ON ALMOST SURE CONVERGENCE OF INFINITE SERIES

by

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ABSTRACT

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This paper focuses upon the question of almost sure convergence of an infinite series \( \sum_{k=1}^{\infty} c_k X_k \), subject to mild restrictions on the growth of the constants \( c_k \) and weak dependence restrictions on the random variables \( X_i \). Two new forms of weak multiplicative dependence are introduced, which apply to a large class of orthogonal sequences and time series \( \{X_i\} \) not satisfying dependence restrictions presently considered in the literature. Examples are the "random telegraph signal", the square of a Gaussian time series, and various moving average models. The dependence restrictions involve product moments \( E(X_{i_1} \cdots X_{i_v}) \) rather than conditional expectations and the like, and are thus relatively easy to check in practice. Our main theorem provides a basic foundation consisting of useful upper bounds of the form \( C \left( \sum_{k=1}^{n} c_k^2 \right)^{v/2} \) for \( E(\sum_{k=1}^{n} c_k X_k)^v \) under either of the two new dependence restrictions introduced. This result is utilized to obtain corresponding "maximal inequalities" which are of general interest. These various results are applied to establish the almost sure convergence of \( \sum_{k=1}^{\infty} c_k X_k \) under mild dependence restrictions and the condition \( \sum_{k=1}^{\infty} c_k^2 < \infty \), and under still milder dependence restrictions and the condition \( \sum_{k=1}^{\infty} c_k^2 \log^2 k < \infty \). Specific applications to time series are considered, with special attention to the "random telegraph signal".
1. Introduction. Let \((\Omega, \mathcal{A})\) be a measurable space, \(P\) a probability measure on \(\mathcal{A}\), and \(\{X_i\}_{i=1}^{\infty}\) a sequence of \(L_2(\Omega, \mathcal{A}, P)\) r.v.'s (random variables). Without loss of generality, it is assumed throughout that

\[
E(X_i) = 0, \quad E(X_i^2) = 1.
\]

The focal question in this paper is whether, for given constants \(\{c_i\}\), the series \(\sum_{k=1}^{\infty} c_k X_k\) converges a.s. (almost surely), i.e., satisfies

\[
P(\omega: \lim_{n \to \infty} \sum_{k=1}^{n} c_k X_k(\omega) \text{ exists and is finite}) = 1.
\]

The conclusion (1.1) entails restriction of the growth of the \(c_i\)'s and restriction of the mutual dependence of the \(X_i\)'s. In Section 2 the nature of these conditions is seen through a brief review of some classical results for independent or orthogonal \(X_i\)'s. Current interest in the question is examined in the contexts of orthogonal series representation of \(L_2\) functions and of infinite moving average representations in time series analysis. Brief indication of further types of application of (1.1) is given also.

In Section 3 several important dependence restrictions apropos to (1.1) are discussed. In particular, two new forms of weak multiplicative dependence are introduced. They are broad enough to include a large class of important sequences \(\{X_i\}\) not satisfying previously considered such restrictions, yet potent enough to yield equally strong conclusions, and they are relatively easy to check in practice. These restrictions involve product moments \(E(X_{i_1} \cdots X_{i_v})\) rather than conditional expectations, conditional probabilities, and the like.

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Our general results are presented in Section 4. A basic foundation is provided by Theorem 4.1, which gives useful upper bounds of the form $c_s (\sum_{k} c_k^2)^{\nu/2}$ for $E(n_1 c_k X_k)^\nu$ under either of the two new dependence restrictions introduced in Section 3. This result is utilized to obtain corresponding "maximal inequalities" (Theorem 4.2, Corollary 4.2) which are of general interest. Corollary 4.2 is applied to establish (Corollary 4.3) the a.s. convergence of $\sum_{k=1}^\infty c_k X_k$ under the condition $\sum_{k=1}^\infty c_k^2 < \infty$ and mild dependence restrictions. An alternative result (Corollary 4.4) obtained directly from Theorem 4.1 allows still milder dependence restrictions but strengthens the condition on $\{c_i\}$.

Specific applications are considered in Section 5, with special attention to the "random telegraph signal" stochastic process. Also treated are the square of a Gaussian time series, moving average models, and Markov chains. The new dependence restrictions introduced in Section 3 are illustrated, thus establishing a basis for application of Corollaries 4.3 and 4.4.

2. Background and motivation. A consequence of Kolmogorov's classical "three series" theorem (Doob (1953), page 111) is that if the $X_i$'s are mutually independent and

$$\sum_{k=1}^\infty c_k^2 < \infty,$$

then (1.1) holds. A further consequence is that, conversely, if the $X_i$'s are independent and uniformly bounded, then (2.1) is necessary for (1.1). Another classical result, due independently to Mensov (1923) and Rademacher (1922), is that if the $X_i$'s are mutually orthogonal, i.e., satisfy

$$E(X_i X_j) = 0 \text{ if } i \neq j,$$
and if

\[ \sum_{k=1}^{\infty} c_k^2 \log^2 k < \infty, \]

then (1.1) holds. Conversely, Tandori (1957) has shown that if the \( c_i \)'s are monotone decreasing but violate (2.3), then there exists a sequence of orthogonal \( X_i \)'s for which the series \( \sum_{k=1}^{\infty} c_k X_k \) a.s. fails to converge. Consequently, for (1.1) to hold for a general class of sequences \( \{X_i\} \), the condition (2.3) cannot be relaxed without imposing dependence restrictions stronger than (or different from) orthogonality, and, on the other hand, the condition (2.1) is about as mild as can be achieved as a trade-off for tightening the dependence restrictions (toward independence).

