is a solution to the
problem.

Conversely, suppose $\bar{x} \in S^*_p$ is a solution to the problem. Then

$$0 = \bar{x}(\cdot)f(\bar{x}, \cdot) \leq z(\cdot)f(\bar{x}, \cdot)$$

(2.12)

for each $z \in S^*_p$, and hence $\bar{x} \in \Lambda(\bar{p})$. 

3. Existence Theorems.

It is seen from Lemma 2.7 that solutions to the problem (2.1) that are elements of $S^*_p$ for some $p$ are also elements of $\Lambda(p)$ and vice-versa. Note that solutions to the problem may exist outside the class $S^*_p$, and we are only establishing sufficient conditions for the existence of a solution in $S^*_p$. We impose further restrictions on $f$. Definition 3.1 introduces three successively stronger versions of monotonicity.

**DEFINITION 3.1.** Let $g : L^\infty_n[0,T] \times [0,T] \to E^n$, then $g$ is:

monotone in its first argument if

$$[x(t)-y(t)][g(x,t)-g(y,t)] \geq 0$$

(3.1)

for all $x,y \in L^\infty_n[0,T]$, $t \in [0,T]$, strictly monotone if

$$[x(t)-y(t)][g(x,t)-g(y,t)] > 0, \text{ if } x(t) \neq y(t);$$

(3.2)

strongly monotone if there exists $k > 0$ such that

$$[x(t)-y(t)][g(x,t)-g(y,t)] \geq k|x(t)-y(t)|^2,$$

(3.2)

where $|\cdot|$ denotes the norm in $E^n$.

**THEOREM 3.2.** Let $f$ be strictly monotone in its first argument. Then there exists at most one solution to (2.1).
PROOF. Suppose \( \bar{x} \) and \( \bar{y} \) are distinct solutions to the problem.
Then for all \( t \in [0,T] \) for which \( \bar{x}(t) \neq \bar{y}(t) \)

\[
0 \geq -\bar{x}(t)f(\bar{y},t) - \bar{y}(t)f(\bar{x},t) \\
= \bar{x}(t)f(\bar{x},t) - \bar{x}(t)f(\bar{y},t) + \bar{y}(t)f(\bar{y},t) - \bar{y}(t)f(\bar{x},t) \\
= [\bar{x}(t) - \bar{y}(t)][f(\bar{x},t) - f(\bar{y},t)] \\
> 0.
\]

The above statement contradicts the strict monotonicity of \( f \) and proves the theorem.

At this point one could conjecture that strict monotonicity of \( f \) is sufficient to ensure the existence of a unique solution. This however, is not so, as can be demonstrated by the following generalization of Karamardian's counter example. Consider \( f(x,t) = -e^{-x(t)} - 1 \), then \( f \) is strictly monotone but \( f(x,t) < 0 \) for all \( x \in L_{n}[0,T] \). Hence there can be no solution to (2.1)

We will prove in Theorem 3.6 that if \( f \) is strongly monotone then a solution, which by virtue of Theorem 3.2 must be unique, does in fact exist. The following two lemmas will be used in the proof of the theorem.

**Lemma 3.3.** If \( f \) is strictly monotone in its first argument then \( \Lambda \) is single valued. Furthermore, if \( f \) is strongly monotone in its first argument then \( \Lambda \), in addition to being single valued, is also continuous on \( P \).
PROOF. Suppose \( f \) is strictly monotone, and for some \( p \in P \) \( \Lambda(p) \) contains two distinct points \( x \) and \( y \). From Theorem 2.5 it follows that

\[
x(\cdot) \geq 0, \quad e x(\cdot) = p(\cdot), \quad f(x, \cdot) \geq \frac{x(\cdot)f(x, \cdot)}{p(\cdot)} \quad e \tag{3.4}
\]

and

\[
y(\cdot) \geq 0, \quad e y(\cdot) = p(\cdot), \quad f(y, \cdot) \geq \frac{x(\cdot)f(x, \cdot)}{p(\cdot)} \quad e. \tag{3.5}
\]

Together, (3.4) and (3.5) yield

\[
y(\cdot)f(x, \cdot) \geq x(\cdot)f(x, \cdot) \tag{3.6}
\]

and

\[
x(\cdot)f(y, \cdot) \geq y(\cdot)f(y, \cdot). \tag{3.7}
\]

Inequalities (3.6) and (3.7) imply

\[
0 \geq x(\cdot)f(x, \cdot) - y(\cdot)f(x, \cdot) + y(\cdot)f(y, \cdot) - x(\cdot)f(y, \cdot) \nonumber = [x(\cdot) - y(\cdot)][f(x, \cdot) - f(y, \cdot)].
\]

This contradicts the assumption of strict monotonicity of \( f \) and proves that \( \Lambda \) is single valued.

