RESPONDENT JEOPARDY IN RANDOMIZED
RESPONSE PROCEDURES

by

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Abstract

Randomized response procedures allegedly provide a respondent in a sample survey with a framework in which he can safely give truthful responses without fear of jeopardizing his own interests. Certain designs, however, offer protection only in a superficial sense. In this article the four major such procedures are re-examined with particular attention toward controlling the jeopardy to the truthful respondent. A measure of jeopardy is defined, the procedures are compared at equal levels of jeopardy and a best design is determined within the class of procedures considered.
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1. INTRODUCTION

1.1 Purpose

Randomized response procedures are used in sample surveys of human populations for estimating the proportion of a population possessing a given attribute. They are most appropriately used when the attribute under study is such that possessors of the attribute (simply called possessors hereafter) show a tendency to conceal having it when confronted with an interviewer's direct question. In typical situations the attribute may concern the respondent's involvement or non-involvement in illegal or socially deviant behavior.

The procedures were introduced to diminish bias due to deliberately misleading responses. They accomplish this by providing the respondent with a framework in which he feels he can respond truthfully without fear of revealing definite information about himself. But, however safe the respondent may be made to feel, there are situations in which he can be jeopardizing his interests by answering truthfully.

The purpose of this article is to re-examine the four major randomized response procedures with particular attention being paid to the protection of the respondent's interests.
1.2 General Discussion

A comprehensive survey of randomized response procedures is found in Greenberg et al. [5]. The modes of operation for all randomized response procedures have a number of features in common. In no such procedure is the respondent confronted with a direct question, the interviewer asking outright, "Are you a possessor?" Instead he is directed to respond to one of a selection of several statements. The respondent determines for himself which statement is chosen for a response from the selection but must do so using a randomizing device provided him. The nature of the selection of statements and of the randomizing devise varies from one procedure to another, and individual particulars for the four major randomized response procedures are described in detail in subsequent sections. In all procedures, the interviewer receives only a response, never learning which statement was responded to. Based upon the bulk of information received, the nature of the randomization procedure and the selection of possible statements to respond to, it is possible to estimate the proportion of possessors in a population.

The respondent's answer cannot be interpreted for certain since the interviewer never learns which statement is chosen. With this reasoning the respondent is made to feel secure that a truthful reply from him will not prejudice his interests (particularly important if he is a possessor) and he is assumed to respond truthfully. Indeed it is essential for the validity of the estimation procedure that this be done since the estimate is obtained under the assumption that a truthful answer is given.

Is it safe for a possessor to answer truthfully? While it is technically true that one cannot for certain interpret the respondent's reply,
one can however make a probability statement interpreting the reply. By
his answer the respondent reclassifies himself in a new population, that
of others who have responded similarly and one in which the (new)
proportion of possessors need not be equal to the proportion in the
population of the survey. In cases where this new proportion is greater
than that for the population the possessor places himself in a potentially
more vulnerable position than before. The collector of the information
could have his records used as a screening device to identify possessors
and in fact with great profit if the new proportion of possessors is
high.

For example, suppose the respondent is to reply to one of the two
statements:

I. I cheated on my income tax last year.

II. My mother's birthdate is in April.

He chooses statement I with probability P and II with probability 1-P,
responding YES if he agrees with the statement and NO if not. If the
proportion of possessors (persons who cheated on their income tax last
year) of the population is .15 and the (new) proportion of possessors
in the set of YES respondents is .9, (as it might be with an injudicious
choice of P) a possessor would have placed himself in a more vulnerable
position with a truthful YES response if the interviewer's records are
later used as a screening device. The set of YES respondents would
clearly be a more fruitful group to investigate for tax evasion than a
randomly chosen subset of the population.

A collector of survey information may not intend that the data he
gathers be used in a screening procedure. However if this happens and if
insufficient attention has been given to the question of respondent jeopardy he will find he has unknowingly misled his respondents into believing they were safe in providing truthful answers. This is a fact which may later be a source of embarrassment for him. In the absence of confidentiality privileges accorded the attorney or clergyman, protection of the interviewer and respondent alike can only come through the judicious choice of a design.

One can, of course, easily conceive of situations in which the problem of respondent jeopardy holds no interest. The attribute under study may be such that no jeopardy occurs in a practical sense. For example, a YES response to the statement, "Sometime during the past year I operated a motor vehicle while under the influence of alcohol," would not jeopardize a respondent's interests beyond mild embarrassment since corroborative evidence necessary for a legal conviction would be, at best, very difficult to obtain. Nevertheless, many situations remain wherein respondent jeopardy is a real issue and it is toward such situations that the results of this article apply.

1.3 Organization of the Article

To establish a measurement of respondent jeopardy one can, for each randomized response procedure and for each jeopardizing response, define a level of jeopardy. See section 2. Two different designs having equal levels of jeopardy are seen to provide the respondent with the same degree of protection whatever the value of the unknown proportion of possessors. In selecting a design when respondent jeopardy is an issue it is proposed here that one should first determine the degree of protection to be afforded the respondent through the choice of a level of jeopardy.
Comparisons to choose a best design should then be made between designs in a class all having that chosen level of jeopardy. Preference is accorded the design which yields an estimate having minimum variance within the class considered.