At this point one may ask: Is there any special interest in obtaining (1.1) under the condition (2.1) while permitting very broad dependence restrictions (perhaps only slightly stronger than orthogonality) on the \( X_i \)'s? The answer is "yes", as will be evident in the following discussion of this question in two particular contexts.

Consider the problem of representing an \( L_2(\Omega, A, \mathbb{P}) \) function \( f \) as a series \( \sum_{k=1}^{\infty} c_k X_k \) in some orthogonal system \( \{X_i\} \) of interest. The constants \( \{c_i\} \) arise through exploitation of the orthogonality \( c_i = E(fX_i) \) and the condition (2.1) arises as a requirement for existence of the series \( \sum_{k=1}^{\infty} c_k X_k \) as a limit in mean square (a limit which equals \( f \) if the system \( \{X_i\} \) is complete). It then becomes natural to inquire whether this limit r.v. exists in the a.s. sense. Indeed, if not, the suggestive notation \( \sum_{k=1}^{\infty} c_k X_k \) is rather misleading.

In particular, the question of a.s. convergence of Fourier series has received considerable attention. Here we take \( \Omega \) to be the interval \([0, 2\pi]\), \( A \) the class of Borel sets in \( \Omega \), \( P \) the (normalized) Lebesgue measure, and \( \{X_i\} \) the orthogonal
system of trigonometric functions \( \cos kw \) and \( \sin kw \) for \( k = 0,1, \ldots \). For the conclusion

\[
(2.4) \quad P(\omega: \sum_{0}^{\infty} (a_k \cos kw + b_k \sin kw) \text{ converges}) = 1 ,
\]

the sufficient condition

\[
(2.5) \quad \sum_{0}^{\infty} (a_k^2 + b_k^2) \log k < \infty
\]

(replacing \( \log^2 k \) in (2.3) by \( \log k \)) was established by Kolmogorov and Seliverstov (1925). The result stood unimproved until, finally, Carleson (1966) established the sufficiency of the condition

\[
(2.6) \quad \sum_{0}^{\infty} (a_k^2 + b_k^2) < \infty
\]

Adopting Carleson's method, Billard (1966, 1967) proved the a.s. convergence of \( \sum_{1}^{\infty} c_k X_k \) under (2.1) for \( \{X_i\} \) the particular orthogonal system of Walsh functions.

A separate line of inquiry, into sequences \( \{X_i\} \) of orthogonal r.v.'s satisfying forms of multiplicative dependence (defined in Section 3), was initiated by Alexits (1961). Carried forward by many contributions, including Alexits and Tandori (1961), Révész (1968), Serfling (1969), and Komlós and Révész (1972), the investigation has recently led to the following result of Komlós (1972).

**Theorem 2.1.** Let \( \{X_i\} \) satisfy (1.0) and, for an even integer \( \nu \geq 4 \),

\[
(2.7) \quad \mathbb{E}(X_i^\nu) \leq K < \infty \quad (i = 1,2, \ldots)
\]

and

\[
(2.8) \quad \mathbb{E}(X_{i_1} X_{i_2} \cdots X_{i_\nu}) = 0 \quad (1 \leq i_1 < i_2 < \cdots < i_\nu)
\]

Then (2.1) implies (1.1).
(The case of greatest practical importance, \( v = 4 \), was given in Komlós and Révész (1972).) Theorem 2.1 contains neither Carleson's nor Billard's results. Our Corollary 4.3 very broadly generalizes Theorem 2.1 but too fails to contain Carleson's and Billard's results.

We now turn to the context of time series and draw attention to the classical result (e.g., see Anderson (1971), §7.6) that if \( \{Y_k\} \) is a weakly stationary sequence having an absolutely continuous spectral distribution function \( F \) on \([-\pi, \pi]\) with positive density \( f \) satisfying \( \int_{-\pi}^{\pi} \log f(\lambda) d\lambda > -\infty \), then \( \{Y_k\} \) may be represented (in the sense of mean square limit) as an "infinite moving average"

\[
Y_n = \sum_{k=0}^{\infty} c_k X_{n-k},
\]

where \( c_0 = 1 \), \( \sum_{k=1}^{\infty} c_k < \infty \), and \( \{X_i\} \) is an orthogonal sequence with \( X_k \) belonging to the closed linear manifold generated (in mean square convergence) by \( \{Y_1, Y_{k-1}, \ldots\} \), each \( k \). Furthermore, in this case, the best linear predictor of \( Y_n \) (in terms of \( X_{n-1}, X_{n-2}, \ldots \)) is found to be

\[
\hat{Y}_n = \sum_{k=1}^{\infty} c_k X_{n-k}.
\]

Under additional restrictions on \( f \) and \( \{c_k\} \), the relation (2.10) may be inverted to express the \( X_i \)'s in terms of the \( Y_i \)'s, i.e., \( X_k = \sum_{j=0}^{\infty} b_j Y_{k-j} \) (with \( b_0 = 1 \)), which in turn yields an expression for the predictor \( \hat{Y}_n \) explicit in \( Y_{n-1}, Y_{n-2}, \ldots \), i.e.,

\[
\hat{Y}_n = -\sum_{k=1}^{\infty} b_k Y_{n-k}.
\]
(In particular, Akutowicz (1957) has established that if $f$ is bounded, then a necessary and sufficient condition for the inversion to hold with $\sum_0^\infty b_k z^k < \infty$ is that the function $\Gamma(z) = \sum_0^\infty c_k z^k$, $|z| < 1$, of a complex argument belong to the Hardy class $H^2$, i.e., be such that $\int_0^{2\pi} |\Gamma(e^{i\lambda})|^{-2} d\lambda$ remains bounded as $\rho \to 1$.)