Suppose now that \( f \) is strongly monotone. Since strong monotonicity implies strict monotonicity, the above proof shows that \( \Lambda \) is single valued. Let \( p_1 \neq p \) be elements of \( P \). From Theorem 3.5 there exist \( x^{p_1} \) and \( x^p \) such that
\[ x^p(\cdot) \geq 0, \quad \text{ex}^p(\cdot) = p(\cdot), \quad f(x^p, \cdot) \geq \frac{x^p(\cdot)f(x^p, \cdot)}{p(\cdot)} e \]

and

\[ x^p(\cdot) \geq 0, \quad \text{ex}^p(\cdot) = p(\cdot), \quad f(x^p, \cdot) \geq \frac{x^p(\cdot)f(x^p, \cdot)}{p(\cdot)} e. \]

The above two systems of inequalities yield

\[ x^p(\cdot)f(x^p, \cdot) \geq [p^p(\cdot)/p(\cdot)] x^p(\cdot)f(x^p, \cdot) \]

and

\[ x^p(\cdot)f(x^p, \cdot) \geq [p(\cdot)/p^p(\cdot)] x^p(\cdot)f(x^p, \cdot). \]

From the strong monotonicity of \( f \) there exists \( k > 0 \) such that

\[ x^p(\cdot)f(x^p, \cdot) - x^p(\cdot)f(x^p, \cdot) + x^p(\cdot)f(x^p, \cdot) - x^p(\cdot)f(x^p, \cdot) \geq k|x^p(\cdot) - x^p(\cdot)|^2. \]

The above three inequalities yield

\[ k |x^p(\cdot) - x^p(\cdot)|^2 \leq |p(\cdot) - p^p(\cdot)| \left[ \frac{x^p(\cdot)f(x^p, \cdot)}{p(\cdot)} - \frac{x^p(\cdot)f(x^p, \cdot)}{p^p(\cdot)} \right] \]

\[ \leq |p(\cdot) - p^p(\cdot)| \left[ \frac{x^p(\cdot)f(x^p, \cdot)}{p(\cdot)} + \frac{p^p(\cdot)f(x^p, \cdot)}{p^p(\cdot)} \right] \]

(3.8)

(3.9)

Since \( f \) is bounded there exists \( M < \infty \) such that

\[ f_1(\cdot, \cdot) < M \]

(3.10)
From (2.8), (3.9), and (3.10) it follows that

\[
\sup_t |x^p(t) - x^1(t)|^2 < \left[ \sup_t |p(t) - p^1(t)| \right] (2M/k). \tag{3.11}
\]

Let \( \epsilon > 0 \) be given and define \( \delta = k \epsilon / 2M \). Then clearly

\[
||x^p - x^1||^2 < \epsilon
\]

whenever \( \sup_t |p(t) - p^1(t)| \leq \delta \) and hence \( \Lambda \) is continuous on \( P \).

In the event that \( \Lambda \) is single valued, that is \( \Lambda(p) = \{x^p\} \), we define \( \Lambda(p,t) = x^p(t) \) for all \( t \in [0,T] \).

**Lemma 3.4.** Let the index set \( I = \{i|f_{1i}(0, \cdot) < 0, \quad i = 1, \ldots, n\} \) be nonempty. Then there exists \( p_o \in P \) such that \( x^{p_o}(\cdot)f(x^{p_o}, \cdot) < 0 \) for all \( x^{p_o} \in \Lambda(p_o) \).

