APPLICATIONS OF MAJORIZATION AND SCHUR FUNCTIONS IN RELIABILITY AND LIFE TESTING

FRANK PROSCHAN*

Abstract. This is an expository paper presenting basic definitions and properties of majorization and Schur functions, and displaying a variety of applications of these concepts in reliability prediction and modelling, and in reliability inference and life testing.

1. Introduction. Certain areas of inequality theory have played a fundamental role in developing new results in reliability theory and life testing. For example, the use of the powerful tool of total positivity to obtain dozens of such applications is well known; see, for example, [4, 5], [9], [3], [21], [1, 2], and the references therein.

Not nearly as well known is the use of the concepts and methods of majorization and Schur functions from inequality theory to obtain bounds, comparisons, and inequalities in reliability and life testing. It is the purpose of this paper:

(a) To present the basic definitions, concepts, and some key theorems in the area of majorization and Schur functions (Section 2); we supplement the formal statements with more intuitive geometric and descriptive exposition.

(b) To present a variety of applications of majorization and Schur functions in reliability prediction and modelling; i.e., probabilistic reliability (Section 3). The applications generally occur in the form

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*Florida State University, Department of Statistics. This research was sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant AFOSR-74-2581A.
of bounds, inequalities, and comparisons. In some cases, the reliability of a system of stochastically unlike components is approximated by the reliability of the same system of stochastically like components (thus permitting a more tractable calculation).

(c) To present a variety of inferential and life testing applications of majorization and Schur function. Examples are unbiased tests for increasing failure rate average, tests for outlying observations, and results concerning exponential life test procedures.

Proofs are not given of the results presented; these may be found in the original papers referenced. Rather the aim is to give a broad sampling of the type of reliability result obtainable by the use of majorization and Schur functions. It is hoped that such a survey will encourage reliability theorists to become more familiar with the relatively little known areas of majorization and Schur functions and their potential for obtaining useful results in reliability and life testing.

2. Majorization and Schur Functions. Given a vector 
\( x = (x_1, \ldots, x_n) \), let \( x_1 \geq x_2 \geq \ldots \geq x_n \) denote a decreasing rearrangement of \( x_1, \ldots, x_n \).

Definition 2.1. A vector \( x \) is said to majorize a vector \( x' \) if

\[
\sum_{i=1}^{j} x[i] \geq \sum_{i=1}^{j} x'[i] \quad \text{for } j = 1, \ldots, n-1 ,
\]

and

\[
\sum_{i=1}^{n} x[i] = \sum_{i=1}^{n} x'[i] ;
\]

we write \( x \gtrsim x' \).

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Although majorization involves comparison of vectors of order $n$, the following characterization shows that we need only consider a pair of coordinates at a time:

**Theorem 2.2** (Hardy, Littlewood, Pólya, 1952, p. 47). Let $x \preceq y$. Then $x'$ can be obtained from $x$ be a finite number of $T$-transformations, where a $T$-transformation changes two coordinates only, and in the following way: If, e.g., coordinates 1 and 2 are being transformed, then $T(a) = (a_1', a_2', a_3, \ldots, a_n)$, where $\max(a_1', a_2') \geq \max(a_1, a_2)$ and $a_1 + a_2 = a_1' + a_2'$.

Majorization represents a partial ordering in $\mathbb{R}^n$, the n-dimensional Euclidean space. A **Schur function** is a function that is monotone with respect to this partial ordering, as stated formally in:

**Definition 2.3.** A function $f$ satisfying the property that $x \preceq x' \implies f(x) \preceq f(x')$ is called a Schur-convex (Schur-concave) function. Functions which are either Schur-convex or Schur-concave are called **Schur functions** (or are said to possess the Schur property).

Note that a Schur function is necessarily, permutation-invariant; i.e., $f(x) = f(x')$ whenever $x'$ is obtained from $x$ by a permutation of coordinates. This follows immediately from the fact that if $x'$ is obtained from $x$ by a permutation of coordinates, then $x \preceq x'$, and so $f(x) \preceq f(x') \preceq f(x')$; i.e., $f(x) = f(x')$.

A useful characterization of Schur functions is provided by the fundamental theorem of Ostrowski [18], which states:

**Theorem 2.4** (Ostrowski). A differentiable, permutation-invariant
function \( f \) on \( \mathbb{R}_n \) is Schur-convex (Schur-concave) if and only if

\[
(2.3) \quad (x_1 - x_2) \left( \frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1} \right) \geq 0 \quad \text{for all } x.
\]

Holding all variables constant except, say \( x_1 \) and \( x_2 \), we present a simple geometrical interpretation of both majorization and Schur-convex functions.