It is seen from (2.10) and (2.11) that the question of a.s. convergence of $\sum_1^\infty c_k \xi_k$ under the condition $\sum_1^\infty c_k^2 < \infty$ may arise in connection not only with orthogonal $\xi_i$'s but also with other types of dependent $\xi_i$'s. Further possibilities in connection with moving averages will be discussed in Section 5.3. Other types of possibilities arise in connection with existence of stochastic integrals over stationary stochastic processes (e.g., see Cramér and Leadbetter (1967), Chapter 5).

In summary, it is of general importance to treat the a.s. convergence of $\sum_1^\infty c_k \xi_k$ under very broad dependence restrictions on $\{\xi_i\}$.

Sometimes the question of a.s. convergence of an infinite series is intermediate to an ultimate goal of establishing a form of the strong law of large numbers, through application of the Kronecker lemma (Révész (1968), page 35),

$$\sum_1^n \mu_i / \lambda_n \text{ convergent } \Rightarrow \left(\sum_1^n \mu_i / \lambda_n\right) / \lambda_n \to 0,$$

for nonnegative $\lambda_i \uparrow \infty$ and arbitrary $\mu_i$. Thus the a.s. convergence of $\sum_1^\infty a_k^{-1} X_k$ serves as a sufficient condition for a.s. convergence of $a_n^{-1} \sum_1^n X_k$ to 0. A related context is that in which the a.s. convergence of $\sum_1^\infty c_k X_k$ is of interest in connection with a sequence $\{X_i\}$ which is a subsequence of some given sequence of fundamental interest (e.g., see Révész (1968), Chapter 5, and Komlós (1974)).
3. On some dependence restrictions of weak multiplicative type. Here we introduce two new dependence restrictions, to be termed \( \alpha \)- and \( \beta \)-multiplicative dependence, under which general properties of sums \( \sum_k X_k \) will be derived in Section 4. In the present section we shall place these restrictions in perspective with various more stringent conditions in use in the literature, namely independence, orthogonality, martingale differences, multiplicativity, weak multiplicativity, and also with a heretofore unnamed condition which we shall term \( \gamma \)-multiplicativity. Important examples of \( \alpha \)-, \( \beta \)-, and \( \gamma \)-multiplicative sequences not satisfying the other conditions will be examined in Section 5.

As observed in Section 2, the condition that a sequence \( \{X_i\} \) of r.v.'s be mutually orthogonal is too weak to imply properties as strong as those implied by mutual independence. For example, given orthogonality, (1.1) holds under (2.3) but not necessarily under (2.1). Thus we seek dependence restrictions of strength greater than orthogonality which still broadly relax the independence assumption yet imply equally strong properties for \( \{X_i\} \).

A productive and widely applicable relaxation of independence is the now classical condition that \( \{X_i\} \) be a sequence of martingale differences:

\[
E(X_i | X_{i-1}, X_{i-2}, \ldots) = 0 \quad (i = 2, 3, \ldots).
\]  

(3.1)

This notion was introduced by Lévy (1937) and its rich potential activated by Doob (1953). The condition is sufficiently strong to yield convergence properties for \( \sum_1^n X_i \) such as the central limit theorem (see Lévy (1937; 1954), Billingsley (1961), Ibragimov (1963), Csörgö (1968), Serfling (1968), Csörgö and Fischler (1970), Heyde and Brown (1970), Brown (1971), and Grams (1972), Chapter III) and full generalizations of classical forms of the law of the iterated logarithm (see Stout (1970a,b; 1974)).
Many important types of orthogonal sequence \( \{X_i\} \) fail to satisfy the stringent conditional expectation restriction (3.1). Thus a weaker condition was introduced by Alexits (1961): a sequence \( \{X_i\} \) is multiplicative if

\[
(3.2) \quad E(X_{i_1} X_{i_2} \cdots X_{i_k}) = 0 \quad (i_1 < i_2 < \ldots < i_k; k = 1, 2, \ldots)
\]

The condition is sufficiently potent to yield "exponential" probability inequalities (Serfling (1969)) and crude forms of the law of the iterated logarithm (Serfling (1969; 1970b), Révész (1973)). However, for certain types of conclusion, condition (3.2) has been found to be unnecessarily stringent. An effective relaxation is: a sequence \( \{X_i\} \) is multiplicative of order \( \nu \), for an even integer \( \nu \), if

\[
(3.3) \quad E(X_{i_1} X_{i_2} \cdots X_{i_\nu}) = 0 \quad (i_1 < i_2 < \ldots < i_\nu)
\]

The case \( \nu = 2 \) corresponds to orthogonality and is thus too weak for our purposes. However, for \( \nu \geq 4 \), we have seen in Theorem 2.1 that condition (3.3), along with the condition \( E(X_i^\nu) \leq K < \infty \) (all \( i \)), suffices for a.s. convergence of

\[
\sum_{k=1}^{\infty} c_k X_k \quad \text{under} \quad \sum_{k=1}^{\infty} c_k^2 < \infty. \quad (\text{Condition (1.0) is implicitly retained.})
\]

As illustrated by the preceding result, condition (3.2) is unnecessarily stringent as regards the focal question of this paper. Furthermore, the scope of application of (3.2) is very narrow. Fiedler and Trautner (1973) show the existence of a bounded complete orthogonal system which does not contain any infinite subsystem satisfying (3.2). Friess and Trautner (1973) show that bounded complete orthogonal systems which do contain such a subsystem are rare in a certain sense. A relaxation of (3.2) in a different direction than (3.3) was introduced by Alexits (1971): a sequence \( \{X_i\} \) is weakly multiplicative if
\[ \sum_{\alpha} |E(X_{i_1} X_{i_2} \cdots X_{i_k})| < \infty, \]

where \( \sum_{\alpha} \) denotes summation over all \( i_1 < i_2 < \cdots < i_k, \ k = 1, 2, \ldots \). The scope of this condition is quite broad, for Alexits (1973) shows that every bounded infinite orthonormal system, even if not complete, contains an infinite subsystem satisfying (3.4). The condition is also rather powerful nevertheless, for, under a similar condition, Révész (1974) establishes a crude law of the iterated logarithm.

