TESTS FOR THE MEAN RESIDUAL LIFE

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SUMMARY

In this paper we develop tests for decreasing mean residual life (DMRL) and new better than used in expectation (NBUE) alternatives. The DMRL statistic is new, and critical constants and a large sample approximation are obtained to make the test readily applicable. The NBUE statistic derived is shown to be equivalent to the total time on test (TTOT) statistic; the latter is ordinarily viewed as a test statistic for increasing failure rate (IFR) alternatives. The DMRL and NBUE tests are investigated on the basis of consistency and asymptotic relative efficiency. Our results lead to a reinterpretation of the TTOT statistic as a test for classes of alternatives larger than the IFR class and including the NBUE class.

Some key words: Inference for life distributions; decreasing mean residual life; new better than used in expectation; increasing failure rate; total time on test statistic; linear functions of order statistics.

1. INTRODUCTION

In the last decade, statisticians have found it useful to categorize life distributions (distributions \( F \) such that \( F(t) = 0 \) for \( t < 0 \)) according to monotonicity properties of the failure rate, the average failure rate, and the mean residual life. Classes that have been considered include:

Increasing failure rate (IFR) class: \( F \) if IFR if \( \frac{F(x + t)}{F(t)} \) is decreasing in \( t \) whenever \( x > 0 \), where \( \overline{F} = 1 - F \) denotes survival probability. Assuming \( F \) has a density \( f \), then \( F \) is IFR if the failure rate \( q(t) = f(t)/\overline{F}(t) \) is increasing for \( 0 \leq t < \infty \).
Increasing failure rate average (IFRA) class: $F$ is IFRA if $[\overline{F}(t)]^{1/t}$ is decreasing in $t > 0$. Assuming $F$ has a density $f$, then $F$ is IFRA if
\[
\frac{1}{t} \int_0^t q(u)du \text{ is increasing for } 0 \leq t < \infty.
\]

New better than used (NBU) class: $F$ is NBU if $\overline{F}(s + t) \leq \overline{F}(s)\overline{F}(t)$ for all $s, t \geq 0$.

Decreasing mean residual life (DMRL) class: $F$ is DMRL if
\[
\frac{\int_s^t \overline{F}(u)du}{\overline{F}(s)} \geq \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)} \quad (1.1)
\]
for all $s \leq t$, $s, t \geq 0$.

New better than used in expectation (NBUE) class: $F$ is NBUE if
\[
\int_0^\infty \frac{\overline{F}(u)du}{\overline{F}(s)} \geq \frac{\int_0^s \overline{F}(u)du}{\overline{F}(s)} \quad (1.2)
\]
for all $s \geq 0$.

It is well known (cf. Bryson and Siddiqui, 1969) that these classes are related through the following implications:

IFR \rightarrow \text{IFRA} \rightarrow \text{DMRL} \leftarrow \text{NBUE} \rightarrow \text{NBUE} \leftarrow \text{NBU} \leftarrow \text{IFR} \leftarrow \text{IFR}

Additional classes of life distributions are obtained for the residual life time at time $t$ as $t \to \infty$ by Balkema and de Haan (1974).
Whereas probabilistic properties of these classes have been discussed extensively in the literature, much less attention has been devoted to statistical inference for these classes. The IFR class has received the most study from the inferential point of view; tests for IFR alternatives include those considered by Proschan and Pyke (1967), Barlow (1968), Barlow and Proschan (1969), Bickel (1969), Bickel and Doksum (1969), and Barlow and Doksum (1972). Tests for IFRA alternatives are developed by Barlow (1968) and tests for NBU alternatives are presented by Hollander and Proschan (1972).

