AN ESTIMATION OF THE ASYMPTOTIC COVARIANCE MATRIX ASSOCIATED WITH BICKEL'S ESTIMATES FOR SHAFT PARAMETERS: THE BIVARIATE CASE

By

Ibrahim A. Ahmad and Pi-Erh Lin†
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The Florida State University
Department of Statistics
Tallahassee, Florida 32306

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ABSTRACT

AN ESTIMATION OF THE ASYMPTOTIC COVARIANCE MATRIX ASSOCIATED WITH BICKEL'S ESTIMATES FOR SHIFT PARAMETERS: THE BIVARIATE CASE

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A nonparametric estimate is proposed for the asymptotic covariance matrix associated with the vector of medians for averages of pairs when estimating the shift parameters in the bivariate one sample case. Sufficient conditions are obtained under which weak and strong consistency of the proposed estimate is established. Using the matrix estimate an approximate $(1 - \alpha)100\%$ confidence region may be constructed for the shift parameters. An example is given using a negative exponential density as kernel function from which the proposed estimate is obtained explicitly. The results presented in this paper may be extended to the general multivariate case with obvious modifications.
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1. Introduction. Let \((X_{11}, x_{21}), \ldots, (X_{1n}, x_{2n})\) be a random sample from a bivariate population with distribution function (df) \(F(x - \theta)\), where \(x' = (x_1, x_2)\), \(\theta' = (\theta_1, \theta_2)\). Assume that \(F\) is continuous and symmetric about \(\theta' = (0,0)\). Let \(F_i(x_i - \theta_1)\) and \(f_i(x_i - \theta_1)\) denote the marginal df and its corresponding probability density function (pdf), with respect to the Lebesgue measure, of \(X_i\) \((i = 1,2)\). Bickel \cite{2} proposed a point estimate for the shift parameter vector \(\theta\) using the medians of averages of pairs given by

\[\hat{\theta}' = (\hat{\theta}_1, \hat{\theta}_2)\]

where

\[(1.1) \quad \hat{\theta}_i = \hat{\theta}_i(X_{i1}, \ldots, X_{in}) = \text{median}_{1 \leq j \leq n} \frac{X_{ij} + X_{ij}}{2}, \quad (i = 1,2)\]

and proved that \(\sqrt{n}(\hat{\theta} - \theta) \sim AN(0, \Sigma)\) where \(\Sigma = (\sigma_{ij})\) with

\[(1.2) \quad \sigma_{ii} = \frac{1}{12\Delta_i^2}, \quad (i = 1,2)\]

and

\[(1.3) \quad \sigma_{12} = \frac{\chi - \gamma}{\Delta_1 \Delta_2}\]

where we have the set

\[\Delta_i = \text{median}_{1 \leq j \leq n} X_{ij} - \text{median}_{1 \leq j \leq n} X_{ij}\]

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\[ \Delta_i = \int f_i^2(x) \, dx \quad (i = 1, 2) \]

and

\[ \gamma = \iint F(x_1, x_2) f_1(x_1) f_2(x_2) \, dx_1 \, dx_2. \]

(Throughout the study the limits of integration will be omitted whenever they extend over the entire real line.) The numerator of \( \sigma_{12} \) is derived from Eq. (3.3) of Bickel [2] upon integration by parts. It should be noted that \( \Sigma \) is invariant with respect to the shift parameters \( \theta_1 \) and \( \theta_2 \). Thus \( F(x_1, x_2) \) and \( f_i(x_i) \) in the expressions of (1.4) and (1.5) may be replaced by \( F(x_1 - \theta_1, x_2 - \theta_2) \) and \( f_i(x_i - \theta_i) \), \( (i = 1, 2) \), respectively.