An analysis following these lines is given for the four major randomized response procedures in the remainder of this article. The level of jeopardy is defined in section 2 where it is also noted that no randomized response design exists for which each possible response fails to increase jeopardy, attesting to the need for an analysis of questions of jeopardy. Sections 3 through 6 contain discussions of the individual procedures. Conclusions are given in section 7 and proofs in section 8.

2. JEOPARDIZING RESPONSES AND THE LEVEL OF JEOPARDY FUNCTION

The four major randomized response procedures discussed in this article have some common conventions and notation. In each procedure only one attribute, A, of a sensitive nature is of interest. A respondent is said to belong to group A if and only if he possesses attribute A. The proportion of possessors is denoted by \( \pi \) and the purpose of the survey is to estimate \( \pi \).

A respondent who provides response R is said to belong to group R, the subpopulation of all persons in the sample giving response R. The probability a respondent in group R belongs to group A is denoted by \( P(A|R) \). A possessor is said to increase his jeopardy with response R if and only if \( P(A|R) > \pi \), in which case the response R is called jeopardizing. A response R is called non-jeopardizing if and only if \( P(A|R) \leq \pi \).

No randomized response procedure exists for which all responses are non-jeopardizing. To see why this is so, consider first a procedure with only two possible responses denoted by \( R_1 \) and \( R_2 \). If both responses are to be
non-jeopardizing one must have $P(A|R_i) \leq \pi$ for $i = 1, 2$. Using Bayes' rule this becomes $P(A|R_1) = P(R_1|A) \pi / [P(R_1)\]^{-1} \leq \pi$ for $i = 1, 2$ or

$$P(R_i|A) \leq P(R_i) \quad i = 1, 2.$$ \hspace{1cm} (2.1)

Adding the two inequalities in (2.1) yields

$$1 = P(R_1|A) + P(R_2|A) \leq P(R_1) + P(R_2) \leq 1.$$ \hspace{1cm} (2.2)

Since strict inequality in (2.2) is impossible, strict equality must hold in both expressions of (2.1). Thus, if a procedure exists for which no increase in jeopardy occurs for either response it must be designed in such a way that the respondent's replies are independent of his membership in group A. But in this case the replies carry no information about event A, and $\pi$ cannot be estimated from the bulk of responses received in the survey. Thus no randomized response design with two possible replies, both non-jeopardizing exists. The argument for designs in which more than two responses are possible (e.g. repeated trials designs described in section 6) proceeds similarly.

Since each randomized response procedure will have at least one jeopardizing response a desired level of confidentiality can only be achieved through an appropriate choice of design parameters.

The quantity $P(A|R)$ can be calculated using Bayes' rule, interpreting $\pi$ as the probability of belonging to group A. One can write:

$$P(A|R) = P(A|R) \pi / [P(R|A)\pi + P(R|A^C)(1-\pi)]^{-1}$$

$$= [1 + (1-\pi)\pi^{-1}g]^{-1}.$$ \hspace{1cm} (2.3)

where $g = P(R|A^C)\pi^{-1}(R|A)$. It is important to note that $g$ is a function
of the parameters in the design, and of R, but not of \( \pi \). The quantity \( P(A|R) \) is of special interest when R is a jeopardizing response. The choice of parameters in a given design determines the value for \( P(A|R) \), which gives a measure for the jeopardy associated with response R. The function \( g \) in (2.3) is called the level of jeopardy function for the given procedure and response R and when evaluated at specific design parameters, the functional value of \( g \) will be called the level of jeopardy for the design and response R. Two designs having the same level of jeopardy afford the same level of respondent protection uniformly in the unknown parameter \( \pi \).

3. THE UNRELATED QUESTION PROCEDURE

The theoretical framework for the unrelated question procedure was developed by Greenberg et al. [4]. In this procedure the respondent randomly chooses one of two statements to which he responds YES or NO. A YES response indicates agreement with the statement and a NO response indicates disagreement. The statements are of the form:

(I) I belong to group A, and

(II) I belong to group B.

Recall that membership in group A means possessing the sensitive attribute. Membership in group B carries no penalty social or legal so respondents will respond to it truthfully. Statement (II) is determined so that the proportion, \( \beta \), of the population belonging to group B is known. (It should be noted that unrelated question procedures in which \( \beta \) is unknown but must be estimated have been proposed but are not considered in this article.

See Greenberg et al. [4], Moors [6] and Dowling and Schactman [3].)
Statement (I) is chosen by the respondent with probability $P$ and statement (II) with probability $1-P$. For a particular unrelated question design two parameters $P$ and $\beta$, (through the appropriate choice of group B) must be determined.