![Diagram](image_url)

**Fig. 2.1.** Two-dimensional geometric properties of majorization and Schur-convexity.

On any line \( x_1 + x_2 = c \) (constant), given a point \( x' \) for which \( x'_1 > x'_2 \), then a second point \( x \) also on the line, but lying to the right (and below) \( x' \), majorizes \( x' \). Moreover, for any Schur-convex function \( f \), we have \( f(x) \geq f(x') \). Thus as \( x \) moves from \( A \) towards \( B \), a Schur-convex function increases, and also successive points
encountered majorize previous points passed through.

In addition, along the line B'B, the Schur function f is symmetric about the point A.

For a Schur-concave function, similar properties hold, with "increasing" and "decreasing" interchanged.

It is well known that the notion of increasing failure rate plays an important role in reliability theory. See, e.g., [6], [3], [5], [10], [12], and the references therein. Recall that a life distribution F with survival probability \( \bar{F} = 1 - F \) has increasing failure rate (IFR) if \( \log \bar{F} \) is concave. A direct connection may readily be established between IFR survival probabilities and Schur-concave functions.

**Remark 2.5.** Let \( \bar{F} \) be a survival probability function, and define \( g(x_1, x_2) = \bar{F}(x_1)\bar{F}(x_2) \) for \( x_1 \geq 0, x_2 \geq 0 \). Then \( g \) Schur-concave \( \iff \bar{F} \) IFR.

It easily follows that \( \bar{F} \) IFR implies that \( h(x) \overset{\text{def}}{=} \prod_{i=1}^{n} \bar{F}(x_i) \) is Schur-concave.

3. **Reliability Prediction and Modelling.** Computing the exact reliability of a system of stochastically unlike components is often difficult, especially if component repair is present. Thus a motivation exists for obtaining bounds on the reliability of systems of unlike components by considering systems of like components.

We consider first k-out-of-n systems. A k-out-of-n system functions if and only if at least \( k \) of the n components function. Let the \( i^{th} \) component have reliability \( p_i \), with corresponding hazard
\[ R_i \overset{\text{def}}{=} -\log p_i, \quad i = 1, \ldots, n. \] Let \( h_k(p) \) denote the reliability of a k-out-of-n system in which component \( i \) has reliability \( p_i \), \( i = 1, \ldots, n \). Then Pledger and Proschan [19] obtain the following comparisons.

**Theorem 3.1.** Let \( R = (R_1, \ldots, R_n) \) be a vector of component hazards which majorizes \( R' = (R'_1, \ldots, R'_n) \), a second vector of component hazards. Then the corresponding reliabilities for a k-out-of-n system satisfy

\[ h_k(p) \geq h_k(p') \quad \text{for} \quad k = 1, \ldots, n-1, \]

and

\[ h_n(p) = h_n(p'). \]

An inductive proof is used to show that the corresponding hazard transform, \(- \log h_k(e^{-R_1}, \ldots, e^{-R_n})\), is a Schur-convex function by verifying the Ostrowski criterion (2.3). (See [8] for a discussion of hazard transforms and their role in system reliability analysis.) Theorem 3.1 can also be obtained as a corollary of a more general result in [23].

A case of special interest occurs when

\[ R'_1 = \ldots = R'_n = \frac{1}{n} \sum_{i=1}^{n} R_i, \]

or equivalently, when
\[ p_1' = \ldots = p_n' = \left( \prod_{i=1}^{n} p_i \right)^{1/n}. \]

Clearly \( R \supseteq R' \), and so by Theorem 2.1, we have:

**Corollary 3.2.** Let \( p \) be a vector of component reliabilities for a \( k \)-out-of-\( n \) system, and let \( p_g = \left( \prod_{i=1}^{n} p_i \right)^{1/n} \) be their geometric mean. Then

\[ h_k(p) \geq h_k(p_g, \ldots, p_g) \quad \text{for} \quad k = 1, \ldots, n-1, \]

and

\[ h_n(p) = h_n(p_g, \ldots, p_g). \]

Conclusions (3.2a,b) give a lower bound for the reliability of \( k \)-out-of-\( n \) systems of unlike components in terms of the corresponding reliability in the case of like components. The bounds are sharp in the following sense. If any value \( p > p_g \) is used for common component reliability, then (3.2a,b) cannot hold for \( k = 1, \ldots, n \).