The literature does not contain developments of tests for DMRL and NBUE alternatives. The purpose of this paper is to develop such tests. The DMRL class arises naturally in medicine, where testing for (and estimation of) mean residual life is of paramount importance. Bryson and Siddiqui find that the DMRL criterion is especially useful in studying empirical data. The NBUE class (called "net decreasing mean residual lifetime" by Bryson and Siddiqui) was investigated by Esary, Marshall, and Proschan (1970, 1973) and shown to play a fundamental role in the study of replacement policies by Marshall and Proschan (1972). Both the DMRL and NBUE classes are readily interpretable, and can easily be applied to physical models in actual experimental settings.

In this paper:

(1) We derive (Section 2) a test of exponentiality versus DMRL alternatives. The test, which is based on a linear function of the order statistics from the sample, is readily applied with the aid of small sample tail probabilities or a large sample approximation.

(2) In Section 3 we derive a test of exponentiality versus NBUE alternatives. The test obtained is the total time on test (TTOT) procedure - ordinarily championed as a test of exponentiality versus IFR or IFRA alternatives
(cf. Barlow (1968), Bickel and Doksum (1969), Barlow and Proschan (1969), and Barlow and Doksum (1972)). In Section 4 - where consistency and asymptotic relative efficiency of the DMRL and TTOT tests are considered - we show that the DMRL test is consistent against DMRL alternatives and the TTOT test is consistent against NBUE alternatives. The consistency of the TTOT procedure, against NBUE alternatives (a much larger class than the IFR class), shows that it is inappropriate to view rejection with the TTOT test as indicative of IFR or IFRA alternatives. A more appropriate conclusion, corresponding to a low significance probability attained by the TTOT test, is that the underlying life distribution is NBUE. In Section 5 we illustrate this point with an example.

2. A TEST FOR DMRL ALTERNATIVES

Recall, \( F \) is said to be DMRL if inequality (1.1) holds for all \( s \leq t \). Inequality (1.1) states that the mean of a unit's remaining life, given that it has survived to time \( s \), is no less than the mean of the unit's remaining life given that it has survived to time \( t \). The boundary members of the DMRL class, obtained when there is equality in (1.1) for all \( s \leq t \), are the exponential distributions.

Consider the problem of testing

\[
H_0: \quad F(t) = 1 - \exp(-\lambda t), \quad t \geq 0, \quad \lambda > 0 \quad (\lambda \text{ unspecified}), \quad (2.1)
\]

versus

\[
H_1: \quad F \text{ is DMRL (and not exponential),} \quad (2.2)
\]
on the basis of a random sample from the life distribution \( F \).

Our test is motivated by considering the following integral as a measure of "DMRL-ness" - for a given \( F \). Let
\[ \Delta(F) = \int \int_{s<t} \left\{ \frac{\int_{s}^{\infty} F(u) du}{F(s)} - \frac{\int_{t}^{\infty} F(u) du}{F(t)} \right\} dF(s) dF(t) \quad (2.3) \]

The parameter \( \Delta(F) \) can be viewed as follows. Define

\[ D(s,t) = \frac{\int_{s}^{\infty} F(u) du}{F(s)} - \frac{\int_{t}^{\infty} F(u) du}{F(t)} \quad (2.4) \]

Note that \( D(s,t) = 0 \) for all \( s \leq t \) if and only if \( H_0 \) is true. Now let \( S, T \) be independent random variables, each with life distribution \( F \). Then

\[ \Delta(F) = E_F \{ I_{\{S<T\}} \cdot D(S,T) \}. \]

For \( s < t \), \( D(s,t) \) is a weighted measure of the deviation from \( H_0 \) (towards \( H_1 \)) and \( \Delta(F) \) is an average value of this deviation. The weights \( F(s) \) and \( F(t) \) represent the proportions of the population still alive at \( s \) and \( t \) respectively, and thus furnishing comparisons concerning the mean residual lifelengths from \( s \) and \( t \) respectively.