For this procedure the probabilities of a YES and of a NO response, events $Y$ and $Y^c$, respectively, are given by

$$P_U(Y) = \pi P + (1-P)\beta$$
$$P_U(Y^c) = P(1-\pi) + (1-P)(1-\beta)$$  \hspace{1cm} (3.1)

The maximum likelihood estimate for $\pi$ (see [4] for details) is given by

$$\hat{\pi}_U = [\kappa n^{-1} - (1-P)\beta]P^{-1}$$ \hspace{1cm} \text{for } P \neq 0

where $n$ is the number of respondents in the sample and $k$ is the number of YES respondents. This estimate is unbiased and its variance can be expressed using (3.1) as:

$$\text{V}(\hat{\pi}_U) = \text{V}(\hat{\pi}_U(P, \beta)) = P_U(Y)P_U(Y^c)[nP^2]^{-1}$$ \hspace{1cm} (3.2)

Using Bayes' rule one finds, using (3.1),

$$P_U(A|Y) = \pi[P + (1-P)\beta][P_U(Y)]^{-1}$$
$$P_U(A|Y^c) = \pi(1-P)(1-\beta)[P_U(Y^c)]^{-1}.$$ \hspace{1cm} (3.3)

By considering the appropriate inequalities, it is easy to see that $P_U(A|Y^c) \leq \pi$ for all choices of $P$ and $\beta$, whereas the inequality $P_U(A|Y) \leq \pi$ holds only when $P = 0$, the value of $P$ for which the information gathered is independent of $A$. Since a response of NO never increases respondent
jeopardy, and a YES response always does, one need only consider $P_U(A|Y)$. Using the expression for $P_U(A|Y)$ in (3.3) and rewriting it in the form of (2.3) one obtains the level of jeopardy function:

$$g_U(P, \beta) = (1-P)\beta[1 + (1-P)\beta]^{-1}.$$  \hspace{1cm} (3.4)

3.1 The Best Unrelated Question (BUQ) Design.

From (3.4) it is clear that different combinations of $P$ and $\beta$ can produce the same level of jeopardy and thereby the same degree of respondent protection. (For example $P_U(A|Y)$ takes on the same value when $P = .5$ and $\beta = .5$ as when $P = 1/6$ and $\beta = .1$.) Among the totality of combinations of $P$ and $\beta$ producing a given index of jeopardy it is possible to find one for which the variance of $\hat{\tau}_U$ is a minimum. For this two lemmas are required. Proofs appear in section 8.

**LEMMA 3.1.** Given $P$, $\beta$ and $\beta'$ where $0 < P, \beta, \beta' < 1$ and $\beta \neq \beta'$, there exists a $P'$, $0 < P' < 1$ such that

$$g_U(P, \beta) = g_U(P', \beta') \hspace{1cm} (3.5)$$

Furthermore, if $\beta \leq \beta'$, then $P \leq P'$.

It is not true that for given $P, P', \beta$ where $0 < P, P', \beta < 1$, that there exists a $\beta'$ $0 < \beta' < 1$ such that (3.5) holds.

**LEMMA 3.2** Let $\beta \leq \beta'$ and $P \leq P'$ be given so that $g_U(P, \beta) = g_U(P', \beta')$. Then $V(\hat{\tau}_U(P, \beta)) \geq V(\hat{\tau}_U(P', \beta'))$.

From these lemmas one concludes that among all unrelated question designs with a given level of jeopardy, the design for which the variance of $\hat{\tau}_U$ is the smallest is the design with $\beta = 1$. That is, for fixed $g$,
\[ V\left[ \hat{\pi}_U(P(1),1) \right] = \min_{\{P, \beta: \ g_U[P(\beta), \beta] = g\}} V\left[ \hat{\pi}_U(P(\beta), \beta) \right]. \]

This will be called the best unrelated question, BUQ, design with a given level of jeopardy. Note that for the BUQ design, using (3.2) and (3.4),

\[ V\left[ \hat{\pi}_U(P,1) \right] = (1 - \pi)[1 - P(1 - \pi)](nP)^{-1} \]

(3.6)

and

\[ g_U(P,1) = 1 - P. \]

(3.7)

It should be emphasized that the value \( \beta = 1 \) is preferred, in contrast to conclusions reached in earlier articles (See [5]), because here comparisons between designs are made at given levels of jeopardy. For fixed \( P \), decreasing \( \beta \) yields a more efficient estimate for \( \pi \) but also increases respondent jeopardy, reflected in an increase in \( P_U(A|Y) \).

There may be some added appeal in using the BUQ design inasmuch as the possessor is aware that the jeopardizing response, YES, is expected whenever question (II) is chosen. This may lessen his suspicions and may increase his willingness to respond truthfully.

It will be seen in subsequent sections that the BUQ design is the most efficient among the four procedures considered in this article when they are compared at equal levels of jeopardy.

4. THE CONTAMINATION PROCEDURE

The contamination procedure was proposed by Boruch [1]. In this procedure the respondent is presented with one statement, "I am a member of group A," and is instructed to lie or tell the truth depending upon
jeopardy, and a YES response always does, one need only consider $P_U(A|Y)$. Using the expression for $P_U(A|Y)$ in (3.3) and rewriting it in the form of (2.3) one obtains the level of jeopardy function:

$$e_U(p, \beta) = (1-p)\beta[p + (1-p)\beta]^{-1}. \quad (3.4)$$

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**Lemma 3.1.** Given $p$, $\beta$ and $\beta'$ where $0 < p, \beta, \beta' < 1$ and $\beta \neq \beta'$, there exists a $p'$, $0 < p' < 1$ such that

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Furthermore, if $\beta \leq \beta'$, then $p \leq p'$.

It is not true that for given $p, p', \beta$ where $0 < p, p', \beta < 1$, that there exists a $\beta'$ $0 < \beta' < 1$ such that (3.5) holds.

**Lemma 3.2** Let $\beta \leq \beta'$ and $p \leq p'$ be given so that $e_U(p, \beta) = e_U(p', \beta')$. Then $\nabla \{\hat{\pi}_U(p, \beta)\} \geq \nabla \{\hat{\pi}_U(p', \beta')\}$.