Additional results of a similar kind are obtained using the odds ratio \( r_i = \frac{1 - p_i}{p_i} \) for component \( i \), \( i = 1, \ldots, n \).

From these results for systems of components having fixed reliabilities (i.e., non time-dependent), Pledger-Proschan [19] derive stochastic comparisons for \( k \)-out-of-\( n \) systems in the time dependent case. Note that the time of failure of a \( k \)-out-of-\( n \) system of independent components with respective life distributions \( F_1, \ldots, F_n \) corresponds to the \((n-k+1)^{th}\) order statistics from the set of underlying
heterogeneous distributions \( \{ F_1, \ldots, F_n \} \). Thus results concerning
the reliability of \( k \)-out-of-\( n \) systems may equivalently be stated in
terms of order statistics from heterogeneous distributions.

Specifically, assume independent observations, one observation
from distribution \( F_i (F'_i) \), \( i = 1, \ldots, n \). The ordered observations
are denoted by \( Y_{(1)} \leq \cdots \leq Y_{(n)} \) \( (Y'_{(1)} \leq \cdots \leq Y'_{(n)}) \). Let \( R_i(t) = -\log F_i(t) \) denote the hazard function of component \( i \), \( i = 1, \ldots, n \).
Let \( X \overset{\text{st}}{=} Y \) mean \( X \) and \( Y \) have the same distribution, and \( X \overset{\text{st}}{=} Y \)
denote the fact that \( P[X > x] > P[Y > x] \) for each real \( x \). From
Theorem 3.1, there follows:

**Theorem 3.3.** Let \( (R_1(t), \ldots, R_n(t)) \overset{\text{st}}{=} (R'_1(t), \ldots, R'_n(t)) \) for
each \( t \geq 0 \). Then

\[
Y_{(1)} \overset{\text{st}}{=} Y'_{(1)} \quad \text{and} \quad Y_{(k)} \overset{\text{st}}{=} Y'_{(k)} \quad \text{for} \quad k = 2, \ldots, n .
\]

A dual result can be obtained by replacing \( F_i \) by \( F'_i \) and
\( P[Y_k > t] \) by \( P[Y_{n-k+1} \leq t] \).

Just as Corollary 3.2 represents a case of special interest ob-
tained from Theorem 3.1, so Corollary 3.4 below represents a case of
special interest obtained from Theorem 3.3.

**Corollary 3.4.** Let \( \overline{F}_i(t) = \cdots = F'_i(t) = \sqrt[n]{ \frac{n}{1} F_i(t) } \) for each\n\( t \geq 0 \). Then \( Y_{(1)} \overset{\text{st}}{=} Y'_{(1)} \), and \( Y_{(k)} \overset{\text{st}}{=} Y'_{(k)} \) \( \text{for} \quad k = 2, \ldots, n \).

Thus Corollary 3.4 gives a conservative bound on the reliability
of \( k \)-out-of-\( n \) systems of stochastically unlike components in terms of
similar systems of stochastically like components. As in Corollary
3.2, the bound is sharp.

Additional bounds and comparisons using majorization are obtained in [19] for the case of proportional hazards, i.e., \( R_i(t) = \lambda_i R(t), \)
i = 1, \ldots, n, and other cases commonly occurring in the reliability context.

In [20], the reliability model is extended to the case in which repair of devices is permitted. Specifically, the following model is assumed. Each of \( n \geq 2 \) machines runs continuously until it fails, at which time repair is initiated. After repair is completed, the machine resumes operation. The operating period for machine \( i \) has distribution \( F_i \), while the repair period has distribution \( G_i \),
i = 1, \ldots, n. All periods are mutually independent. Letting \( N(t) \) denote the number of machines operating at time \( t \), we note that \( \{N(t), t \geq 0\} \) may be a very unwieldy stochastic process since we have not assumed that \( F_1 = \ldots = F_n \) nor that \( G_1 = \ldots = G_n \).

A typical comparison of systems in [20] utilizing majorization theory is the following:

**Theorem 3.5.** Assume either that \( N(0) = N'(0) = n \), or that \( N(0) = N'(0) = 0 \).

(a) For \( t \geq 0 \), let \( F_i(t) = 1 - e^{-\lambda_i t}, F'_i(t) = 1 - e^{-\lambda'_i t} \) for \( i = 1, \ldots, n \), \( G_i(t) = \ldots = G_n(t) = 1 - e^{-\rho t} \), and \( \lambda \geq \lambda' \). Then \( N(t) \leq N'(t) \) for each \( t \geq 0 \).