The sample analogue of the parameter \( \Delta(F) \) forms the basis for our proposed test. (See, for example, Crouse (1966) for applications of this approach to a number of testing problems.) Substituting the empirical distribution function \( F_n \) for \( F \) in (2.3) we obtain the statistic \( T \):

\[ T = n^{-2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{n-j}{n} \frac{\tau(X_n) - \tau(X_i)}{n} - \frac{n-i}{n} \frac{\tau(X_i) - \tau(X_j)}{n} \quad (2.5) \]

where \( X_1 < X_2 < \ldots < X_n \) are the order statistics of the sample and \( \tau(X_i) \), the total time on test to \( X_i \), is defined as

\[ \tau(X_i) = \sum_{j=1}^{i} D_j \quad (2.6) \]
where

\[ D_j = (n - j + 1)(X_j - X_{j-1}) , \quad j = 1, \ldots, n \quad (2.7) \]

In (2.7) we take \( X_0 = 0 \). The \( D \)'s are known as the normalized sample spacings, or alternatively, as the total times on test between successive failures.

A statistic that is asymptotically equivalent to \( T \), but that should be slightly better for small \( n \) due to its use of the \( i = 0 \) term missing from the first summation of (2.5), is

\[
V = n^{-2} \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \frac{(n-j)}{n^2} \frac{\tau(X_i) - \tau(X_j)}{n} \frac{n - i}{n} \frac{\tau(X_i) - \tau(X_j)}{n} . \quad (2.8)
\]

In comparison with \( T \), \( V \) has the advantage that it uses the information in the sample concerning lifelength measured from time 0, i.e., the lifelength of a new item.

Substituting the right-hand-side of (2.6) for the various total time on test \( \tau \)'s appearing in (2.8), and reversing the order of the triple summations, yields (after straightforward but tedious algebra)

\[
V = n^{-4} \sum_{i=1}^{n} c_{in} X_i , \quad (2.9)
\]

where

\[
c_{in} = \left(4i^3/3\right) - 4ni^2 + 3n^2i - (n^3/2) + (n^2/2) - (i^2/2) + (i/6) . \quad (2.10)
\]

From (2.9) and (2.10) we see that

\[
V \sim \frac{1}{n} \sum_{i=1}^{n} J_{1}(\frac{i}{n})X_i ,
\]

where
\[ J_1(u) = (4u^3/3) - 4u^2 + 3u - \frac{1}{2} . \quad (2.11) \]

Since \( J_1(u) \) is bounded and continuous on the unit interval, we can apply Theorems 2 and 3 of Stigler (1974) to obtain

**THEOREM 2.1.** Assume that \( \int x^2 dF(x) < \infty \). Then \( \sigma^2(J_1,F) > 0 \) implies

\[
L(n^{\frac{1}{b}}[V - \mu(J_1,F)]) \rightarrow N(0,\sigma^2(J_1,F)) ,
\]

where

\[
\mu(J_1,F) = \int x J_1(F(x)) dF(x) , \quad (2.12)
\]

\[
\sigma^2(J_1,F) = \iint J_1(F(x)) J_1(F(y)) [F(\min(x,y)) - F(x)F(y)] dx dy , \quad (2.13)
\]

and \( J_1 \) is given by (2.11).

The distribution of \( V \) is not scale invariant. In order to make our test scale invariant we use the test statistic:

\[
V^* = \frac{V}{\tau(X_n/n)} = \frac{V}{X} . \quad (2.14)
\]

From Theorem 2.1 we can immediately obtain, using Slutsky's theorem,

**COROLLARY 2.1.** Under the assumptions of Theorem 2.1,

\[
L(n^{\frac{1}{b}}[V^* - \frac{\mu(J_1,F)}{\mu(F)}]) \rightarrow N(0,\sigma^2(J_1,F)/\mu^2(F)) ,
\]

where \( \mu(F) \) is the mean of the distribution \( F \).