From these lemmas one concludes that among all unrelated question designs with a given level of jeopardy, the design for which the variance of $\hat{\pi}_U$ is the smallest is the design with $\beta = 1$. That is, for fixed $g$,
\[ V\left( \hat{\pi}_U(P(1),1) \right) = \min \{ (P, \beta): \ g_U[P(\beta), \beta] = g \} \]

This will be called the best unrelated question, BUQ, design with a given level of jeopardy. Note that for the BUQ design, using (3.2) and (3.4),

\[ V\left( \hat{\pi}_U(P,1) \right) = (1 - \pi)[1 - P(1 - \pi)](nP)^{-1} \]

and

\[ g_U(P,1) = 1 - P. \] (3.7)

It should be emphasized that the value \( \beta = 1 \) is preferred, in contrast to conclusions reached in earlier articles (See [5]), because here comparisons between designs are made at given levels of jeopardy. For fixed \( P \), decreasing \( \beta \) yields a more efficient estimate for \( \pi \) but also increases respondent jeopardy, reflected in an increase in \( P_U(A|Y) \).

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It will be seen in subsequent sections that the BUQ design is the most efficient among the four procedures considered in this article when they are compared at equal levels of jeopardy.

4. THE CONTAMINATION PROCEDURE

The contamination procedure was proposed by Boruch [1]. In this procedure the respondent is presented with one statement, "I am a member of group A," and is instructed to lie or tell the truth depending upon
the outcome of a randomizing procedure. The two parameters in this
design are the probabilities for false positive and false negative
statements. They are denoted by \( \phi_p = P(Y|A^c) \) and \( \phi_n = P(Y^c|A) \). Note that

\[
P_C(Y) = \pi(1 - \phi_n) + (1 - \pi)\phi_p, \\
and \\
P_C(Y^c) = \pi\phi_n + (1 - \pi)(1 - \phi_p).
\]

The maximum likelihood estimate is given by

\[
\hat{\pi}^c = [kn^{-1} - \phi_p][1 - \phi_n - \phi_p]^{-1} \text{ for } \phi_n + \phi_p \neq 1,
\]

where \( n \) is the number of respondents in the survey and \( k \) is the number
of YES respondents. This estimate is unbiased and has variance (using (4.1))

\[
V(\hat{\pi}^c) = V(\hat{\pi}^c(\phi_p, \phi_n)) = P_C(Y)P_C(Y^c)[n(1 - \phi_p - \phi_n)^2]^{-1}.
\]

Using Bayes' rule one finds

\[
P_C(A|Y) = \pi(1 - \phi_n)[P_C(Y)]^{-1}
\]

and

\[
P_C(A|Y^c) = \pi\phi_n[P_C(Y^c)]^{-1}.
\]

It is easily seen from (4.3) that \( P_C(A|Y) \leq \pi, [P_C(A|Y^c) \leq \pi] \) if and only
if \( \phi_p + \phi_n \geq 1, (\phi_p + \phi_n \leq 1) \). Thus exactly one response is jeopardizing
and for purposes of the following discussion the case where \( \phi_n + \phi_p > 1 \),
where a NO response is jeopardizing, is considered. Then from (4.3)
one can write \( P_C(A|Y^c) \) in the form (2.3) where the level of jeopardy
function is given by

\[
\phi_C = \phi_C(\phi_p, \phi_n) = (1 - \phi_p)\phi_n^{-1}.
\]
In the case under consideration, $0 \leq g_C(\phi_p, \phi_n) \leq 1$. For fixed $\phi_n$ and $g$, $0 \leq g, \phi_n \leq 1$, there exists a $\phi_p$, $0 \leq \phi_p \leq 1$ such that $g_C(\phi_p, \phi_n) = g$ and $1 \leq \phi_p + \phi_n$. From (4.2) and (4.3) one can rewrite

$$V(\hat{\pi}^C(\phi_p, \phi_n)) = [(1 - \pi)g_C + \pi][\phi_p^{-1} - \pi - (1 - \pi)g_C][(g_C - 1)^2 n]^{-1} \quad (4.5)$$

When considered as a function of $\phi_n$, $0 \leq \phi_n \leq 1$, $\phi_n + \phi_p > 1$, with $\phi_p$ varying so as to cause $g_C(\phi_p, \phi_n) = g_C$ to remain constant, $V(\hat{\pi}^C(\phi_p, \phi_n))$ takes on its minimum when $\phi_n = 1$, as is immediate from (4.5). Thus, for a fixed level of jeopardy, the best contamination design in the case $\phi_p + \phi_n > 1$ is the one for which $\phi_n = 1$. In this design, a possessor always responds NO and a non-possessor responds NO with probability $1 - \phi_p$ and YES with probability $\phi_p$. (The case $\phi_p + \phi_n < 1$ is symmetric to the one under discussion and yields a best design when $\phi_n = 0$.)