When \( F \) is specified, an approximate \((1 - \alpha)100\%\) confidence region for \( \theta \) may be constructed as

\[ n(\hat{\theta} - \theta) \Sigma^{-1} (\hat{\theta} - \theta) \leq \chi^2_1(1 - \alpha), \]

where \( \chi^2_1(1 - \alpha) \) is the \((1 - \alpha)100\%\) percentile of a Chi-square distribution with one degree of freedom. If, on the other hand, \( F \) is not specified, a consistent estimate \( \hat{\Sigma} \) for \( \Sigma \) is desirable so that the expression (1.6) with \( \hat{\Sigma} \) replacing \( \Sigma \) will provide an approximate \((1 - \alpha)100\%\) confidence region for \( \theta \).

It is the purpose of this paper to propose a smooth consistent estimate \( \hat{\Sigma} = (\hat{\sigma}_{ij}) \) for \( \Sigma \). Since \( \Sigma \) is a function of the unknown df and pdf, a natural estimate of \( \Sigma \) may be obtained by replacing these unknown quantities with their respective estimates. As noted earlier \( \Sigma \) is invariant with respect to \( \theta \), we will write \( \hat{\chi} \) for \( \chi - \theta \) in the expressions of df and pdf.

In this paper we will propose a nonparametric estimate of \( \Sigma \) using the kernel method of density and distribution estimates developed by Rosenblatt [6], Parzen [5], Nadaraya [4], and Cacoullos [3], among others.
A kernel function \( k(u) \) is a known pdf satisfying the following conditions:

\[
\sup_{-\infty < u < \infty} k(u) < \infty \quad \text{and} \quad \lim_{|u| \to \infty} |u|k(u) = 0.
\]

Using a kernel function \( k_i \) and a sequence of positive real numbers \( \{a_n\} \), \( a_n \to 0 \) as \( n \to \infty \), an estimate of \( f_i(x) \) is given by

\[
\hat{f}_i(x) = \frac{1}{n} \int k_i \left( \frac{x - u}{a_n} \right) dF_{i_n}(x) = \frac{1}{n} \sum_{j=1}^{n} k_i \left( \frac{x - X_{ij}}{a_n} \right),
\]

where \( F_{i_n}(x) \) denotes the marginal empirical df of \( X_i \) (\( i = 1,2 \)). To estimate \( F(x_1, x_2) \), let \( K_i \) denote the df corresponding to \( k_i \) (\( i = 1,2 \)). Then the kernel estimate of \( F \) is given by

\[
\hat{F}(x_1, x_2) = \int k_i \left( \frac{x_1 - u_1}{a_n} \right) \int k_2 \left( \frac{x_2 - u_2}{a_n} \right) dF_n(u_1, u_2)
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} K_i \left( \frac{x_1 - X_{1j}}{a_n} \right) K_2 \left( \frac{x_2 - X_{2j}}{a_n} \right),
\]

where \( F_n(u_1, u_2) \) denotes the joint empirical df of \( X_1 \) and \( X_2 \). Using (1.8) and (1.9) we propose to estimate \( \sigma_{ij} \) by

\[
\hat{\sigma}_{ii} = \frac{1}{12 \hat{\Delta}_i^2}, \quad (i = 1,2),
\]

and

\[
\hat{\sigma}_{12} = \frac{\frac{1}{2} - \hat{\gamma}}{\hat{\Delta}_1 \hat{\Delta}_2},
\]

where
(1.12) \[ \hat{\Lambda}_i = \int \hat{f}_i^2(x) \, dx \]

and

(1.13) \[ \hat{\gamma} = \int \int \hat{f}(x_1, x_2) \hat{f}_1(x_1) \hat{f}_2(x_2) \, dx_1 \, dx_2 \]

The estimate \( \hat{\Lambda}_i \) was proposed by Bhattacharyya and Roussas [1] for the functional \( \Lambda_i \). They showed that \( E|\hat{\Lambda}_i - \Lambda_i| \to 0 \) as \( n \to \infty \) if \( na_n \to \infty \) as \( n \to \infty \).