When doing calculations under \( H_0 \) we can, since \( V^* \) is scale invariant, take \( F \) to be exponential with scale parameter \( \lambda = 1 \). For \( F_0(x) = 1 - \exp(-x) \), we find \( \mu(J_1,F_0) = 0, \ \sigma^2(J_1,F_0) = 1/210, \ \mu(F_0) = 1 \), and thus we may state
COROLLARY 2.2. Under $H_0$,

$$L(n^{1/2} V^*) \to N(0,1/210) .$$

Using Corollary 2.2, a large sample $\alpha$ level test of $H_0$ versus $H_1$ rejects $H_0$ if $210n^{1/2} V^*$ exceeds $z_\alpha$, the upper $\alpha$ percentile point of the standardized normal distribution. Whereas significantly large values of $V^*$ indicate DMRL alternatives, significantly small values of $V^*$ suggest increasing mean residual life (IMRL) alternatives. (F is IMRL if, for all $s \leq t$, inequality (1.1) holds with "$z" replaced by "s".)

The small sample null distribution of the statistic $V^*$ is given in Table 1. For $n = 2(1)20(5)50$, Table 1 contains lower and upper percentile points (based on Monte Carlo sampling with 20,000 replications each) in the .01, .05, and .10 regions.
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3. A TEST FOR NBUE ALTERNATIVES

Recall that $F$ is NBUE if inequality (1.2) holds for all $s \geq 0$. That is, if for all $s \geq 0$, the mean residual life at time $s$ is not greater than the mean life of a new item (i.e., the mean residual life at time 0). The boundary members of the NBUE class, obtained when there is equality in (1.2) for all $s \geq 0$, are again the exponential distributions. We now consider the problem of testing $H_0$ (2.1) versus

$$H_2: F \text{ is NBUE (and not exponential)} \quad (3.1)$$

on the basis of a random sample from $F$. Analogous to the development in Section 2, the parameter

$$\gamma(F) = \int_0^\infty \bar{F}(s) \left[ \int_0^s \frac{\bar{F}(u)}{\bar{F}(s)} du \right] dF(s) \quad (3.2)$$

is taken as a measure of the "NBUE-ness" of the distribution $F$. The sample counterpart to $\gamma(F)$, obtained by substituting the empirical distribution function $\bar{F}_n$ for $F$ in (3.2), is

$$K = n^{-1} \sum_{i=1}^{n-1} \left\{ \frac{(n-i)}{n} \bar{X} - \frac{j=i+1}{n} \frac{(X_j - X_1)}{n} \right\} \quad , \quad (3.3)$$

where $X_1 < X_2 < \ldots < X_n$ are the order statistics. Interchanging the order of summation, direct algebra yields

$$K = n^{-2} \sum_{i=1}^{n} \frac{3n}{2} - 2i + \frac{1}{2} X_i \quad . \quad (3.4)$$

As in Section 2, in order to make $K$ scale invariant, we studentize it and consider
\[ K^* = \frac{K}{X} . \] (3.5)

The \( K^* \) statistic can be seen to be simply a linear function of the TTOT statistic \( \sum_{i=1}^{n-1} \tau(X_i)/\tau(X_n) \) considered and tabled by Barlow (1968).

The stochastic process analogue of the TTOT statistic was developed by Cox (1955) and Cox and Lewis (1966) for testing that the underlying process is a homogeneous Poisson process against alternatives where the event rate of the process is increasing.

The asymptotic normality of the TTOT statistic is well-known (cf. Nadler and Eilbott (1967) and Barlow and Doksum (1972)). We record the result here for convenience. Note that, from (3.4), \( K \sim (n)^{-1} \sum_{i=1}^{n} J_2(\frac{i}{n})X_i \), where

\[ J_2(u) = (3/2) - 2u . \] (3.6)

We can again apply Stigler's results. Assume \( \int x^2 dF(x) < \infty \). Then \( \sigma^2(J_2,F) > 0 \) implies

\[ L(n^{\frac{1}{2}}K^* - \frac{\mu(J_2,F)}{\mu(F)}) \rightarrow N(0,\sigma^2(J_2,F)/\mu^2(F)) , \] (3.7)

where \( \mu(F) \) is the mean of \( F \), \( \mu(J_2,F) \) and \( \sigma^2(J_2,F) \) are given by (2.12) and (2.13) with \( J_1 \) replaced by \( J_2 \). In particular, under \( H_0 \), \( L(n^{\frac{1}{2}}K^*) \rightarrow N(0,\frac{1}{12}) \).