When $\phi_n = 1$, one has from (4.4) and (4.5)

and

$$g_C(\phi_p, 1) = 1 - \phi_p \quad (4.6)$$

$$V(\hat{\pi}^C(\phi_p, 1)) = (1 - \pi)[1 - \phi_p(1 - \pi)](n \phi_p)^{-1} \quad (4.7)$$

4.1 The Best Contamination Design vs. the BUQ Design.

With respect to the criterion for comparing designs, as proposed in this article, the BUQ design and the best contamination design are equivalent. That is, for a given level of jeopardy the estimates for the two designs have equal variances. To see this one need only identify the parameter $P$ in the BUQ design with $\phi_p$ in the best contamination design and compare (3.7) with (4.6) and (3.6) with (4.7). Furthermore under identical survey conditions either design will yield the same estimate.
5. THE WARNER PROCEDURE

In 1969 Warner [7] proposed the first randomized response procedure. In his procedure the respondent randomly chooses one of two statements for a response:

(I) I belong to group A,

(II) I do not belong to group A.

A YES response indicates agreement with the statement and a NO response indicates disagreement. The respondent chooses statement (I) with probability $P$ and (II) with probability $1 - P$, and $P$ is the only parameter to be determined in the procedure. One can easily calculate the probabilities of YES and NO responses (events $Y$ and $Y^C$ respectively) as

$$P_W(Y) = P\pi + (1 - P)(1 - \pi)$$

$$P_W(Y^C) = P(1 - \pi) + (1 - P)\pi.$$  \hspace{1cm} (5.1)

Warner has given the maximum likelihood estimate for $\pi$,

$$\hat{\pi}_W = \frac{kn^{-1} - (1 - P)(2P - 1)^{-1}}{n(2P - 1)^2} \text{ for } P \neq 1/2$$

where $n$ is the number of respondents in the survey and $k$ is the number of YES responses. The estimate, $\hat{\pi}$, is unbiased and has variance, using (5.1), given by

$$V(\hat{\pi}_W) = P_W(Y)P_W(Y^C)[n(2P - 1)^2]^{-1}$$  \hspace{1cm} (5.2)

Using Bayes' rule one finds
\[ P_w(A|Y) = [1 + (1 - \pi)\pi^{-1}g_1(P)]^{-1} \]  

(5.3a)

and

\[ P_w(A|Y^c) = [1 + (1 - \pi)\pi^{-1}g_2(P)]^{-1} \]  

(5.3b)

where the jeopardy functions for responses \( Y \) and \( Y^c \) are, respectively,

\[ g_1(P) = (1 - P)P^{-1} \]  

(5.4a)

\[ g_2(P) = P(1 - P)^{-1} \]  

(5.4b)

A YES, (NO) response will be non-jeopardizing if \( P_w(A|Y) \leq \pi \), \( (P_w(A|Y^c) \leq \pi) \).

It is easily seen that this occurs if and only if \( P \leq 1/2, (P \geq 1/2) \).

Note that a value of \( P = 1/2 \) (not permitted) is the value for which responses are independent of the event \( A \).

5.1 The Warner Design vs. The BUQ Design.

To compare the Warner Design with \( P > 1/2 \) with the BUQ Design, use (3.6) and (5.2) to write

\[
\frac{V(\tilde{\pi}_w)}{V(\tilde{\pi}_u)} = \frac{[P\pi + (1 - P)(1 - \pi)][1 - P\pi - (1 - P)(1 - \pi)]nP'}{[1 - P'(1 - \pi)(1 - \pi)n(2P - 1)^2} \]  

(5.5)

Where \( P \) and \( P' \) are the parameters in the Warner Design and the BUQ Designs, respectively. Comparisons are made between designs with the same levels of jeopardy. Equating levels of jeopardy using (5.4a) and (3.7) one has \( (1 - P)P^{-1} = 1 - P' \) or

\[ P' = P^{-1}(2P - 1) \]  

where \( P > 1/2 \)  

(5.6)

Substituting for \( P' \) from (5.6) in (5.5) yields
\[
\frac{V(\hat{\pi}_W)}{V(\hat{\pi}_U)} = \frac{P + \pi - 2P\pi}{P + \pi - 2P\pi - (1 - P)} > 1, \quad P > 1/2.
\]

Note that the denominator in the middle term above is positive since it equals \((1 - \pi)(2P - 1)\). Hence the ratio is seen to be greater than 1. Hence \(\hat{\pi}_U\) is a more efficient estimate than \(\hat{\pi}_W\) for \(\pi\).

A similar argument can be carried out for the case where \(P < 1/2\) in the Warner Design. Hence the BUQ Design always yields a more efficient estimate than does the Warner Design in all comparisons at equal levels of jeopardy, and for all values of \(\pi\).

6. REPEATED TRIALS PROCEDURES

A variety of procedures involving repeated trials with each respondent have been proposed in an effort to find more efficient estimates for \(\pi\). One of these variants was introduced by Chow [2].