**Proof.** Suppose \( i_o \in I \) and define \( N_{p_o} = \{x|\exists x \in L^1_{n+[0,T]}, \quad x(\cdot) \geq 0, \quad ex(\cdot) \leq p_o(\cdot)\} \). The continuity of \( f \) implies the existence of a \( p_o \in P \) such that for each \( x \in N_{p_o}, f_{1i_o}(x, \cdot) < 0 \). By Corollary 2.6 \( \Lambda(p_o) \) is nonempty and each \( x^{p_o} \in \Lambda(p_o) \) satisfies

\[
x^{p_o}(\cdot) \geq 0, \quad ex^{p_o}(\cdot) = p_o(\cdot), \quad f(x^{p_o}, \cdot) \geq \frac{x^{p_o}(\cdot)f(x^{p_o}, \cdot)}{p_o(\cdot)} \quad \text{e. (3.13)}
\]

Hence each \( x^{p_o} \) is also an element of \( N_{p_o} \). The desired result follows immediately from (3.13).
REMARK. 3.5. The importance of the above lemma lies not in the fact that $x^0(f(x^0, \cdot)) < 0$ for some $x^0$, but rather that such an $x^0$ exists in $\Lambda(p_o)$ for some $p_o$. If $f$ is strictly monotone then such an $x^0$ is also unique by Lemma 3.3.

Additional restrictions on $f$ to ensure the existence of a solution are needed. They are stated in the following:

CONSTRAINT QUALIFICATION 3.6. It will be assumed that for each $i$, $i = 1, \ldots, n$, either $f_i(0, t) \geq 0$ or $f_i(0, t) < 0$ for all $t \in [0, t]$.

THEOREM 3.7. Let $f$ be strongly monotone in its first argument and assume Constraint Qualification 3.6. Then a unique solution to (2.1) exists.

PROOF. Case I. Suppose that for each $i$, $f_i(0, \cdot) \geq 0$. Then $\bar{x} = 0$ is a solution which by Theorem 3.2 is unique.

Case II. Suppose that for each $i$, $f_i(0, \cdot) < 0$. From strong monotonicity of $f$ and Lemma 3.3 $\Lambda$ is single valued and continuous on $P$. Hence from the continuity of $f$ it follows that the composite function $\gamma : P \times [0, T] \rightarrow E$, defined by $\gamma(p, t) = \Lambda(p, t)f(\Lambda(p), t), t \in [0, T]$, is also continuous on $P$. Lemma 3.4 implies the existence of a $p_o \in P$ for which $\gamma(p_o, \cdot) < 0$.

Consider then all $p(\cdot) > p_o(\cdot)$. Theorem 2.5 yields

$$\Lambda(p_o, \cdot) \geq 0, \quad e\Lambda(p_o, \cdot) = p_o(\cdot), \quad f(\Lambda(p_o), \cdot) \geq \frac{\Lambda(p_o, \cdot)f(\Lambda(p_o), \cdot)}{p_o(\cdot)}$$

(3.14)

and
\[ \lambda(p, \cdot) \geq 0, \quad e^{\lambda(p, \cdot)} = p(\cdot), \quad \mathbb{E}(\lambda(p, \cdot)) \geq \mathbb{E}(p, \cdot) - \mathbb{E}(\lambda(p, \cdot)) \]

(3.15)

Strong monotonicity along with (3.14) and (3.15) results in

\[ [p(\cdot) - p_0(\cdot)] [\gamma(p, \cdot)/p(\cdot)] \geq [p(\cdot) - p_0(\cdot)] [\gamma(p_o, \cdot)/p_o(\cdot)] + k|x^p(\cdot) - x^{p_0}(\cdot)|^2 \]

(3.16)

for some positive real \( k \). Schwartz's inequality applied to (3.14) and (3.15) yields

\[ |x^p(\cdot) - x^{p_0}(\cdot)|^2 \geq [p(\cdot) - p_0(\cdot)]^2/n. \]

(3.17)

Substitution of (3.17) into (3.16) gives

\[ \gamma(p, \cdot)/p(\cdot) \geq \gamma(p_o, \cdot)/p_o(\cdot) + [p(\cdot) - p_0(\cdot)](k/n). \]

(3.18)

Define \( p^*(\cdot) = p_o(\cdot) - \frac{\gamma o(\cdot)}{\gamma_p(\cdot)} \). Clearly \( p^*(\cdot) > p_0(\cdot) \).

Substituting \( p^* \) into (3.18) for \( p \) we obtain

\[ \gamma(p, \cdot)/p^*(\cdot) \geq 0, \]

or equivalently \( \gamma(p^*, \cdot) \geq 0 \). Since \( P \) is connected,

\[ \gamma(p_o, \cdot) < 0 \leq \gamma(p^*, \cdot), \]
and \( \gamma \) is continuous on \( P \), there must exist a \( \tilde{p} \in P \), \( p_0(\cdot) < \tilde{p}(\cdot) \leq p^*(\cdot) \), with the property

\[
\gamma(\tilde{p}, \cdot) = 0. \tag{3.20}
\]

Then \( \Lambda(\tilde{p}) \) satisfies (2.1) and is unique by Lemma 3.3. This proves the theorem.