In Section 2, some asymptotic behavior of the estimates \( \hat{\Lambda}_i \) and \( \hat{\gamma} \) will be presented. In particular, the strong consistency of \( \hat{\Lambda}_i \) and \( \hat{\gamma} \) are obtained. These results are then used in Section 3 to establish the weak and strong consistency of \( \hat{\sigma}_{ij} \) \( (i,j = 1,2) \). The explicit expressions for \( \hat{\Lambda}_i \) and \( \hat{\gamma} \), using a negative exponential pdf as the kernel function, are presented in Section 4.

2. Consistency of the functional estimates \( \hat{\Lambda}_i \) and \( \hat{\gamma} \). Recall the definition of \( \Lambda_i \), \( \gamma \), \( \hat{\Lambda}_i \), and \( \hat{\gamma} \). The density functional \( \Lambda_i \) is important in the study of asymptotic efficiency for certain nonparametric tests of hypothesis such as location shift, regression, dependence, and analysis of variance, etc. We will establish, in this section, weak as well as strong consistency of \( \hat{\Lambda}_i \) \( (i = 1,2) \) and \( \hat{\gamma} \).

THEOREM 2.1. Let \( na_n \to \infty \) as \( n \to \infty \). Then

(2.1) \[ E|\hat{\Lambda}_i - \Lambda_i| \to 0 \] as \( n \to \infty \) \( (i = 1,2) \).

PROOF. See Theorem 2.2 of Bhattacharyya and Roussas [1].
THEOREM 2.2. Let \( k_i \) \((i = 1, 2)\) be a kernel function with bounded variation. If, for any \( c > 0, \sum_{n=1}^{\infty} \exp(-cn^2) < \infty, \) then

\[
\hat{\Delta}_i \to \Delta_i \quad \text{with probability one as } n \to \infty \quad (i = 1, 2).
\]

PROOF. Theorem 2.1 implies that \( E\hat{\Delta}_i \to \Delta_i \) as \( n \to \infty. \) Thus it suffices to show that, for \( i = 1, 2, \)

\[
|\hat{\Delta}_i - E\hat{\Delta}_i| \to 0 \quad \text{with probability one as } n \to \infty.
\]

Now the left hand side of (2.3) is bounded above by

\[
|\int \hat{F}_i^2(x)dx - E\hat{F}_i^2(x)dx| + |\int \hat{F}_i^2(x)dx - E\hat{F}_i^2(x)dx| \\
\leq \sup_{-\infty < x < \infty} |\hat{f}_i(x) - E\hat{f}_i(x)| \left( \int \hat{f}_i(x)dx + \int E\hat{f}_i(x)dx \right) + \int \text{Var}[\hat{f}_i(x)]dx \\
\leq 2 \sup_{-\infty < x < \infty} |\hat{f}_i(x) - E\hat{f}_i(x)| + \int \text{Var}[\hat{f}_i(x)]dx.
\]

The first term of the last expression follows from the fact that \( \int \hat{f}_i(x)dx = \int E\hat{f}_i(x)dx = 1. \) The second term is approximately equal to \( \left( na_n \right)^{-1} \int f_i(x)dx \int k_i^2(x)dx \) (see Theorem 2A of Parzen [5]) which converges to 0 as \( na_n \to \infty \) and \( n \to \infty. \) Thus, given \( \epsilon > 0, \) there exists a positive integer \( N \) such that, for all \( n \geq N, \) we have

\[
\int \text{Var}[\hat{f}_i(x)]dx < \frac{\epsilon}{2}.
\]

Hence, for \( \epsilon > 0 \) and all \( n \geq N, \) we have

\[
P[|\hat{\Delta}_i - E\hat{\Delta}_i| > \epsilon] \leq P[ \sup_{-\infty < x < \infty} |\hat{f}_i(x) - E\hat{f}_i(x)| > \frac{\epsilon}{4} ]
\]

\[
\leq C \exp(-\beta na_n^2), \quad (0 < C < \infty)
\]

(2.4)
where \( \beta = \alpha(\epsilon/4\mu_1)^2 \) with \( 0 < \alpha \leq 2 \) and \( \mu_1 \) being the total variation of \( k_i \). The last inequality of (2.4) follows from Eq. (6) of Nadaraya [4]. Therefore,
\[
\sum_{n=N}^{\infty} P[|\hat{A}_i - E\hat{A}_i| > \epsilon] < \infty ,
\]
which, in conjunction with the Borel-Cantelli lemma, establishes (2.3) and hence the theorem. \( \Box \)

The next two theorems will deal with the weak and strong consistency of \( \hat{\gamma} \).