Yet even the relaxations (3.3) and (3.4) are overly restrictive, excluding in particular various potential applications in the time series context. We now introduce two dependence restrictions, closely related to each other but taking somewhat different directions, each of which relaxes (3.3).

A sequence \( \{X_i\} \) (satisfying (1.0)) is \( \alpha \)-multiplicative of order \( \nu \), for an even integer \( \nu \), if

\[ |E(X_{i_1} X_{i_2} \cdots X_{i_\nu})| \leq \alpha(i_2 - i_1, i_3 - i_2, \ldots, i_\nu - i_{\nu-1}) (i_1 < i_2 < \cdots < i_\nu), \]

where

\[ \alpha = \sum_{j_1 = 1}^{\infty} \sum_{j_2 = 1}^{\infty} \cdots \sum_{j_{\nu-1} = 1}^{\infty} \alpha(j_1, j_2, \ldots, j_{\nu-1}) < \infty. \]

The case \( \nu = 2 \) represents a simple relaxation of orthogonality and was introduced and utilized in Serfling (1970b), Corollary 2.2.1, in deriving the following strong law of large numbers: if \( \{\xi_i\} \) satisfies \( E(\xi_i) = 0, \ E(\xi_i^2) = \sigma_i^2 \) with \( \sum \sigma_k \leq 2 \log k < \infty \), and \( E(\xi_i \xi_j) \leq \rho_{ij} \sigma_i \sigma_j \) with \( \rho_{ij} \geq 0 \) and \( \sum_{k=1}^{\infty} \rho_k < \infty \), then \( n^{-1} \sum_{i=1}^{n} \xi_i \) converges a.s. to 0. This result reduces, in the case of orthogonal \( \xi_i \)'s, to a classical SLLN. These remarks serve to illustrate the strengths and limitations of the case \( \nu = 2 \). It is sufficiently powerful to yield conclusions implied by orthogonality, but, being weaker than orthogonality, it suffers the same
limitations as orthogonality itself. For the cases \( v \geq 4 \), the \( \alpha \)-multiplicative condition is considerably more powerful than orthogonality (although it does not imply orthogonality). Indeed, it is essentially as effective as condition (3.3), for we shall show in Corollary 4.3 that (3.5) may be substituted for (2.8) in Theorem 2.1. Among the cases \( v \geq 4 \), the most important is \( v = 4 \), the case most amenable to practical verification.

A sequence \( \{X_i\} \) (satisfying (1.0)) is \( \beta \)-multiplicative of order \( v \), for an even integer \( v \), if

\[
(3.6a) \ |E(X_{i_1} X_{i_2} \cdots X_{i_v})| \leq \min\{\beta(i_2-i_1), \beta(i_4-i_3), \ldots, \beta(i_v-i_{v-1})\} \ (i_1 < i_2 < \ldots < i_v),
\]

where \( \beta(*) \) is a function satisfying

\[
(3.6b) \quad \beta = \sum_{j=1}^{\infty} j^{v/2-1} \beta(j) < \infty.
\]

For \( v = 2 \) this condition coincides with the \( \alpha \)-multiplicative condition and thus, as discussed above, represents a relaxation of orthogonality having the same effectiveness and limitations. For \( v \geq 4 \), the condition differs from \( \alpha \)-multiplicativity in nature and scope but is similar in role and effectiveness. For example, our Corollary 4.3 will also justify substitution of (3.6) for (2.8) in Theorem 2.1. The important case \( v = 4 \) of (3.6) is contained in a set of conditions introduced and utilized by Révész (1969), to whom we thus give credit for innovation of the \( \beta \)-multiplicative notion and stimulation of our own interest in it.

The two dependence restrictions just considered are seemingly in the category of orthogonality-related conditions. But also they are closely related to a dependence restriction which has arisen in a quite different context, time series
analysis, and for a quite different purpose, constraining the non-Gaussianity of the observed
We shall now introduce this other condition. It is confined to 4-th order and, instead of directly restricting the product moments $E(X_{i_1}X_{i_2}X_{i_3}X_{i_4})$, it restricts their "non-Gaussian parts". The condition is based on the well-known fact (Anderson (1958), page 39) that for $(X_{i_1},X_{i_2},X_{i_3},X_{i_4})$ multivariate Normal with mean vector 0, the moment $E(X_{i_1}X_{i_2}X_{i_3}X_{i_4})$ is given by

\[(3.7) \quad E_{i_1}E_1E_{i_2}E_{i_3}E_{i_4} = E(X_{i_1}X_{i_2}X_{i_3}X_{i_4})E(X_{i_1}X_{i_2})E(X_{i_1}X_{i_3})E(X_{i_1}X_{i_4})E(X_{i_2}X_{i_3})E(X_{i_2}X_{i_4})E(X_{i_3}X_{i_4}) .\]

We shall call a sequence $\{X_i\}$ (satisfying (1.0)) $\gamma$-multiplicative (of order 4) if

\[(3.8a) \quad |E(X_{i_1}X_{i_2}X_{i_3}X_{i_4}) - E_{G}(X_{i_1}X_{i_2}X_{i_3}X_{i_4})| \leq \gamma^{i_2-i_1,i_3-i_2,i_4-i_3}(i_1 < i_2 < i_3 < i_4) ,\]

where

\[(3.8b) \quad \gamma = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \sum_{j_3=1}^{\infty} \gamma(j_1,j_2,j_3) < \infty .\]