Significantly large values of \( K^* \) suggest NBUE alternatives; significantly small values suggest new worse than used in expectation (NWUE) alternatives. (\( F \) is NWUE if, for all \( s \geq 0 \), inequality (1.2) holds with "\( \geq \" replaced by "\( < \".) Using the asymptotic normality of \( K^* \), a large sample \( \alpha \) level test of \( H_0 \) versus \( H_2 \) rejects \( H_0 \) if \( (12n)^{\frac{1}{2}}K^* \) exceeds \( z_\alpha \). Similarly, the approximate \( \alpha \) level test of \( H_0 \) versus \( F \) NWUE rejects when \( (12n)^{\frac{1}{2}}K^* \leq -z_\alpha \).
We do not need to furnish new small sample critical points of $K^*$ because Barlow (1968, Table 3) has tabulated percentile points of the statistic
\[
\left[ \sum_{i=1}^{n-1} \frac{\tau(X_i)}{\tau(X_n)} \right]; \text{ the latter is seen to be equal to } (nK^*) + (n - 1)/2.
\]

The development here, which led us to the TTOT statistic suggests that the TTOT statistic can detect, in addition to IFR alternatives, the larger class of NBUE alternatives. This is made more precise in the next section.

4. CONSISTENCY AND ASYMPTOTIC RELATIVE EFFICIENCY
OF THE DMRL AND NBUE TESTS

We first obtain consistency of the DMRL test against DMRL alternatives.

THEOREM 4.1. In addition to the assumptions of Theorem 2.1, assume $F$ is continuous, DMRL, and not exponential. Then the DMRL test is consistent.

Proof. From Corollary 1, and the fact that $\mu(J_1, F_0) = 0$, where $F_0$ denotes the exponential distribution, it follows that the DMRL test (the test which rejects for large values of $V^*$) will be consistent if and only if $\mu(J_1, F) > 0$. We first show that $F$ DMRL and not exponential implies that $\Delta(F) > 0$. We then note that $\Delta(F) = \mu(J_1, F)$, to complete the proof.

Recall $D(s, t)$, defined by (2.4); $D(s, t) = \int_s^t \left( \frac{\bar{F}(s)}{\bar{F}(t)} - \frac{\bar{F}(s)}{\bar{F}(t)} \right)$. Then $F$ DMRL implies $D(s, t) \geq 0$ for $s < t$. Since $F$ is not exponential, there exists $(s_0, t_0)$ with $s_0 < t_0$ such that $D(s_0, t_0) > 0$. Note that $\bar{F}(s_0) \geq \bar{F}(t_0) > 0$, since $s_0 < t_0$, and since $\bar{F}(t_0) = 0$ would imply $D(s_0, t_0) = 0$. Since $F$ is continuous and $\int_x^\infty \bar{F}$ is continuous, then $D(s, t)$ is continuous. Thus, there exists a $\delta_1$ such that $D(s, t) > 0$ for all $(s, t)$ such that $|s - s_0| < \delta_1$, $|t - t_0| < \delta_1$. Since $F$ is continuous, there exists a $\delta_2$ such that $|\bar{F}(s) - \bar{F}(s_0)| > 0$ and $|\bar{F}(t) - \bar{F}(t_0)| > 0$ for all $|s - s_0| < \delta_2$, $|t - t_0| < \delta_2$. Let
\[ \delta = \min(\delta_1, \delta_2). \] Then for \( |s - s_0| < \delta, \ |t - t_0| < \delta, \ D(s, t) > 0 \) and
\[ |\overline{F}(s) - \overline{F}(s_0)| > 0, \ |\overline{F}(t) - \overline{F}(t_0)| > 0. \] It follows that \( \Delta(F) > 0. \) The proof is completed by directly evaluating \( \Delta(F) \) and \( \mu(J_1, F) \) (the former via repeated integration by parts) and showing