**THEOREM 2.3.** If \( na_n \to \infty \) as \( n \to \infty \), then

\[
E|\hat{\gamma} - \gamma| \to 0 \quad \text{as} \quad n \to \infty .
\]

**PROOF.** The theorem implies that \( \hat{\gamma} \to \gamma \) weakly as \( n \to \infty \). Now

\[
|\hat{\gamma} - \gamma| \leq \left| \iint \left[ \hat{f}_1(x_1, x_2) - f_1(x_1, x_2) \right] f_1(x_1) f_2(x_2) dx_1 dx_2 \right| \\
+ \left| \iint \left[ \hat{f}_1(x_1) - f_1(x_1) \right] f_2(x_1, x_2) dx_1 dx_2 \right| \\
+ \left| \iint \left[ \hat{f}_2(x_2) - f_2(x_2) \right] f_1(x_1, x_2) dx_1 dx_2 \right| \\
\leq \sup_{-\infty < x_1, x_2 < \infty} |\hat{F}(x_1, x_2) - F(x_1, x_2)| \\
+ \int |\hat{f}_1(x_1) - f_1(x_1)| dx_1 + \int |\hat{f}_2(x_2) - f_2(x_2)| dx_2 \\
= C_{1n} + C_{2n} + C_{3n} , \quad \text{say} .
\]
The second inequality of (2.6) follows from the facts that $F(x_1, x_2)$ is a df (hence, $F \leq 1$) and that $\int f_1(x_i)dx_i = 1$ for $i = 1, 2$. The theorem will be proved by showing that $E C_{rn} \to 0$ as $n \to \infty$ for $r = 1, 2, 3$. To show $E C_{1n} \to 0$ note that

$$C_{1n} \leq \sup_{-\infty < x_1, x_2 < \infty} |\hat{F}(x_1, x_2) - EF(x_1, x_2)|$$

(2.7)

$$+ \sup_{-\infty < x_1, x_2 < \infty} |EF(x_1, x_2) - F(x_1, x_2)| .$$

We will evaluate the first term on the right hand side of (2.7), and show that it converges to 0 with probability one as $n \to \infty$. Let $F_{21}(\cdot | x_1)$ denote the conditional df of $X_2$, given $X_1 = x_1$ and recall that $F_1(x_1)$ is the marginal df of $X_1$. Then

(2.8) \[ EF(x_1, x_2) = \iint K_1 \left( \frac{x_1 - u_1}{a_n} \right) K_2 \left( \frac{x_2 - u_2}{a_n} \right) dF(u_1, u_2) \]

$$= \int K_1 \left( \frac{x_1 - u_1}{a_n} \right) \left( \int K_2 \left( \frac{x_2 - u_2}{a_n} \right) dF_{21}(u_2 | u_1) \right) dF_1(u_1)$$

$$= \int \left\{ \int K_1 \left( \frac{x_1 - u_1}{a_n} \right) dF_{21}(u_2 | u_1) \right\} dK_2 \left( \frac{x_2 - u_2}{a_n} \right)$$

$$= \iint F(u_1, u_2) dK_1 \left( \frac{x_1 - u_1}{a_n} \right) dK_2 \left( \frac{x_2 - u_2}{a_n} \right) .$$