This condition originated in the context of sequences $\{X_i\}$ wide sense stationary of order 4, i.e., such that (in addition to (1.0)) the quantities $E(X_{i_1}X_{i_1+a})$, $E(X_{i_1}X_{i_1+a+b})$, and $E(X_{i_1}X_{i_1+a+b+c})$ depend only on $a$, $b$ and $c$. In such a context, the left hand side of (3.8a) was introduced by Magness (1954) as a measure of deviation between the 4-th order moment properties of a sequence $\{X_i\}$ and a parallel Gaussian sequence. A summability condition (3.8b) was added in order to justify certain Fourier transform operations. Parzen (1957) exploited condition (3.8) in deriving asymptotic properties of spectral estimates for a one-dimensional time series, and Rosenblatt (1959) exploited it in an extended form in the more general context of asymptotic properties of cospectra and quadrature spectra estimates for multi-dimensional time series. Further, Rosenblatt (1959, 1962) elucidated the nature of (3.8) as a form of dependence restriction.
For orthogonal sequences \( \{X_i\} \), \( \alpha \)- and \( \gamma \)-multiplicativity (of order 4) coincide. More generally, it is seen easily that the \( \alpha \) and \( \gamma \) conditions hold simultaneously if and only if

\[
(3.9a) \quad |E_{G}(X_{i_1} X_{i_2} X_{i_3} X_{i_4})| \leq \gamma^{*}(i_2-i_1, i_3-i_2, i_4-i_3) \quad (i_1 < i_2 < i_3 < i_4),
\]

where

\[
(3.9b) \quad \gamma^{*} = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \sum_{j_3=1}^{\infty} \gamma^{*}(j_1,j_2,j_3) < \infty.
\]

Thus one way to establish \( \alpha \)-multiplicativity of order 4 is to verify the \( \gamma \) and \( \gamma^{*} \) conditions. The quantity \( E_{G}(X_{i_1} X_{i_2} X_{i_3} X_{i_4}) \) being a function simply of covariances \( E(X_{i j}) \), it is relatively easy to check whether (3.9) holds for a given sequence \( \{X_i\} \).

Two stochastic processes considered by Magness (1954) in illustration of his measure of "non-Gaussian part", the "random telegraph signal" and the square of a Gaussian time series, turn out to be both \( \beta \)- and \( \gamma \)-multiplicative but not \( \alpha \)-multiplicative. Both of these examples fail to satisfy more stringent restrictions such as (3.3). We shall examine these and other examples in Section 5.

4. **General results.** We consider sequences \( \{X_i\} \) which are either \( \alpha \)- or \( \beta \)-multiplicative of order \( \nu \). Useful upper bounds of the form \( C \left( \sum_{k=1}^{\infty} c_k^2 \right)^{\nu/2} \) for \( E(\sum_{k=1}^{\infty} c_k X_k)^{\nu} \) are provided in Theorem 4.1 and Corollary 4.1. Theorem 4.1 is a basic foundation to the sequel. Related maximal inequalities of general interest are provided in Theorem 4.2 and Corollary 4.2. Finally, it is shown that \( \sum_{k=1}^{\infty} c_k X_k \) converges a.s. under the condition \( \sum_{k=1}^{\infty} c_k^2 < \infty \) in the case \( \nu \geq 4 \) (Corollary 4.3) and under the condition \( \sum_{k=1}^{\infty} c_k^2 \log^2 k < \infty \) in the case \( \nu = 2 \) (Corollary 4.4).
Several preliminary lemmas will be needed. The first states some well-known, and easily proved, numerical inequalities.

**Lemma 4.1.** Let \( \{a_i\} \) be nonnegative constants. Let \( 0 \leq p \leq 1 \). Then

\[
\left( \sum_{i=1}^{n} a_i \right)^p \leq \sum_{i=1}^{n} a_i^p
\]

and

\[
\prod_{i=1}^{n} a_i \leq \frac{1}{n} \sum_{i=1}^{n} a_i^n.
\]

Some intermediate steps in our development will involve various sums of the form

\[
w_j = \sum_{k=1}^{n} y_k^j, \quad j \geq 1,
\]

generated by \( n \) given numbers \( y_1, \ldots, y_n \). We shall use the fact that any sum of the form

\[
\sum_{(i_1, \ldots, i_m)} r_1^{i_1} \cdots r_m^{i_m},
\]

where \( m \) and \( r_1, \ldots, r_m \) are fixed positive integers and \( \sum_{(i_1, \ldots, i_m)} \) denotes summation over all \( m \)-tuples \( (i_1, \ldots, i_m) \) of distinct integers from the set \( \{1, \ldots, n\} \), may be represented in terms of the sums \( w_1, \ldots, w_{r_1+\ldots+r_m} \), where the representation does not depend upon \( n \). For example, making use of

\[
\left[ \sum_{i=1}^{n} a_i \right] \left( \sum_{j=1}^{n} b_j \right) = \sum_{i \neq j} a_i b_j + \sum_{i=1}^{n} a_i b_i,
\]

we obtain
\[
\Sigma(2) y_{i_1} y_{i_2} = w_1 r_1 r_2 - w_1 r_1 r_2
\]

Likewise may be found

\[
\Sigma(3) y_{i_1} y_{i_2} y_{i_3} = w_1 r_1 w_2 r_2 r_3 - w_1 r_1 r_2 r_3 - w_1 r_1 r_2 r_3 - w_1 r_2 r_3 r_1 + 2w_1 r_1 r_2 r_3
\]

In general, the m-fold symmetric function (4.4) may be handled by reduction to lower order cases, as described in Burnside and Panton (1899). We shall be concerned only with the case \(r_1 = \cdots = r_m = 1\) in (4.4), for which the representation may be stated in a convenient form as follows. Put \(w_0 = 1\) and

\[(4.7) I_m = \{(i_1, \ldots, i_m) \mid \text{each } i_j \geq 0; \text{at least one } i_j \geq 2; i_1 + \cdots + i_m = m\}\]

and denote summation over \((i_1, \ldots, i_m) \in I_m\) by \(\Sigma(I_m)\).