\[ \Delta(F) = \mu(J_1, F) = -6^{-1}\mu(F) + 2^{-1}\int F^2(x)dx - 3^{-1}\int F^4(x)dx. \] (4.1)

Theorem 4.1 states that the class of F's for which the DMRL test is consistent contains the continuous DMRL distributions. Equation (4.1) can be used to characterize the class of F's for which the DMRL test is consistent. Let \( \mu_{r,n}(F) \) denote the expected value of the \( r \)th order statistic in a random sample of size \( n \) from F. Then the right-hand-side of (4.1) is seen to be equal to
\[ -6^{-1}\mu_{1,1}(F) + 2^{-1}\mu_{1,2}(F) - 3^{-1}\mu_{1,4}(F). \] Hence we have shown that (subject to the assumptions of Theorem 2.1) the DMRL test is consistent if and only if
\[ \mu_{1,2}(F)/2 > (\mu_{1,1}(F)/6) + (\mu_{1,4}(F)/3). \]

We next turn to the consistency of the NBUE (TTOT) test. From (3.7) we have (subject to the assumptions of Theorem 2.1) that the NBUE-TTOT test (the test which rejects for large values of \( K^* \)) will be consistent if and only if \( \mu(J_2, F) > 0. \) The result that F IFR implies \( \mu(J_2, F) > 0 \) (and thus the consistency of the NBUE-TTOT test against IFR alternatives) is a special case of a more general result of Barlow and Doksum (1972). Theorem 4.2 below shows that the consistency class of the NBUE-TTOT test also contains the (continuous) NBUE distributions.

**THEOREM 4.2.** In addition to the assumptions of Theorem 2.1 (with \( J_2 \) replacing \( J_1 \)), assume F is continuous, NBUE, and not exponential. Then the NBUE-TTOT test is consistent.
Proof. We show that $F$ NBUE, continuous, and not exponential implies that $\gamma(F) > 0$. The proof is then completed by showing $\gamma(F) = \mu(J_2,F)$. Let

$$D(s) = (\bar{F}(s)\int_0^s \bar{F}) - \int_0^s \bar{F}.$$ 

Then $\gamma(F) = \int_0^s D(s)dF(s)$. Since $\bar{F}$ is NBUE, then $D(s) \geq 0$ for all $s \geq 0$. Since $F$ is not exponential, there exists an $s_0$ such that $D(s_0) > 0$. Now, $\int_0^s \bar{F}$ is continuous and so is $\bar{F}(s)$ by hypothesis. Thus $D(s)$ is continuous. Let $s_1 = \inf\{s: D(s) = 0 \text{ and } s > s_0\}$. If $D(s) > 0$ for every $s \geq s_0$, define $s_1 = \infty$. Note that $\bar{F}(s_0) > 0$, since otherwise $D(s_0) = 0$. On the interval $[s_0, s_1]$, $F$ is not constant, since $\bar{F}(s_0) = \bar{F}(s_1) = c$, (say), implies $0 = D(s_1) = c\int_0^{s_1} \bar{F} - \int_0^{s_1} \bar{F} > c\int_0^{s_1} \bar{F} - \int_0^{s_0} \bar{F} = D(s_0) > 0$; the inequality holds since $\int_0^{s_1} \bar{F} = \int_0^{s_0} \bar{F} + \int_{s_0}^{s_1} \bar{F} = c(s_1 - s_0) + \int_{s_0}^{s_1} \bar{F} > \int_0^{s_0} \bar{F}$.