(b) For \( t \geq 0 \), let \( G_i(t) = 1 - e^{-\lambda_i t}, G'_i(t) = 1 - e^{-\lambda'_i t} \) for \( i = 1, \ldots, n \), \( F_i(t) = \ldots = F_n(t) = 1 - e^{-\lambda t}, \rho \geq \rho' \). Then
\( N'(t) \geq^{st} N(t) \) for each \( t \geq 0 \).

Note that Theorem 3.5 provides a stochastic comparison at each fixed point in time. A much stronger stochastic type of comparison is actually possible under a slightly stronger hypothesis. We need the following definition.

**Definition 3.6.** Stochastic process \( \{X(t), t \geq 0\} \) is stochastically larger than stochastic process \( \{Y(t), t \geq 0\} \) (written \( X(t), t \geq 0 \) \( \geq^{st} \) \( Y(t), t \geq 0 \)) if \( (X(t_1), \ldots, X(t_n)) \geq^{st} (Y(t_1), \ldots, Y(t_n)) \) for every choice of \( 0 < t_1 < \ldots < t_n, n = 1, 2, \ldots \).

Pledger and Proschan [20] prove:

**Theorem 3.7.** Assume either that \( N(0) = N'(0) = n, \) or that \( N(0) = N'(0) = 0. \)

(a) For \( t \geq 0, \) let \( F_i(t) = 1 - e^{-\lambda_i t} \) for \( i = 1, \ldots, n, \)
\( F'(t) = 1 - e^{-\lambda' t}, \) where \( \lambda' = \frac{1}{n} \sum \lambda_i, \) and \( G_i(t) = \ldots = G_n(t) = 1 - e^{-\rho t}. \) Then \( \{N(t), t \geq 0\} \geq^{st} \{N'(t), t \geq 0\}. \)

(b) For \( t \geq 0, \) let \( G_i(t) = 1 - e^{-\rho_i t} \) for \( i = 1, \ldots, n, \)
\( G'(t) = 1 - e^{-\rho' t}, \) where \( \rho' = \frac{1}{n} \sum \rho_i, \) and \( F_i(t) = \ldots = F_n(t) = 1 - e^{-\lambda t}. \) Then \( \{N'(t), t \geq 0\} \geq^{st} \{N(t), t \geq 0\}. \)

Typical reliability applications of Theorem 3.7 consist of bounds for such functionals as:

(i) \( \int_0^M N(t) \, dt, \) the total up-time during \([0,M], M > 0; \)

(ii) \( \int_0^M N(t) e^{-at} \, dt \) for \( a > 0, \) the total discounted up-time during
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\[ [0, M]; \]

(iii) \( \min \{ \inf \{ t : N(t) \leq k \}, M \} \), the first time during \([0, M]\) that the number of functioning machines drops to \( k \) (arbitrarily taken as \( M \), if it never does in \([0, M]\)).

(iv) \( \int_{0}^{M} I_{\{ t : N(t) \geq k \}}(s)ds \), the total time in \([0, M]\) during which at least \( k \) machines are operating.

Next we consider a different kind of application of majorization.

In [14], the following theorem is proved:

Theorem 3.8. Let \( X_1, \ldots, X_n \) be exchangeable random variables, \( a \geq b \), and \( \phi \) be a continuous, convex, permutation-invariant function. Then

\[
(3.3) \quad E\phi(a_1 X_1, \ldots, a_n X_n) \geq E\phi(b_1 X_1, \ldots, b_n X_n).
\]

If \( \phi \) is strictly convex, equality occurs only if the \( X_i \) are all 0 with probability 1 or if \( b \) is a rearrangement of \( a \).

A corollary of Theorem 3.8 yields a reliability application.

Corollary 3.9. Let \( X_1, \ldots, X_n \) be exchangeable random variables. Then \( \frac{1}{n} E \max(0, X_1, \ldots, X_n) \) is nonincreasing in \( n \).

To obtain a reliability application, identify \( \max(X_1, \ldots, X_n) \) as the lifelength of a parallel system in which component \( i \) has life-length \( X_i \); component lifelengths are not necessarily independent, but are exchangeable. The conclusion states that the mean system lifelength of a parallel system increases more slowly than the number of components.
Note however that Corollary 3.9 is actually more general, since it applies to random variables on the whole real line, and not just nonnegative random variables.