**LEMMA 4.2.** There exist integers \(d(i_1, \ldots, i_m)\) for \((i_1, \ldots, i_m) \in I_m\)

such that

\[(4.8) w_m = \Sigma(m) y_{i_1} \cdots y_{i_m} + \Sigma(I_m) d(i_1, \ldots, i_m) w_{i_1} \cdots w_{i_m}\]

In particular, (4.8) give for \(m = 2\) and 3 the following special cases of (4.5) and (4.6):

\[(4.9) w_1^2 = \Sigma(2) y_i y_j + w_2\]

and

\[(4.10) w_1^3 = \Sigma(3) y_i y_j y_k + 3w_1 w_2 - 2w_3\]

For later reference, we define

\[(4.11) D_m = \Sigma(I_m) |d(i_1, \ldots, i_m)|\]

and note that \(D_m \geq 1\) and depends only on \(m\).
Consider now a collection of r.v.'s $X_1, \ldots, X_n$ and a collection of constants $c_1, \ldots, c_n$ and put

$$s_j = \sum_{k=1}^{n} (c_k x_k)^j, \quad j \geq 1. \quad (4.12)$$

The foregoing results may be applied with $y_i = c_i x_i$ and thus $w_i = s_i$, and we shall utilize the representation given by Lemma 4.2, with $m$ replaced by an even integer $\nu$. Thus

$$s_1^\nu = t_1 + z_1, \quad (4.13)$$

where

$$t_1 = \sum_{i=1}^{\nu} c_{1}^{i_1} \cdots c_{\nu}^{i_\nu} x_{i_1} \cdots x_{i_\nu} \quad (4.14)$$

and

$$z_1 = \sum_{i=1}^{\nu} s_{i_1} \cdots s_{i_\nu} \quad (4.15)$$

We shall pass to upper bounds for $E(s_1^\nu)$ by treating $E(t_1)$ and $E(z_1)$ separately. The following result deals with $E(z_1)$ without restriction of the dependence of the $X_i$'s. Previous forms of the result are contained in Serfling (1969) and Komlós (1972).

**Lemma 4.3.** Let $X_1, \ldots, X_n$ satisfy (1.0) and, for an even integer $\nu$, $E(X_1^\nu) \leq K_{\nu,n} < \infty$, $1 \leq i \leq n$. Then there exists an integer $h$, $0 \leq h \leq \nu - 2$, such that

$$E(z_1) \leq D_{\nu} K_{\nu,n} [E(s_1^\nu)]^{h/\nu} \left( \sum_{k=1}^{n} c_k^2 \right)^{(\nu-h)/2} \quad (4.16)$$
PROOF. Consider \((i_1, \ldots, i_v) \in I_v\). Let \(t = t(i_1, \ldots, i_v)\) denote the number of \(i_j\)'s equal to 1, and note that \(0 \leq t \leq v-2\) must hold. Now observe that, for \(j \geq 2\), we have, by (4.1),

\[
|S_i|^{2/j} \leq \left( \sum_{k=1}^{n} |c_k x_k|^j \right)^{2/j} \leq \sum_{k=1}^{n} (c_k x_k)^2 = S_2,
\]

i.e.,

\[
|S_i| \leq S_2^{j/2}.
\]

Hence

\[\tag{4.17}
|S_{i_1} \cdots S_{i_v}| \leq |S_1|^{t} S_2^{(v-t)/2},
\]

and thus, by the Hölder inequality,

\[\tag{4.18}
E|S_{i_1} \cdots S_{i_v}| \leq [E(S_1^v)]^{t/v} [E(S_2^{v/2})]^{(v-t)/v}.
\]

Now let \(h\) be the value of \(t\), \(0 \leq t \leq v-2\), which maximizes the right hand side of (4.18). Then

\[\tag{4.19}
E(Z_v) \leq D_v [E(S_1^v)]^{h/v} [E(S_2^{v/2})]^{(v-h)/v}.
\]

Finally, by the Minkowski inequality,

\[
[E(S_2^{v/2})]^{2/v} \leq \sum_{k=1}^{n} c_k^2 [E(x_k^v)]^{2/v},
\]

so that

\[\tag{4.20}
[E(S_2^{v/2})]^{(v-h)/v} \leq K_{v,n} \left( \sum_{k=1}^{n} c_k^2 \right)^{(v-h)/2}.
\]

Combining (4.19) and (4.20), we have (4.16).\]
The next two lemmas treat $E(T_v)$, under general restrictions of the $\alpha$- and $\beta$-multiplicative type, respectively.