Similarly,

(2.9) \[ \hat{F}(x_1, x_2) = \iint F_n (u_1, u_2) dK_1 \left( \frac{x_1 - u_1}{a_n} \right) dK_2 \left( \frac{x_2 - u_2}{a_n} \right) . \]
Therefore, making use of (2.8) and (2.9), we have
\[ \sup_{-\infty < x_1, x_2 < \infty} \left| \hat{F}(x_1, x_2) - \hat{E}F(x_1, x_2) \right| \]
(2.10)
\[ \leq \sup_{-\infty < x_1, x_2 < \infty} \left| F_n(x_1, x_2) - F(x_1, x_2) \right| + 0 \]
with probability one as \( n \to \infty \), by Glivenko-Cantelli lemma. Moreover, since
\[ |F_n(x_1, x_2) - F(x_1, x_2)| \leq 2 \]
for all \( x_1 \) and \( x_2 \), we conclude that
\[ E \left\{ \sup_{x_1, x_2} \left| \hat{F}(x_1, x_2) - \hat{E}F(x_1, x_2) \right| \right\} \to 0 \]
as \( n \to \infty \). The second term on the right hand side of (2.7) converges to 0 as \( n \to \infty \) by Theorem 2.1 of Cacoullos [3] with a slight modification taking into account the uniform continuity of \( f \). This convergence result can also be obtained directly from (2.8). Next, we will show that \( E C_{2n} \to 0 \) as \( n \to \infty \). It follows from Theorem 1A of Parzen [5] that, at each continuity point \( x \) of \( f_1 \), \( E|\hat{f}_1(x) - f_1(x)| \to 0 \) as \( na_n \to 0 \) and \( n \to \infty \). Moreover, \( E|\hat{f}_1(x) - f_1(x)| \leq E f_1(x) + f_1(x) \) which is integrable and converges to the integrable function \( 2f_1(x) \) at each continuity point \( x \) of \( f_1 \), the Lebesgue dominated convergence theorem applies. Therefore \( E C_{2n} \to 0 \) as \( n \to \infty \). Similarly, \( E C_{3n} \to 0 \) as \( n \to \infty \).\]

**THEOREM 2.4.** Let \( k_i \) \( (i = 1, 2) \) be a kernel function with bounded variation. If, for any \( c > 0 \), \( \sum_{n=1}^{\infty} \exp(-cna_n^2) < \infty \), then
(2.11) \[ \hat{\gamma} + \gamma \] with probability one as \( n \to \infty \).
PROOF. It is established in Theorem 2.3 that $C_{1n} \to 0 \text{ with probability one as } n \to \infty$. All is remained is to show that, for $r = 2, 3$,

$$C_{rn} \to 0 \text{ with probability one as } n \to \infty.$$ 

(2.12)

We will establish (2.12) only for $r = 2$. The same proof applies for $r = 3$ with obvious modifications. Now

$$C_{2n} \leq \int |\hat{f}_1(x) - E\hat{f}_1(x)| \, dx + \int |E\hat{f}_1(x) - f_1(x)| \, dx$$

$$\leq \int |\hat{f}_1(x) - E\hat{f}_1(x)| \, dx + E \, C_{2n}.$$ 

Since $E \, C_{2n} \to 0$ as $n \to \infty$ (obtained in the proof of Theorem 2.3), (2.12) with $r = 2$ will be established using the Borel-Cantelli lemma if we can show that, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P[\int |\hat{f}_1(x) - E\hat{f}_1(x)| \, dx > \varepsilon] < \infty.$$ 

(2.13)

Let $\delta > 0$ be any (large) real number and define the open interval $A_\delta = (-\delta, \delta)$. Then

$$\int |\hat{f}_1(x) - E\hat{f}_1(x)| \, dx = \int_{A_\delta} + \int_{A_\delta^c}$$

$$\leq \int_{A_\delta} |\hat{f}_1(x) - E\hat{f}_1(x)| \, dx + \int_{A_\delta^c} |\hat{f}_1(x) - E\hat{f}_1(x)| \, dx + \int_{A_\delta^c} \hat{f}_1(x) \, dx$$

$$= \int_{A_\delta} |\hat{f}_1(x) - E\hat{f}_1(x)| \, dx + |1 - \int_{A_\delta} [\hat{f}_1(x) - E\hat{f}_1(x)] \, dx - \int_{A_\delta} \hat{f}_1(x) \, dx|$$

$$+ \int_{A_\delta^c} \hat{f}_1(x) \, dx$$

$$\leq 2\int_{A_\delta} |\hat{f}_1(x) - E\hat{f}_1(x)| \, dx + |1 - \int_{A_\delta} \hat{f}_1(x) \, dx| + \int_{A_\delta^c} \hat{f}_1(x) \, dx$$