Additional results concerning exchangeable random variables which are positively dependent by mixture are obtained by Shaked [24].

**Definition 3.10.** Let \( X_1, \ldots, X_n \) be exchangeable random variables with joint distribution \( F(x_1, \ldots, x_n) \). Let \( F \) be representable as a mixture as follows:

\[
F(x_1, \ldots, x_n) = \int G_\alpha(x_1)G_\alpha(x_2)\cdots G_\alpha(x_n) dH(\alpha),
\]

where \( G_\alpha \) is a distribution function indexed by \( \alpha \) and \( H \) is a distribution function for the index \( \alpha \). Then the random vector \( (X_1, \ldots, X_n) \) and its distribution function \( F \) are said to be **positively dependent by mixture** (we write \( PDM(X_1, \ldots, X_n) \)).

Shaked derives the following majorization comparison:

**Theorem 3.11.** Let \( X_1, \ldots, X_n \) be PDM and let \( Y_1, \ldots, Y_n \) be independent and identically distributed (i.i.d.) random variables, with each \( X_i \) and each \( Y_i \) having the same univariate marginal distribution. Let \( X(1) \leq \cdots \leq X(n) \) denote the order statistics of \( (X_1, \ldots, X_n) \) and \( Y(1) \leq \cdots \leq Y(n) \) denote the order statistics of \( (Y_1, \ldots, Y_n) \). Then

\[
(F_{Y(1)}(x), \ldots, F_{Y(n)}(x)) \preceq (F_{X(1)}(x), \ldots, F_{X(n)}(x))
\]

for each real \( x \), and as a consequence,
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(3.6) \[ \left( E Y_{(1)}, \ldots, E Y_{(n)} \right) \preceq \left( E X_{(1)}, \ldots, E X_{(n)} \right). \]

Reliability applications of Theorem 3.11 may readily be found.

Thus from (3.5) we conclude that:

For a series (parallel) system of components, system life span length is stochastically larger (smaller) in the PDM case than in the i.i.d. case.

Another type of application occurs in the area of shock models. A rather comprehensive treatment of univariate shock models is given in [9], [1, 2], [16], [15]; the first three papers emphasize preservation of such reliability properties as IFR, IFRA, etc. An interesting multivariate extension involving the preservation of Schur-concavity and Schur-convexity is given in [22]. The multivariate model is as follows.

Assume that shocks of type \( i \) occur according to a Poisson process having rate \( \lambda_i, i = 1, \ldots, n \), with the \( n \) processes mutually independent. Assume that the probability of surviving the joint occurrence of \( k_1 \) shocks of type 1, \( \ldots, k_n \) shocks of type \( n \) is \( F_{k_1, \ldots, k_n} \). Then the probability \( \overline{F}(\lambda) \) of surviving till time \( t \) is given by

(3.7) \[ \overline{F}_t(\lambda) = \sum_{k_n=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} \prod_{i=1}^{n} \left[ e^{-\lambda_i t} \left( \frac{k_i}{\lambda_i} \right)^{k_i} / k_i! \right] \frac{F_{k_1, \ldots, k_n}}{k_1, \ldots, k_n}. \]

Proshchan and Sethuraman [22] derive the following preservation
property for Schur functions:

**Theorem 3.12.** Let $\tilde{F}_{k_1, \ldots, k_n}$ be Schur-concave (Schur-convex).

Then for fixed $t > 0$, $H_t(\lambda)$ given in (3.7) is Schur-concave (Schur-convex) in $\lambda$.

4. Applications of Majorization and Schur Functions to Reliability Inference and Life Testing. In Section 3 we presented some sample applications of majorization in probabilistic reliability prediction and bounding. In the present section, we display some typical applications of majorization in the statistical aspects of reliability and life testing.

To explain the first application, we need to present some terminology and notation. A life distribution $F$ with hazard function $R(t) = -\log F(t)$ is said to have an **increasing failure rate average (IFRA)** if $\frac{1}{t} R(t)$ is increasing in $t \geq 0$; i.e., $R(t)$ is starshaped.

The class of IFRA distributions plays a fundamental role in reliability modelling. For example, Birnbaum, Esary, and Marshall [7] show that **coherent systems of independent IFR components** do not necessarily have an IFR life distribution, but **do have an IFRA life distribution**.