4. Differentiable \( f \).

In this section \( f \) is assumed to be Fréchet differentiable. Conditions for existence of a solution to (2.1) will be given in terms of \( \nabla f(x, t) \), the \( n \times n \) gradient matrix of \( f \) with respect to its first argument evaluated at \( x \). Three successively stronger versions of positive definiteness are introduced. The notion of strong positive definiteness was first formulated by Karamardian; ours is a direct generalization of his concept to the space \( L_0^n[0, T] \).

**Definition 4.1.** The \( n \times n \) matrix \( M(x, t) \), with elements \( m_{ij}(x, t) \in L_0^n[0, T] \), \( i, j = 1, \ldots, n \), is positive semidefinite if for all \( x, y \in L_0^n[0, T] \) and \( t \in [0, T] \)

\[
y(t)M(x, t)y(t) \geq 0; \tag{4.1}
\]

positive definite if for all \( x, y \in L_0^n[0, T] \) and \( t \in [0, T] \)

\[
y(t)M(x, t)y(t) > 0 \tag{4.2}
\]
whenever \( y(t) \neq 0 \); strongly positive definite if there exists \( k > 0 \) such that for all \( x, y \in L_\infty^0[0,T] \) and \( t \in [0,T] \)

\[
y(t)M(x, t)y(t) \geq k|y(t)|^2.
\] (4.3)

The following lemma establishes the correspondence between different notions of monotonicity and definiteness. Its result in conjunction with Theorem 3.7 will immediately yield Theorem 4.3.

**Lemma 4.2.** Let \( f \) be continuously Fréchet differentiable in its first argument. Then the following statements hold:

i) \( f \) is monotone if \( \nabla f \) is positive semidefinite,

ii) \( f \) is strictly monotone if \( \nabla f \) is positive definite,

iii) \( f \) is strongly monotone if \( \nabla f \) is strongly positive definite.

**Proof.** Let \( x \) and \( y \) be distinct points in \( L_\infty^0[0,T] \) and define for all \( 0 \leq \alpha \leq 1 \) and \( t \in [0,T] \).

\[
\psi(\alpha, t) = f(\alpha x + (1-\alpha)y, t)[x(t) - y(t)].
\] (4.4)

From the differentiability of \( f \) and (4.4) it follows that

\[
\psi(1, \cdot) - \psi(0, \cdot) = [f(x, \cdot) - f(y, \cdot)][x(\cdot) - y(\cdot)]
\]

and

\[
\frac{d}{d\alpha} \psi(\alpha, \cdot) = \psi'(\alpha, \cdot) = [x(\cdot) - y(\cdot)]\nabla f(x_{\alpha}, \cdot)[x(\cdot) - y(\cdot)],
\]
where $x_\alpha = \alpha x + (1 - \alpha)y$. An application of the mean-value theorem yields

$$
\psi(1, \cdot) - \psi(0, \cdot) = \psi'(\alpha^*, \cdot) = [x(\cdot) - y(\cdot)] \nabla f(x_{\alpha^*, \cdot})[x(\cdot) - y(\cdot)]
$$

for some $0 < \alpha^* < 1$. Hence the lemma follows.

**THEOREM 4.3.** Suppose $\nabla f$ is strongly positive definite and Constraint Qualification 3.6 holds. Then a unique solution to the problem exists.

In the special case when $f$ is linear in $x$, that is, of the form $f(x, \cdot) = M(\cdot)x(\cdot) + b(\cdot)$ where $M$ is an $n \times n$ matrix and $b$ is an $n$-vector, the following results obtain.

**COROLLARY 4.4.** Let $f$ be of the form $f(x, \cdot) = M(\cdot)x(\cdot) + b(\cdot)$. Then the following statements hold:

i) $f$ is monotone if and only if $M$ is positive semidefinite;

ii) $f$ is strictly monotone if and only if $M$ is positive definite

iii) $f$ is strongly monotone if and only if $M$ is strongly positive definite.

**COROLLARY 4.5.** Suppose $M$ is strongly positive definite and Constraint Qualification 3.6 holds. Then a unique solution to the problem exists.