From the contradiction, we conclude $F$ is not constant on $[s_1, s_0]$. Since $D(s) > 0$ for $s_1 < s < s_0$, $\bar{F}(s_0) > \bar{F}(s_1)$, and since $F$ is continuous, then $\gamma(F) > 0$.

To see that $\gamma(F) = \mu(J_2,F)$, note that

$$\gamma(F) = \mu(F) \cdot \int_0^\infty \bar{F}(s)dF(s) - \int_0^\infty \int_0^x \bar{F}(x)dF(s)dx$$

$$= \frac{\mu(F)}{2} - \int_0^\infty \bar{F}(x)F(x)dx$$

$$= \frac{-\mu(F)}{2} + \int_0^\infty \bar{F}^2(x)$$

and

$$\mu(J_2,F) = \frac{-\mu(F)}{2} + 2\int_0^\infty x\bar{F}(x)dF(x)$$

$$= \frac{-\mu(F)}{2} + \int_0^\infty \bar{F}^2(x)dx.$$
Observe that we have really shown more than just consistency against continuous NBUE alternatives. We have shown that, subject to the assumptions of Theorem 2.1, the NBUE-TTOT test is consistent if and only if $u_{1,2}(F) > u(F)/2$.

We next compare the DMRL and NBUE-TTOT tests on the basis of Pitman asymptotic relative efficiency. For a sequence \( \{F_\theta\}_n \) of alternatives with \( \theta_n = \theta_0 + cn^{-\frac{2}{k}} \), where \( c \) is an arbitrary positive constant and \( F_{\theta_0} \) is exponential, the asymptotic relative efficiency of \( V^\ast \) with respect to \( K^\ast \) is found to be

\[
e_F(V^\ast, K^\ast) = \frac{(35/2)\{m_1'(\theta_0)/m_2'(\theta_0)\}^2}{\text{Var}_0(V^\ast)}, \tag{4.2}
\]

where \( m_i(\theta) = \mu(J_i, F_\theta)/\mu(F_\theta) \), \( i = 1, 2 \), are the asymptotic means of \( V^\ast \) and \( K^\ast \) for the alternative \( F_\theta \), the factor \( 35/2 \) in (4.2) equals \( \lim_n \{\text{Var}_0(K^\ast)/\text{Var}_0(V^\ast)\} \), and \( m_1'(\theta_0) = \{d/d\theta)m_1(\theta)\}_{\theta=\theta_0} \). We have evaluated (4.2) for linear failure rate, Makeham, and Weibull alternatives given respectively by \( F_1(x) = 1 - \exp(-x(1 + \theta x^2/2)), \theta \geq 0, x \geq 0 \), \( F_2(x) = 1 - \exp(-x(1 + e^{-x} - 1)), \theta \geq 0, x \geq 0 \), and \( F_3(x) = 1 - \exp(-x^\theta), \theta \geq 1, x \geq 0 \). For \( F_1 \) and \( F_2 \), \( H_0 \) is achieved at \( \theta = \theta_0 = 0 \) and for \( F_3 \), \( H_0 \) is achieved at \( \theta = \theta_0 = 1 \). Direct calculations yield

\[
e_{F_1}(V^\ast, K^\ast) = 1.094, \quad e_{F_2}(V^\ast, K^\ast) = .700, \quad e_{F_3}(V^\ast, K^\ast) = .486. \tag{4.3}
\]