(Actually, the components can be IFRA.) In the shock model context, Esary, Marshall, and Proschan [9] show that with additive damages, the **number $N$ of shocks required to exceed a specified critical damage level is a discrete IFRA random variable**; i.e., $P^{1/k}[N > k]$ is decreasing in $k = 1, 2, \ldots$.

Thus, it is of importance to develop tests for IFRA, the usual
formulation being

\[ H_0: \ F \text{ is exponential ,} \]

vs.

\[ H_1: \ F \text{ is IFRA, but not exponential ,} \]

where \ F \ is the unknown distribution being sampled.

We need the following concept introduced by Proschan and Sethuraman [22].

**Definition 4.1.** A random vector \( X \) stochastically majorizes a random vector \( X' \) if \( f(X) \leq f(X') \) for each Schur-convex function \( f \). We write \( X \preceq_m X' \).

Marshall, Olkin, and Proschan [13] implicitly show the following (as a corollary of a more general result):

**Theorem 4.2.** Let \( X_{[1]} \geq \ldots \geq X_{[n]} \) be the (reverse) order statistics from \ F, \ IFRA, \ and \( X_{[1]} \geq \ldots \geq X_{[n]} \) the corresponding order statistics from \( G, \) exponential. Then

\[
\left( \frac{X_{[1]}}{EX_1}, \ldots, \frac{X_{[n]}}{EX_1} \right) \preceq_m \left( \frac{X'_{[1]}}{EX_1}, \ldots, \frac{X'_{[n]}}{EX_1} \right).
\]

*(4.1)*

**Statement** (4.1) may be used to develop unbiased tests of \( H_0 \) vs. \( H_1 \). By Definition 4.1, if \( h \) is any Schur-convex function, then

\[
h\left( \frac{X_{[1]}}{EX_1}, \ldots, \frac{X_{[n]}}{EX_1} \right) \preceq h\left( \frac{X'_{[1]}}{EX_1}, \ldots, \frac{X'_{[n]}}{EX_1} \right).
\]

(One choice mentioned in [13] is the test statistic \( \xi(X_1 - \bar{X})^2 / \bar{X}^2 \), proportional to

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the squared sample coefficient of variation.) To carry out the test at the $\alpha$ level of significance, determine $k_{\alpha}$ such that

$$P[h\left(\frac{X_{[1]}}{\bar{X}}, \ldots, \frac{X_{[n]}}{\bar{X}}\right) \leq k_{\alpha}] = \alpha$$

when $F$ is exponential, and reject $H_0$ when the $h$ statistic $\leq k_{\alpha}$.

For a second inferential application of Theorem 4.2, consider the problem of testing for outliers when the distribution $F$ is known to be IFRA. Suppose that $X_{[1]} \geq \ldots \geq X_{[n]}$ are reverse order statistics from $F$, except possibly that $X_{[1]}$ is an outlier; i.e., $X_{[1]}$ does not arise as an observation from $F$, but from $F_1 \leq F$, $F_1 \neq F$.

A natural test of the hypothesis that $X_{[1]} \geq \ldots \geq X_{[n]}$ all come from $F$ is to reject the hypothesis if $X_{[1]}/\bar{X}$ is too large. If $F$ is unknown, the distribution of this statistic is unavailable, but since $F$ is IFRA, it follows from (4.2) below with $a_1 = 1$,

$$a_2 = \ldots = a_n = 0 \text{ that } \frac{X_{[1]}}{\bar{X}} \leq \frac{Y_{[1]}}{\bar{Y}}$$

where $Y_{[1]} \geq \ldots \geq Y_{[n]}$ are the reverse order statistics from an exponential distribution.

To control the type I error, note that,

$$P[X_{[1]}/\bar{X} > k_{\alpha}] \leq P[Y_{[1]}/\bar{Y} > k_{\alpha}] = \alpha$$

The inequality (4.2) needed above is:

(4.2) $$\sum_{i=1}^{n} a_i \frac{X_{[i]}}{\bar{X}} \leq \sum_{i=1}^{n} a_i \frac{Y_{[i]}}{\bar{Y}}$$

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where \( X_1 \geq \ldots \geq X_n \) are reverse order statistics from \( F \), IFRA, 
\( Y_1 \geq \ldots \geq Y_n \) are reverse order statistics from an exponential, 
and \( a_1 \geq \ldots \geq a_n \). Inequality (4.2) may be obtained from Theorem 4.2 
by choosing the Schur-convex function

\[
\psi(z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i / z_i, \quad a_1 \geq \ldots \geq a_n.
\]

Similarly, the test statistic \( X_n / \bar{X} \) can be used when \( X_n \) is 
suspected of being an outlier. More generally, a test against the 
possibility that \( X_1, \ldots, X_r \) and \( X_s, \ldots, X_n \) are all out-
liers uses the statistic \( (X_1 + \ldots + X_r - X_s - \ldots - X_n) / \bar{X} \).