(2.14)
\[ \begin{align*}
&= 2 \int_{A_\delta} \left| \hat{f}_1(x) - \hat{E}_f_1(x) \right| dx + 2 \int_{A_\delta^C} \hat{E}_f_1(x) dx \\
&\leq 4\delta \sup_{-\infty < x < \infty} \left| \hat{f}_1(x) - \hat{E}_f_1(x) \right| + 2 \int_{A_\delta^C} \hat{E}_f_1(x) dx.
\end{align*} \]

Since \( \hat{E}_f_1(x) \) is a pdf for each \( n \), the integral \( \int_{A_\delta^C} \hat{E}_f_1(x) dx \) can be made arbitrary small by choosing \( \delta \) sufficiently large. In fact,

\[ \int_{A_\delta^C} \hat{E}_f_1(x) dx = \int_{A_\delta^C} \int_{A_\delta} \frac{f_1(x - a_n u) k(u) du}{dx} dx. \]

(2.15)

\[ = \int k(u) \left[ 1 - F_1(\delta - a_n u) + F_1(-\delta - a_n u) \right] du \]

\[ = 1 - F_1(\delta) + F_1(-\delta) + a_n \left[ f_1(\delta) - f_1(-\delta) \right] \int k(u) du. \]

The last expression of (2.15) is obtained by applying the Taylor expansion to \( F_1 \). Thus, given \( \epsilon > 0 \), there exists a positive integer \( N \) and a positive constant \( \delta^* = \delta^*(\epsilon, N) \) such that, for all \( n \geq N \) and \( \delta \geq \delta^* \), we have

(2.16)

\[ \int_{A_\delta^C} \hat{E}_f_1(x) dx \leq \frac{\epsilon}{4}. \]

Now choose any \( \delta \) (not depending on \( n \)), \( \delta \geq \delta^* \), then for all \( n \geq N \), we have

\[ P \left[ \int \left| \hat{f}_1(x) - \hat{E}_f_1(x) \right| dx > \epsilon \right] \leq P \left[ \sup_{-\infty < x < \infty} \left| \hat{f}_1(x) - \hat{E}_f_1(x) \right| > \frac{\epsilon}{8\delta} \right]. \]

(2.17)

\[ \leq C \exp \left\{ -\alpha \left( \frac{\epsilon}{8\delta} \right)^2 \frac{na_1^2}{n} \right\}, \quad (0 < \alpha \leq 2, \ 0 < C < \infty) . \]
The last inequality follows from Eq. (6) of Nadaraya [4] where \( u_1 \) is the total variation of \( k_1 \). Therefore

\[
\sum_{n=N}^{\infty} P[|\hat{f}_1(x) - E\hat{f}_1(x)| dx > \epsilon] < C_{n=N}^{\infty} \exp(-c n a_n^2),
\]

which is finite by assumption with \( c = \alpha[\varepsilon/(8\delta u_1)]^2, \quad 0 < \alpha \leq 2 \). This establishes (2.13) and hence (2.12). \( \square \)

3. **Consistency of the variance-covariance estimates.** Recall that \( \hat{\sigma}_{ii} = 1/(12 \hat{\Delta}_1^2) \) and \( \hat{\sigma}_{12} = (\hat{\gamma} - \hat{\gamma})(\hat{\Delta}_1 \hat{\Delta}_2) \). The consistency of \( \hat{\sigma}_{ij} \) follows from Theorems 2.1, 2.2, 2.3, and 2.4 in conjunction with the Slutsky lemma. The results are summarized in the following theorems without proofs.