**Lemma 4.4.** Let $X_1, \ldots, X_n$ satisfy (1.0) and, for an even integer $v$,

(3.5a). Then

\[(4.21) \quad |E(T_v)| \leq a_n v! \left( \sum_{k=1}^{n} c_k^2 \right)^{v/2}, \]

where

\[(4.22) \quad a_n = \sum_{j_1=1}^{n-v+1} \sum_{j_2=1}^{n-v+2-j_1} \cdots \sum_{j_{v-1}=1}^{n-1-j_1-\cdots-j_{v-2}} a(j_1, j_2, \ldots, j_{v-1}). \]

**Proof.** By (3.5a) and the definition of $T_v$,

\[|E(T_v)| \leq v! \sum_{1 \leq i_1 < \cdots < i_v \leq n} |c_{i_1} \cdots c_{i_v}| |E(X_{i_1} \cdots X_{i_v})| \]

\[\leq v! \sum_{1 \leq i_1 < \cdots < i_v \leq n} |c_{i_1} \cdots c_{i_v}| |\alpha(i_2 - i_1, i_3 - i_2, \ldots, i_v - i_{v-1})| \]

\[= v! \sum_{j_1=1}^{n-v+1} \sum_{j_2=1}^{n-v+2-j_1} \cdots \sum_{j_{v-1}=1}^{n-1-j_1-\cdots-j_{v-2}} \sum_{k=1}^{n-j_1-\cdots-j_{v-1}} \]

\[(4.23) \quad |c_k c_{k+j_1} \cdots c_{k+j_1+\cdots+j_{v-1}}| |\alpha(j_1, \ldots, j_{v-1})|. \]

By (4.2),

\[\sum_{k=1}^{n-j_1-\cdots-j_{v-1}} |c_k c_{k+j_1} \cdots c_{k+j_1+\cdots+j_{v-1}}| \leq \frac{1}{v} \sum_{k=1}^{n-j_1-\cdots-j_{v-1}} \sum_{i=1}^{v} c_k^{v} c_{k+j_1+\cdots+j_{i-1}} \]

\[(4.24) \quad \leq \frac{1}{v} \sum_{i=1}^{v} \sum_{k=1}^{n} c_k^{v} = \sum_{k=1}^{n} c_k^{v}. \]

But (4.1) implies $\sum_{k=1}^{n} c_k^{v} \leq \left( \sum_{k=1}^{n} \frac{c_k^2}{2} \right)^{v/2}$. Hence, by (4.23) and (4.24), we have

(4.21). \(\square\)
Note that the dependence restriction imposed by Lemma 4.4 is weaker than strict \( \alpha \)-multiplicativity, since the summability condition (3.5b) is not required. The analogue of Lemma 4.4 for a similarly broadened form of \( \beta \)-multiplicativity is given by the following result.

**Lemma 4.5.** Let \( X_1, \ldots, X_n \) satisfy (1.0) and, for an even integer \( \nu \), (3.6a). Then

\[
|E(T_{\nu})| \leq \beta_n \left( \frac{\nu!}{(\nu/2 - 1)!} \right) \left( \prod_{k=1}^{n} c_k \right)^{\nu/2},
\]

where

\[
\beta_n = \sum_{j=1}^{n} j^{\nu/2 - 1} \beta(j).
\]

**Proof.** By (3.6a) and the definition of \( T_{\nu} \), and using the inequality \( 2ab \leq a^2 + b^2 \), we have, writing \( m \) for \( \nu/2 \) where convenient,

\[
|E(T_{\nu})| \leq \nu! \sum_{1 \leq i_1 < \cdots < i_{\nu} \leq n} |c_{i_1} \cdots c_{i_{\nu}}| |E(X_{i_1} \cdots X_{i_{\nu}})|
\]

\[
\leq \nu! \sum_{1 \leq i_1 < \cdots < i_{\nu} \leq n} |c_{i_1} \cdots c_{i_{\nu}}| \min\{\beta(i_2 - i_1), \beta(i_4 - i_3), \ldots, \beta(i_{\nu} - i_{\nu-1})\}
\]

\[
\leq \nu! 2^{-m} \sum_{1 \leq i_1 < \cdots < i_{\nu} \leq n} (c_{i_1}^2 + c_{i_2}^2)(c_{i_3}^2 + c_{i_4}^2) \cdots (c_{i_{\nu}}^2 + c_{i_{\nu-1}}^2)
\]

\[
\times \min\{\beta(i_2 - i_1), \beta(i_4 - i_3), \ldots, \beta(i_{\nu} - i_{\nu-1})\}
\]

\[
\leq \nu! 2^{-m} \sum_{1 \leq i_1 < \cdots < i_{\nu} \leq n} \sum_{j_1 = 1}^{2} \sum_{j_2 = 3}^{4} \cdots \sum_{j_{m-1} = 1}^{\nu} \sum_{j_m = 1}^{\nu-1} \frac{\nu!}{(\nu - m)!} \left( \frac{1}{j_1} \cdots \frac{1}{j_{m-1}} \right) \beta_j \beta_{j_m}
\]

\[
\times \min\{\beta(i_2 - i_1), \beta(i_4 - i_3), \ldots, \beta(i_{\nu} - i_{\nu-1})\}
\]

\[
(4.27) = \nu! 2^{-m} \sum_{j_1 = 1}^{2} \sum_{j_2 = 3}^{4} \cdots \sum_{j_{m-1} = 1}^{\nu} \frac{\nu!}{(\nu - m)!} \beta_{j_1} \cdots \beta_{j_m}.
\]
where

\[(4.28) \quad B_{j_1, \ldots, j_{m}} = \sum_{1 \leq i_1 < \cdots < i_{v} \leq n} c_{i_1}^2 c_{i_2}^2 \cdots c_{i_{v}}^2 \times \min\{\beta(i_2-i_1), \beta(i_4-i_3), \ldots, \beta(i_{v}-i_{v-1})\}.\]

As an example of the technique used to bound the terms $B_{j_1, \ldots, j_{m}}$, consider