Next we present some results useful in life testing derived in 
[5] by majorization methods and the Ostrowski Theorem 2.4 above. 
Order statistics \( X_1 \leq \ldots \leq X_n \) are from distribution \( F \), while 
order statistics \( Y_1 \leq \ldots \leq Y_n \) are from distribution \( G \), where 
\( F \) and \( G \) are life distributions. We shall first state the theorem 
and then interpret its applications in life testing.

**Theorem 4.3.** Let \( G^{-1}F \) be starshaped on the support of \( F \),
\( F(0) = G(0) = 0 \), and \( EX = EY \). Then

\[
\begin{align*}
(1) & \sum_{i=1}^{r} EY(i) / EY(i) \text{ and } \sum_{i=1}^{r} (n-i+1)E(Y(i)-Y(i-1)) / \sum_{i=1}^{r} (n-i+1)E(X(i)-X(i-1)) \\
& \text{are increasing in } r(1 \leq r \leq n); \\
(ii) & (EY(n), EY(n-1), \ldots, EY(1)) \supset (EX(n), EX(n-1), \ldots, EX(1)) \text{ and }
\end{align*}
\]
\[ \sum_{i=1}^{r} (n-i+1)E(X_{(i)} - X_{(i-1)}) \geq \sum_{i=1}^{r} (n-i+1)E(Y_{(i)} - Y_{(i-1)}) \quad \text{for } 1 \leq r \leq n; \]

(iii) \[ \sum_{i=1}^{n} a_i (n-i+1)E(X_{(i)} - X_{(i-1)}) \geq \sum_{i=1}^{n} a_i (n-i+1)E(Y_{(i)} - Y_{(i-1)}) \quad \text{for} \]

\[ a_1 \geq a_2 \geq \ldots \geq a_n. \]

Note the following special cases in the reliability field of the hypothesis, \( G^{-1}F \) starshaped:

(a) \( G \) exponential, \( F \) IFRA;

(b) \( G \) Weibull with shape parameter \( a_1 \), \( F \) Weibull with shape parameter \( a_2 > a_1 \);

(c) \( G \) gamma with shape parameter \( a_1 \), \( F \) gamma with shape parameter \( a_2 > a_1 \).

The quantity \( \sum_{i=1}^{r} (n-i+1)(X_{(i)} - X_{(i-1)}) \) appearing in (i) and (ii) above represents the observed time on test at the \( r \)th failure. Recall that \( \hat{\theta}_{r,n} = \) observed time on test divided by the observed number of failures is the estimate of mean life generally used in estimating an exponential mean. Thus one consequence of the second half of (i) is that if the underlying distribution is IFRA with mean \( \theta \) and the exponential estimate is used, then the bias \( \hat{\theta}_{r,n} - \theta \) tends to diminish as the degree of censorship diminishes, i.e., as \( r \) increases.

The bias in censored sampling from an IFRA distribution is nonnegative; i.e., \( \hat{\theta}_{r,n} - \theta \geq 0 \), with \( \hat{\theta}_{n,n} - \theta = 0 \).

From statement (ii) we deduce that if \( h \) is any Schur-convex
function then \( h(EY(1), \ldots, EY(n)) \geq h(EX(1), \ldots, EX(n)) \).

The quantity \((n-i+1)(X_{(i)}-X_{(i-1)})\) appearing throughout Theorem 4.3 represents the time on test observed between the \(i-1\)st and \(i\)th failures; it is also referred to as the \(i\)th normalized spacing.

Statement (iii) asserts that comparisons can be made concerning weighted normalized spacings coming from a pair of distributions, one being starshaped with respect to the other (e.g., an IFRA distribution with respect to an exponential distribution).

Acknowledgment. I wish to thank R. E. Barlow, A. W. Marshall, S. E. Nevius, and J. Sethuraman for their help.

REFERENCES


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