In the Chow procedure, referred to hereafter as the repeated trials procedure, the respondent is given a spherical bottle with a narrow neck. The bottle contains \(K\) balls, \(M\) red and \(K - M\) white. The respondent shakes the bottle thoroughly and tilts it up to allow exactly \(k\) balls to drop into the neck of the bottle. Without letting the interviewer know what balls are showing in the neck, the respondent states the number of red balls if he belongs to group A or the number of white balls if he does not. One requires that \(k \leq \min (M, K - M)\). The response of the \(i^{th}\) person is denoted by \(Z_i\) and one can calculate

\[
P_R(Z_i = m) = \frac{\binom{M}{m} \binom{K - M}{k - m}}{\binom{K}{k}} + (1 - \pi) \binom{K - M}{m} \binom{M}{k - m} \quad (6.1)
\]
for \( m = 0, 1, \ldots, k \). The maximum likelihood estimate for \( \pi \) (See [2] for details) is given by

\[
\hat{\pi}_R = \frac{\bar{z} - (1 - MK^{-1})}{2 MK^{-1} - 1}, \quad MK^{-1} \neq 1/2
\]

where \( \bar{z} = (kn)^{-1} \sum_{i=1}^{n} z_i \) and \( n \) is the number of respondents in the sample.

Moreover

\[
V(\hat{\pi}_R) = \frac{\pi(1 - \pi)}{n} + \frac{(K - k) MK^{-1}(1 - MK^{-1})}{(K - 1)kn(2MK^{-1} - 1)^2} \tag{6.2}
\]

Using Bayes' rule one finds that

\[
P_R(A | Z = m) = \left[ \binom{M}{m} \binom{K - M}{k - m} \pi[P_R(Z = m)]^{-1}
\right]^{-1} \tag{6.3}
\]

\[
= \left[ 1 + g_R(K, M, k, m)(1 - \pi)\pi^{-1} \right]^{-1}
\]

where

\[
g_R(K, M, k, m) = \left[ \binom{M}{m} \binom{M}{k - m} \right]^{-1} \binom{K - M}{m} \binom{K - M}{k - m}
\]

Two symmetric cases exist, \( M > K - M \) and \( M < K - M \). For purposes of this discussion the case \( M > K - M \) is chosen. In this case any response of \( m > \frac{k}{4}k \) is jeopardizing as is seen in the following lemma, the proof of which is given in section 8.
Lemma 6.1. For the repeated trials procedure with \( M > K - M \).

(a) \( P_R(A|Z = m) > \pi \) whenever \( m > k - m \),

(b) \( P_R(A|Z = m) < \pi \) whenever \( m < k - m \), and

(c) \( P_R(A|Z = m) = \pi \) whenever \( m = k - m \).

If one is to give the problem of confidentiality of responses first priority one presumably should design the procedure to guard against the worst that can happen. The response \( m = k \) yields the greatest jeopardy in the case under consideration as in noted in Lemma 6.2.

Lemma 6.2. In the repeated trials procedure with \( M > K - M \), \( P_R(A|Z = k) = \max_{0 \leq m \leq k} P_R(A|Z = m) \).

For purposes of comparison below the level of jeopardy function corresponding to the response \( Z = k \) is given, using (6.4) as:

\[
ge_R(K,M,k,k) = \left( \begin{array}{c} K - M \\ k \end{array} \right) \left( \begin{array}{c} M \\ k \end{array} \right)^{-1} \tag{6.5}\]

6.1 The Repeated Trials Design vs. the BUQ Design.

To compare the BUQ Design with the repeated Trials Design with \( M > K - M \) use (6.2) and (3.6) to write

\[
\frac{V(\pi_R^\circ)}{V(\pi_U^\circ)} = \frac{\pi(1 - \pi)n^{-1} + (K - k)MK^{-1}(1 - MK^{-1})[(K - 1)kn(2MK^{-1} - 1)^2]^{-1}}{(1 - \pi)[1 - P(1 - \pi)][nP]^{-1}} \tag{6.6}\]

Comparisons are made between designs with the same levels of jeopardy.

Equating levels of jeopardy using (6.5) and (3.7) one has

\[
1 - P = \left( \begin{array}{c} K - M \\ k \end{array} \right) \left( \begin{array}{c} M \\ k \end{array} \right)^{-1} \tag{6.7}\]
Substitution for $P$ in (6.6) from (6.7) yields

$$\frac{V(\hat{\pi}^*_R)}{V(\hat{\pi}^*_U)} = \frac{1}{1 - \pi} \cdot \frac{(K - k)M(K - M)}{k(K - 1)(2M - K)^2} \left\{ \begin{bmatrix} M \\ k \end{bmatrix} \begin{bmatrix} K - M \\ k \end{bmatrix}^{-1} \right\}. $$

To show this ratio is greater than 1, it is sufficient to prove the following lemma:

**Lemma 6.3.** For any integers $k$, $M$, $K$, such that $1 \leq k \leq K - M < M < K$,

$$\frac{(K - k)M(K - M)}{k(K - 1)(2M - K)^2} \left\{ \begin{bmatrix} M \\ k \end{bmatrix} \begin{bmatrix} K - M \\ k \end{bmatrix}^{-1} \right\} > 1.$$

An indication of the proof of this lemma appears in section 8.

The comparison with a repeated Trials Design with $M < K - M$ is carried out similarly. Hence the EUQ Design always yields a more efficient estimate than does any repeated trials design.

7. CONCLUSIONS

The results of this article apply to randomized response procedures in surveys for which protection of the respondents' interests is a major concern. Using the level of jeopardy function as defined in section 2, one has a means of quantifying the degree of jeopardy a respondent places himself in with a given response. The discussion of what level of protection is to be offered is a subjective one and should be given first priority. Table 1 gives a tabulation of $P(A|R)$ for selected values of $\pi$ and $g$. The table enables one to compare the proportion of possessors in the population, $\pi = P(A)$, with the proportion of possessors in the set of respondents giving response $R_1$, $P(A|R)$. For example, in a design with
a choice of \( g = .3 \), if \( \pi = .05 \) (unknown, of course) by a response \( R \)
the respondent reclassifies himself in a set in which the proportion of
possessors is \( .15 \).