**THEOREM 3.1.** If \( n a_n \to \infty \) as \( n \to \infty \), then, for \( i,j = 1,2 \),

\[
(3.1) \quad \hat{\sigma}_{ij} \to \sigma_{ij} \quad \text{in probability as } n \to \infty.
\]

**THEOREM 3.2.** Let \( k_i \) (\( i = 1,2 \)) be a kernel function with bounded variation. If, for any \( c > 0 \), \( \sum_{n=1}^{\infty} \exp(-c n a_n^2) < \infty \), then, for \( i,j = 1,2 \),

\[
(3.2) \quad \hat{\sigma}_{ij} \to \sigma_{ij} \quad \text{with probability one as } n \to \infty.
\]

4. **An example.** In this section, we will obtain explicit expressions for \( \hat{\Delta}_1 \) and \( \hat{\gamma} \) taking a negative exponential pdf as the kernel function \( k_i \) (\( i = 1,2 \)). That is, for \( i = 1,2 \),

\[
k_i(u) = \exp(-u) \quad \text{if} \quad u \geq 0
\]

\[
= 0 \quad \text{otherwise,}
\]

(4.1)
and

\[ K_i(u) = 1 - \exp(-u) \quad \text{if} \quad u \geq 0 \]

\[ = 0 \quad \text{otherwise} \quad . \]

Using (4.1) and (4.2) the expressions for \( \hat{\Lambda}_i \) and \( \hat{\gamma} \) are obtained as follows: For \( \hat{\Lambda}_i \), we have,

\[
\hat{\Lambda}_i = \frac{1}{(na_n)^2} \int [\sum_{j=1}^{n} \exp\left(-\frac{x - x_{ij}}{a_n}\right)]^2 \, dx
\]

\[
= \frac{1}{na_n} \int_{X_i}^{\infty} \exp\left[-\frac{2(x - x_{il})}{a_n}\right] \, dx
\]

\[
+ \frac{1}{(na_n)^2} \sum_{j \neq i} \max(x_{ij}, x_{il}) \int_{X_i}^{\infty} \exp\left[-\frac{2x - (x_{ij} + x_{il})}{a_n}\right] \, dx
\]

\[
- \frac{1}{2na_n} + \frac{1}{2n^2 a_n} \sum_{j \neq i} \exp\left[-\frac{|x_{ij} - x_{il}|}{a_n}\right] .
\]

For \( \hat{\gamma} \), we have

\[
\hat{\gamma} = \frac{1}{n} \sum_{j=1}^{n} \int \left\{ \left[ 1 - \exp\left(-\frac{t - x_{1j}}{a_n}\right) \right] \left[ 1 - \exp\left(-\frac{s - x_{2j}}{a_n}\right) \right] \right\}
\]

\[
\times \frac{1}{(na_n)^2} \left[ \sum_{j=1}^{n} \exp\left(-\frac{t - x_{1j}}{a_n}\right) \right] \left[ \sum_{j=1}^{n} \exp\left(-\frac{s - x_{2j}}{a_n}\right) \right] \, dt \, ds
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \left\{ \int \left[ 1 - \exp\left(-\frac{t - x_{1j}}{a_n}\right) \right] \frac{1}{na_n} \sum_{j=1}^{n} \exp\left(-\frac{t - x_{1j}}{a_n}\right) \, dt \right\}
\]

\[
\times \frac{1}{na_n} \left[ \int \left[ 1 - \exp\left(-\frac{s - x_{2j}}{a_n}\right) \right] \sum_{j=1}^{n} \exp\left(-\frac{s - x_{2j}}{a_n}\right) \, ds \right]
\]
\begin{equation}
(4.4) \quad \frac{1}{n} \sum_{j=1}^{n} \left[ 1 - \frac{1}{2n} \sum_{j' = 1}^{n} \exp \left( - \frac{|x_{1j} - x_{1j'}|}{a_n} \right) \right] \\
\times \left[ 1 - \frac{1}{2n} \sum_{j'' = 1}^{n} \exp \left( - \frac{|x_{2j} - x_{2j''}|}{a_n} \right) \right].
\end{equation}

REFERENCES