Of course in this case Constraint Qualification 3.6 is equivalent to the statement: for each $i$, $i = 1, \ldots, n$, either $b_i(t) \geq 0$ or $b_i(t) < 0$ for all $t \in [0,T]$. It should also be noted that the results in this special case follow directly from those of Karamardian (1969) since we may view the problem as a finite dimensional case for each $t$ in $[0,T]$. That is find $n$-vector $\bar{x}_t$ satisfying

$$
\bar{x}_t \geq 0 \quad f(\bar{x}_t)_t \geq 0 \quad \bar{x}_t f(\bar{x}_t)_t = 0.
$$
If a unique solution \( \bar{x}_t \) exists for almost all \( t \) in \([0,T]\) then a unique solution \( x^* \) to (2.1) exists where \( x^*(t) = \bar{x}_t \).

5. Applications: Linear and Nonlinear Programming.

Traditionally linear and nonlinear programming problems have been presented in the form of a primal and a dual program. Sufficient conditions for existence of a solution were obtained for both primal and dual problems. Furthermore Kuhn-Tucker type theorems provided necessary and sufficient conditions for the existence of a solution to the primal. Most of the regularity conditions needed to ensure a solution were aimed at the individual functions involved in the optimizing function and the constraints. No uniqueness results were obtained.

We will demonstrate that the complementarity problem (2.1) serves as a unifying concept by enabling us to obtain solutions to both primal and dual problems simultaneously. The uniqueness of these solutions is ensured. Necessary and sufficient conditions are given for their existence. The regularity conditions imposed are directed at the constraints rather than individual functions.

Consider then the following linear problem presented in Levinson (1966)

**Primal Problem I.**

Maximize

\[
\ell(z) = \int_0^T c(t)z(t)dt \tag{5.1}
\]

subject to

\[
z(t) \geq 0 \tag{5.2}
\]
and

\[ 0 \leq \psi(x, t) = -B(t)z(t) + a(t) + \int_{0}^{t} K(t, s)z(s)ds \] (5.3)

for all \( t \in [0,T] \). Here \( c, z \in L_{m}^{\infty}[0,T], a \in L_{p}^{\infty}[0,T], \) and \( B \) and \( K \) are bounded measurable \( m \times p \) matrices on \([0,T]\) and \([0,T] \times [0,T]\) respectively.

Dual Problem I

Minimize

\[ L(w) = \int_{0}^{T} a(t)w(t)dt \] (5.4)

subject to

\[ w(t) \geq 0 \] (5.5)

and

\[ 0 \leq \Psi(w, t) = B^{T}(t)w(t) - c(t) - \int_{t}^{T} K^{T}(s,t)w(s)ds. \] (5.6)

To write (5.1) - (5.6) in the form of (2.1) the following identifications are made:

\[ q(t) = [a(t); c(t)], \] (5.7)

\[ M(t) = \begin{bmatrix} -B(t) & 0 \\ -B^{T}(t) & 0 \end{bmatrix}, \] (5.8)

and
\[
H(t,s) = \begin{bmatrix}
0 & \mathbb{1}_{[s \leq t]} K(t,s) \\
-\mathbb{1}_{[s > t]} K^T(s,t) & 0
\end{bmatrix}
\] (5.9)

Here \( I \) is the usual indicator function, \( q \) is an \( n \times 1 \) vector, \( M \) and \( H \) are \( n \times n \) matrices, and \( n = m + p \). If we let

\[
x(t) = \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}
\] (5.10)

and

\[
f(x, t) = q(t) + M(t)x(t) + \int_0^T H(t,s)x(s)ds
\] (5.11)

then constraints (5.2), (5.3), (5.5), and (5.6) may be written as

\[
x(t) \geq 0, \quad f(x, t) \geq 0
\]

for all \( t \in [0, T] \).

The following well known lemma is an easy consequence of Fubini's Theorem.

**Lemma 5.1.**

\[
\int_0^T z(t)[\psi(t, w(t)) + c(t)]dt = \int_0^T w(t)[a(t) - \psi(z, t)]dt.
\]

Weak duality is proven in Lemma 5.2.