**TABLE 1**

\[
P(A|R) \text{ FOR SELECTED VALUES OF } \pi \text{ AND } g.
\]

<table>
<thead>
<tr>
<th>Index of Jeopardy, ( g )</th>
<th>Proportion of Possessors, ( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.05</td>
</tr>
<tr>
<td>.9</td>
<td>.06</td>
</tr>
<tr>
<td>.8</td>
<td>.06</td>
</tr>
<tr>
<td>.7</td>
<td>.07</td>
</tr>
<tr>
<td>.6</td>
<td>.08</td>
</tr>
<tr>
<td>.5</td>
<td>.10</td>
</tr>
<tr>
<td>.4</td>
<td>.12</td>
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<tr>
<td>.3</td>
<td>.15</td>
</tr>
<tr>
<td>.2</td>
<td>.21</td>
</tr>
<tr>
<td>.1</td>
<td>.35</td>
</tr>
</tbody>
</table>

It has been shown that among the four major randomized response
procedures the BUQ design yields the most efficient estimate for \( \hat{\pi} \) among
designs having equal levels of jeopardy. This is found to be so whatever the value of \( g \) and whatever the value of \( \pi \). It may be necessary
to consider the efficiency of \( \hat{\pi}_U(P,1) \) in determining the level of jeopardy
for a survey. Table 2 gives the standard deviation for \( \hat{\pi}_U(P,1) \) for \( n = 100 \).
and for selected values of $g$ and $\pi$. It is not surprising that the efficiency of $\hat{\pi}(P,1)$ decreases as the protection to the respondent increases.

**TABLE 2**

\[
\left\{\frac{\hat{\pi}(P,1)}{\sqrt{n}}\right\}^{1/2} \text{ FOR SELECTED VALUES OF } \pi \text{ AND } g, \; n = 100.
\]

<table>
<thead>
<tr>
<th>Index of Jeopardy $g$</th>
<th>Proportion of Possessors, $\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.05</td>
</tr>
<tr>
<td>.9</td>
<td>.2933</td>
</tr>
<tr>
<td>.8</td>
<td>.1962</td>
</tr>
<tr>
<td>.7</td>
<td>.1503</td>
</tr>
<tr>
<td>.6</td>
<td>.1212</td>
</tr>
<tr>
<td>.5</td>
<td>.1000</td>
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<tr>
<td>.4</td>
<td>.0825</td>
</tr>
<tr>
<td>.3</td>
<td>.0671</td>
</tr>
<tr>
<td>.2</td>
<td>.0539</td>
</tr>
<tr>
<td>.1</td>
<td>.0378</td>
</tr>
</tbody>
</table>

Finally it must be emphasized that the estimate $\hat{\pi}_U$ in the BUQ design fares very poorly in comparison with the estimate $\hat{\pi}_d$ for $\pi$ using the direct question approach. The standard deviation for $\hat{\pi}_d$ is given by $\left\{\pi(1 - \pi)n^{-1}\right\}^{1/2}$. Thus for $\pi = .1$ and $n = 100$, the standard deviation for $\hat{\pi}_d$ equals .03 which is considerably less than any figure in the second column of table 2. One pays twice for using this approach (in terms of using an inefficient estimate), once for using a randomized
response procedure, and again for protecting the respondents' interests. It is clear that while it is possible to determine randomized response designs which afford more than superficial respondent protection it is also clear that the approach of this article is to be recommended only when respondent jeopardy is of serious concern.
8. PROOFS

Proofs or indications of proofs for the Lemmas in sections 3 and 6 are given in this section. The reader is referred to the appropriate earlier section for statements of the lemmas.

8.1 Proof of Lemma 3.1.

Using (3.4) set \( g(P, \beta) = g(P', \beta') \) and solve for \( P' \) to get

\[
P' = \beta' P [\beta' P + \beta(1 - P)]^{-1}.
\] (8.1)

From the above it is clear that \( 0 \leq P' \leq 1 \). For the second statement, suppose \( \beta \leq \beta' \). Then \( \beta(1 - P) + \beta' P < \beta' \) or \( \beta'[\beta' P + \beta(1 - P)]^{-1} > 1 \).

Thus, using (8.1) one sees \( P' > P \) as required.

8.2 Proof of Lemma 3.2.

Write the expression for \( D = \text{Var} \pi_U (P, \beta)[\text{Var} \pi_U (P', \beta')][^{-1} \) using (3.1) and (3.2). Then substitute for \( P' \) using (8.1). After algebraic simplification, this yields

\[
D = \frac{\beta'(1 - P \pi - (1 - P)\beta)}{\beta'(1 - P \pi - (1 - P)\beta) + (\beta - \beta')(1 - P)}
\]

Since \( \beta' > \beta \) and the numerator above is positive, \( D > 1 \) as was to be proved.