5. AN APPLICATION OF THE DMRL AND NBUE TESTS

Bryson and Siddiqui (1969) have analyzed survival time data from a study of Siddiqui and Gehan (1966). The data are survival times (in days from diagnosis) of patients suffering from chronic granulocytic leukemia. Here \( n = 43 \) and the order statistics \( X_1 < X_2 < \ldots < X_{43} \) are: 7, 47, 58, 74, 177, 232, 273, 285, 317, 429, 440, 445, 455, 468, 495, 497, 532, 571, 579, 581, 650, 702, 715, 779, 881, 900, 930, 968, 1077, 1109, 1314, 1334, 1367, 1534, 1712, 1784, 1877, 1886, 2045, 2056, 2260, 2429, 2509.
Bryson and Siddiqui estimated the average failure rate \( \frac{1}{t} \int_0^t q(u)du \)

\( = -\frac{\log(F(t))}{t} \) and the mean residual life \( \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)} \) using the Siddiqui-Gehan data. Letting \( S_t \) denote the number of survivors at time \( t \) out of an initial sample of size \( n \), their sample estimators, for the average failure rate and mean residual life, were respectively, \( -t^{-1} \log(S_t/n) \)

and \( \left( S_t \right)^{-1} \sum_{j=1}^t (t_j - t) \), where \( t_j \) denotes the survival time of the \( j \)th element in the group of those having survived to time \( t \). Figure 1 of their paper gives the sample average failure rate and Figure 2, the sample mean residual life.

Applying the DMRL statistic \( V^* \) and the NBUE-TTOT statistic \( K^* \) to the data we find \( V^* = .015 \) corresponding to a \( P \) value of .08 (normal approximation) and \( K^* = .072 \) corresponding to a \( P \) value of .05 (normal approximation). These objective test results are in agreement with the visual indications one gets from Figure 2 of Bryson and Siddiqui. Namely, the sample mean residual life shows a general (but not strictly) decreasing trend, and whereas \( F \) DMRL and \( F \) NBUE is suggested, the NBUE property appears to be the more appropriate decision.

Figure 1 of Bryson and Siddiqui leaves much doubt as to whether \( F \) is an IFR distribution (or even an IFRA distribution). Yet the TTOT statistic is significant at \( P = .05 \). This example clearly illustrates our earlier warning. Namely, whereas in the past, significantly large values of the TTOT statistic were thought to be indicative of an IFR or IFRA alternative, a more appropriate conclusion (for a large value of \( K^* \)) is that the data suggest the presence of an NBUE distribution.
6. COMMENTS AND CONCLUSIONS

**DMRL testing problem:** This paper provides a new test of $F$ exponential versus $F$ DMRL (and not exponential). Critical constants (Table 1) and a large sample approximation make the test readily applicable for detecting DMRL (reject for large values of $V^*$) or IMRL (reject for small values of $V^*$) alternatives. (We note here that Bryson (1974) has suggested a test for IMRL alternatives. He only gives critical values for $n = 10, 15, 20, 25$ and $30$, and does not derive the asymptotic distribution of his test statistic.)

**NBUE testing problem:** The TTOT statistic arose in a natural way, as a test statistic for NBUE alternatives and was found to be consistent against (continuous) NBUE alternatives. The TTOT statistic has been shown to have good efficiency properties when competing with IFR tests (by Bickel and Doksum, 1969), when competing with NBU tests (by Hollander and Proschan, 1972), and when competing with DMRL tests (Section 4 of this paper). The additional usefulness of the TTOT statistic - demonstrated in this article - should also lead to a reinterpretation of the statistic as a test for classes of alternatives larger than the IFR class and including the NBUE class.

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In this paper we develop tests for decreasing mean residual life (DMRL) and new better than used in expectation (NBUE) alternatives. The DMRL statistic is new, and critical constants and a large sample approximation are obtained to make the test readily applicable. The NBUE statistic derived is shown to be equivalent to the total time on test (TTOT) statistic; the latter is ordinarily viewed as a test statistic for increasing failure rate (IFR) alternatives. The DMRL and NBUE tests are investigated on the basis of consistency and asymptotic relative efficiency. Our results lead to a reinterpretation of the TTOT statistic as a test for classes of alternatives larger than the IFR class and including the NBUE class.