**Lemma 5.2.** Let \( z \) and \( w \) be feasible for Primal and Dual Problems I, respectively, that is they satisfy (5.2), (5.3), (5.5), and (5.6). Then \( \mathcal{L}(z) \leq \mathcal{L}(w) \).
PROOF. Let \( z \) and \( w \) be feasible. Then

\[
\xi(z) = \int_0^T z(t)c(t)dt \\
\leq \int_0^T z(t)[\psi(w, t) + c(t)]dt \\
= \int_0^T w(t)[a(t) - \psi(z, t)] dt \\
\leq \int_0^T w(t)a(t)dt = L(w),
\]

where the second equality follows from Lemma 5.1.

A strong version of the Kuhn-Tucker Conditions is presented in Theorem 5.3.

THEOREM 5.3. For \( \bar{z} \) and \( \bar{w} \) to be unique optimal solutions to Primal Problem I and Dual Problem I, and \( L(\bar{w}) = \xi(\bar{z}) \), it is necessary and sufficient that there exists a unique \( \bar{x} \) satisfying

\[
\bar{x}(\cdot) \geq 0, \quad f(\bar{x}, \cdot) \geq 0, \quad x(\cdot)f(x, \cdot) = 0. \tag{5.12}
\]

PROOF. (Necessity) Let \( \bar{x} \) be a unique solution of (5.12). Then

\[
0 = \int_0^T \bar{x}(t)f(\bar{x}, t)dt \\
= \int_0^T [\bar{w}(t)\psi(z, t) + z(t)\psi(w, t)]dt \\
= \int_0^T [\bar{w}(t)a(t) - \bar{z}(t)c(t)]dt + \int_0^T [\bar{w}(t)[\psi(z, t) - a(t)] + z(t)[\psi(w, t) + c(t)]]dt
\]
\[
= \int_{0}^{T} [w(t)a(t) - z(t)c(t)]dt
\]

\[= L(\hat{w}) - \ell(\hat{z}).\]

Hence by Lemma 5.2 \( \hat{w} \) and \( \hat{z} \) are optimal, their uniqueness follows from that of \( \tilde{x} \).

(Sufficiency) Let \( \hat{w} \) and \( \hat{z} \) be unique optimal solutions to Dual and Primal Problems I with \( L(\hat{w}) = \ell(\hat{z}) \). Then an argument identical to the one above proves that \( f(\hat{x}, \cdot)\hat{x}(\cdot) = 0 \). Hence \( \hat{x} \) is a solution of (5.12). Its uniqueness follows from that of \( \hat{z} \) and \( \hat{w} \).

The following corollary is an easy consequence of Theorem 3.7.

**COROLLARY 5.4.** Let \( H \) and \( q \), defined by (5.9) and (5.7) satisfy

i) there exists \( k > 0 \) such that for all \( t \in [0, T] \) and \( x \in L^\infty_n[0, T] \)

\[x(t) \int_{0}^{T} H(t, s)x(s)ds \geq k|x(t)|^2\]

and

ii) for each \( i, i = 1, \ldots, n \), either \( q_i(t) \geq 0 \) or \( q_i(t) < 0 \) for all \( t \in [0, T] \).

Then unique optimal solutions \( \tilde{z}, \tilde{w} \) for Primal and Dual Problems I exist and \( \ell(\tilde{z}) = L(\tilde{w}) \).

It should be noted that there are no restrictions on \( B \) whatsoever and the solutions are guaranteed to be unique.
Analogous to the case of the linear problem discussed above we now present a nonlinear problem, first studied by Farr and Hanson (1974a).

Primal Problem II

Maximize

\[ \mathcal{J}(z) = \int_0^T \phi(z(t)) \, dt \]  \hfill (5.13)

subject to

\[ z(t) \geq 0 \]  \hfill (5.14)

and

\[ 0 \leq \gamma(z, t) = c(t) - h(z, t) + \int_0^t K(t, s) g(z(s)) \, ds \]  \hfill (5.15)

Dual Problem II

Minimize

\[ L(u, w) = \mathcal{J}(u) + \int_0^T w(t) \gamma(u, t) \, dt + \int_0^T u(t) \Gamma(u, w, t) \, dt. \]  \hfill (5.16)

subject to

\[ u(t), \ w(t) \geq 0 \]  \hfill (5.17)

and

\[ 0 \leq \Gamma(u, w, t) = -\nabla \phi(u(t)) + [\nabla h(u(t))] w(t) - \int_t^T \nabla g(u(t)) ] K(t, s) w(s) \, ds. \]  \hfill (5.18)