8.3 Proof of Lemma 6.1.

For part (a) use (6.3) and routine algebra to see that \( P_R(A | Z = m) > \pi \) holds if and only if

\[
\left[ \begin{array}{c} m \\ k - m \\ M \\ k - m \\ m \\ M \\ k - m \\ m \\ k - m \\ m \\ k - m \\ 1 \end{array} \right] > 0.
\] (8.2)
The proof of part (a) is complete if we can show (8.2) holds for $M > K - M$ and $m > k - m$. Since $M > K - M$, $\left[ M - (k - m) \right]_{2m - k} > \left[ K - M - (k - m) \right]_{2m - k}$ where $(a)_b \equiv a(a - 1) \cdots (a - b + 1)$. Thus

\[
(K - M)_{k - m} (M)_{k - m} \left[ M - (k - m) \right]_{2m - k} >
\]

\[
(M)_{k - m} [K - M - (k - m)]_{2m - k} (K - M)_{k - m}
\]
or

\[
(K - M)_{k - m} (M)_{m} > (M)_{k - m} (K - M)_{m}.
\]

From this (8.2) follows directly. Part (b) is proved similarly. Part (c) follows by substitution of $m = k - m$ in (6.3).

8.4 Proof of Lemma 6.2.

From (6.3) and (6.4) one sees the lemma is proved if it is shown that $H \equiv g_R(K,M,k,m + 1) e_R^{-1}(K,M,k,m) < 1$.

Note that

\[
H = \frac{\left( \begin{array}{c} K - M \\ m + 1 \end{array} \right) \left( \begin{array}{c} M \\ k - m - 1 \end{array} \right) (M)_{m} (K - M)_{k - m}}{\left( \begin{array}{c} M \\ m + 1 \end{array} \right) \left( \begin{array}{c} K - M \\ k - m - 1 \end{array} \right) (K - M)_{m} (M)_{k - m}} = \frac{(K - M - k + m + 1)(K - M - m)}{(M - m)(M - k - m + 1)}.
\]

Since $K - M < M$, it follows that

$K - M - (k - m) + 1 < M - (k - m) + 1$ and

$K - M - m < M - m$. Thus $H < 1$.

8.5 Proof of Lemma 6.3.

Let $k, M, K$ be integers with $0 < k \leq K - M < M < K$.

Let $L = \frac{M(K - M)}{(2M - K)^2(K - 1)}$ and $h(k) = \left( \begin{array}{c} M \\ k \end{array} \right) \left[ \begin{array}{c} K - M \\ k \end{array} \right]^{-1}$.
One must show that \((Kk^{-1} - 1) L(h(k) - 1) > 1\) or

\[h(k) > 1 + [L(Kk^{-1} - 1)]^{-1}\] for \(1 \leq k \leq K - M\). \hspace{1cm} (8.3)

This can be proved inductively. The case for \(k = 1\) is straightforward.

Assume (8.3) holds for arbitrary \(k\). Note that

\[h(k + 1) = \frac{M - k}{K - M - k} h(k) > \frac{M - k}{K - M - k} \left(1 + [L(Kk^{-1} - 1)]^{-1}\right)\]

The induction step (and the Lemma) is proved if it is shown that

\[\frac{M - k}{K - M - k} \left(1 + \frac{1}{L(Kk^{-1} - 1)}\right) > 1 + \frac{1}{L[(K + 1)^{-1} - 1]}\] \hspace{1cm} (8.4)

for \(1 \leq k \leq K - M - 1\). Inequality (8.4) holds if and only if

\[
f(k) \equiv (L - 1)(2M - K)k^2 + [L(2M - K)(1 - 2K) - (2M - K)(1 - K) + K]k + L(2M - K)K(K - 1) - K(K - M) > 0\] \hspace{1cm} (8.5)

for \(1 \leq k \leq K - M - 1\).

To determine for what values of \(k\), \(f(k) > 0\), one can analyze \(f\) as a quadratic expression of a continuous variable \(x\) of the form \(f(x) = ax^2 + bx + c\). If \(L > 1\), two possibilities exist. Either \(f(x)\) has no real roots in which case \(f(x) > 0\) for all \(x\) (in which case (8.5) holds) or \(f(x)\) has two positive real roots. One then notes \(K - M < -b/2a\) and
\( f(K - M) > 0 \). Thus \( f(k) > 0 \) for \( 1 \leq k \leq K - M \) as required.

If \( L < 1 \), \( f(x) \) has two real roots, one positive and one negative. Since \( f(0) > 0 \) and \( f(K - M) > 0 \) one concludes that \( f(k) > 0 \) for \( 1 \leq k \leq K - M \) as required.

Finally if \( L = 1 \), then \( f(x) \) has one real root. Since \( f(0) > 0 \), \( f(K - M) > 0 \), \( f(k) > 0 \) for \( 0 < k < K - M \) as required and the Lemma is proved.
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Respondent Jeopardy in Randomized Response Procedures

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March, 1975

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Randomized response techniques
Sample surveys
Confidentiality
Respondent jeopardy

Randomized response procedures allegedly provide a respondent in a sample survey with a framework in which he can safely give truthful responses without fear of jeopardizing his own interests. Certain designs, however,
20. ABSTRACT (Continued)

offer protection only in a superficial sense. In this particular article the four major such procedures are re-examined with particular attention toward controlling the jeopardy to the truthful respondent. A measure of jeopardy is defined, the procedures are compared at equal levels of jeopardy and a best design is determined within the class of procedures considered.