Here \( \phi : L^\infty_p[0, T] \to E, \ w, \ c \in L^\infty_m[0, T], \ u, \ z \in L^\infty_p[0, T], \ h : L^\infty_p[0, T] \to L^\infty_m[0, T], \ g : L^\infty_p[0, T] \to L^\infty_p[0, T], \) and \( K \) is an \( m \times p \) bounded measurable matrix on \( [0, T] \times [0, T]. \)

Let \( n = m + p \) and define

\[ x(t) = [w(t) \ | \ u(t)] \]  \hfill (5.19)
and

\[ f(x,t) = [\gamma(u,t), \Gamma(u,w,t)], \quad (5.20) \]

where now \( x \in \mathbb{L}^\infty_0[0,T] \) and \( f : \mathbb{L}^\infty_0[0,T] \times [0,T] \to \mathbb{L}^\infty_0[0,T] \).

In what follows the assumption of weak duality is made. Thus a larger class of functions can be considered. Specific regularity conditions to satisfy weak duality are presented in Larson and Hanson (1973). These results are included here for completeness.

**ASSUMPTION 5.5.** It will be assumed that for all \( z \) and \((u,w)\), feasible solutions for Primal and Dual Problems II, \( \ell(z) \leq L(u,w) \).

**THEOREM 5.6.** (Larson and Hanson (1973)). Let \( \phi \) and \( g \) be concave and \( h \) convex. Suppose \( z(t) \) and \((u(t), w(t))\) are feasible solutions for Primal and Dual Problems II respectively. Then \( \ell(z) \leq L(u,w) \).

**PROOF.** By the concavity of \( \phi \) and \( g \) and the convexity of \( h \) we have

\[
\ell(z) - L(u,w) = \int_0^T [\phi(z(t)) - \phi(u(t)) - w(t)\gamma(u,t) - u(t)\Gamma(u,w,t)]dt
\]

\[
\leq -\int_0^T z(t)\Gamma(u,w,t)dt - \int_0^T w(t)\gamma(u,t)dt.
\]

The right hand side of the last inequality is nonpositive by constraints (5.14), (5.15), (5.17) and (5.18). Hence the theorem follows.
THEOREM 5.7. For \( \bar{z} \) and \((\bar{z}, \bar{w})\) to be unique optimal solutions to Primal Problem II and Dual Problems II, and \( L(\bar{x}, \bar{w}) = \lambda(\bar{z}) \), it is necessary and sufficient that there exists a unique \( \bar{x} \) satisfying

\[
\bar{x}(\cdot) \geq 0, \quad f(\bar{x}, \cdot) \geq 0, \quad \bar{x}(\cdot)f(\bar{x}, \cdot) = 0 \tag{5.21}
\]

PROOF. (Necessity) Let \( \bar{x} \) be a unique solution of (5.21). Then

\[
0 = \int_0^T \bar{x}(t)f(\bar{x}, t)dt
\]

\[
= \int_0^T \bar{z}(t)\Gamma(\bar{z}, \bar{w}, t)dt + \int_0^T \bar{w}(t)\gamma(\bar{z}, t)dt.
\]

and hence both integrals in the last equality vanish. From Assumption 5.5 and (5.16) \( L(\bar{z}, \bar{w}) = \lambda(\bar{z}) \) and \( \bar{z} \) and \( \bar{w} \) are optimal. The uniqueness of \( \bar{z} \) and \( \bar{w} \) follows from that of \( \bar{x} \).

(Sufficiency). Suppose \( \bar{z} \) and \( \bar{w} \) are unique optimal solutions and \( L(\bar{z}, \bar{w}) = \lambda(\bar{z}) \). An argument similar to the one used above shows that \( \bar{x} \) is a unique solution of (5.12).

COROLLARY 5.8. Suppose that for all \( t \in [0,T] \) either \( f(0,t) \geq 0 \) or \( f(0,t) < 0 \). Furthermore, assume that \( f \) is strongly monotone and continuous in its first argument. Then unique optimal solutions \( \bar{z}, \bar{w} \) for Primal and Dual Problems II exist and \( L(\bar{z}, \bar{w}) = \lambda(\bar{z}) \).

The proof of the corollary follows from Theorem 3.7. Its importance lies in the fact that regularity conditions are not imposed on individual functions in the constraints, but rather only on the constraints \( \Gamma \) and \( \gamma \) as the whole.
Bibliography