$B_{1, 3, 5, \ldots, v-1}$. The other $2^m - 1$ terms can be handled in exactly the same fashion. Define

\[J_{\ell} = \{(i_1, \ldots, i_{v}) : 1 \leq i_1 < \cdots < i_{v} \leq n \text{ and} \]

\[i_{2\ell} - i_{2\ell-1} = \max\{(i_2-i_1), (i_4-i_3), \ldots, (i_{v}-i_{v-1})\}\]

for $\ell = 1, 2, \ldots, m$, and denote summation over $(i_1, \ldots, i_{v}) \in J_{\ell}$ by $E(J_{\ell})$. Then

\[B_{1, 3, 5, \ldots, v-1} \leq \sum_{\ell=1}^{m} \sum_{J_{\ell}} (J_{\ell}) c_{i_1}^2 c_{i_2}^2 \cdots c_{i_{v}}^2 \beta(i_{2\ell} - i_{2\ell-1}) \]

\[= \sum_{\ell=1}^{m} \sum_{i_1=1}^{n-v+1} \sum_{i_3=i_1+2}^{n-v+3} \cdots \sum_{i_{v-1}=i_{v-3}+2}^{n-1} c_{i_1}^2 c_{i_3}^2 \cdots c_{i_{v-1}}^2 \times \]

\[\times \sum_{i_2=i_1+1}^{i_3-1} \sum_{i_4=i_3+1}^{i_5-1} \cdots \sum_{i_{v}=i_{v-1}+1}^{n} \beta(i_{2\ell} - i_{2\ell-1}) \]

\[(i_{2\ell} - i_{2\ell-1}) = \max\{(i_2-i_1), (i_4-i_3), \ldots, (i_{v}-i_{v-1})\}\]

\[\leq \sum_{\ell=1}^{m} \sum_{1 \leq i_1 < i_3 < \cdots < i_{v} \leq n} c_{i_1}^2 c_{i_3}^2 \cdots c_{i_{v}}^2 \beta(i_{2\ell+1} - i_{2\ell-1}) \]

\[\times \min\{i_3-i_1-1, i_2-c_{i_{2\ell+1}-i_{2\ell-1}}, \ldots, \min(i_{v-1}, i_{2\ell+1}-i_{2\ell-1})\} \beta(i_{2\ell+1} - i_{2\ell-1})\]

\[\times \min(i_5-i_3-1, i_{2\ell+1}-i_{2\ell-1}) \cdots \min(n-i_{v-1}, i_{2\ell+1}-i_{2\ell-1}) \beta(i_{2\ell+1} - i_{2\ell-1})\]
\[
\sum_{\ell=1}^{m} \sum_{1 \leq i_1 < i_3 < \cdots < i_{v-1} \leq n} c_{i_1 i_3} c_{i_3 i_{v-1}} c_{i_{v-1} i_2 i_1} \sum_{i_2 \ell, i_1 \ell, \ell - 1} (i_2 \ell - i_1 \ell - 1)^{m-1} \beta(i_2 \ell - i_1 \ell - 1)
\]

\[
\leq \beta_n \sum_{\ell=1}^{m} \sum_{1 \leq i_1 < i_3 < \cdots < i_{v-1} \leq n} c_{i_1 i_3} c_{i_3 i_{v-1}} c_{i_{v-1} i_2 i_1}
\]

\[
\leq \beta_n (m!)^{-1} \sum_{\ell=1}^{m} \left( \sum_{k=1}^{\frac{n}{2}} c_{k} \right)^{m},
\]

i.e.,

\[
(4.29) \quad B_{1,3,5,\ldots,v-1} \leq \beta_n \frac{1}{(v/2 - 1)!} \left( \sum_{k=1}^{\frac{n}{2}} c_k \right)^{v/2}.
\]

The same bound is obtained for the other \(2^m - 1\) terms in (4.27), so that (4.25) follows.\[\]

The case \(v = 4\) of Lemma 4.5 has in effect been established by Révész (1969), as may be seen from a careful scrutiny of the proof of his Theorem MM-3. We have utilized his method of proof.

We now are prepared to derive the most fundamental result of the present paper, namely, upper bounds of the form \(A(\sum_{i=1}^{n} c_i^2)^{v/2}\) for \(E(\sum_{i=1}^{n} c_i X_i)^v\), under dependence restrictions of either the \(\alpha\)- or \(\beta\)-multiplicative type.

**THEOREM 4.1.** Let \(X_1, \ldots, X_n\) satisfy (1.0) and, for an even integer \(v\),

\[
(4.30) \quad \sup_{1 \leq i \leq n} E(X_i^v) = K_{v,n} < \infty.
\]

(i) \(\alpha\)-multiplicative case: if

\[
(4.31) \quad |E(X_{i_1} \cdots X_{i_v})| \leq \alpha(i_2 - i_1, i_3 - i_2, \ldots, i_v - i_{v-1}) (1 \leq i_1 < i_2 < \cdots < i_v \leq n),
\]

(ii) \(\beta\)-multiplicative case: if

\[
(4.32) \quad |E(X_{i_1} \cdots X_{i_v})| \leq \beta(i_2 - i_1, i_3 - i_2, \ldots, i_v - i_{v-1}) (1 \leq i_1 < i_2 < \cdots < i_v \leq n),
\]

(iii) \(\gamma\)-multiplicative case: if

\[
(4.33) \quad |E(X_{i_1} \cdots X_{i_v})| \leq \gamma(i_2 - i_1, i_3 - i_2, \ldots, i_v - i_{v-1}) (1 \leq i_1 < i_2 < \cdots < i_v \leq n),
\]

(iv) \(\delta\)-multiplicative case: if

\[
(4.34) \quad |E(X_{i_1} \cdots X_{i_v})| \leq \delta(i_2 - i_1, i_3 - i_2, \ldots, i_v - i_{v-1}) (1 \leq i_1 < i_2 < \cdots < i_v \leq n).
\]
then

\[
(4.32) \quad \mathbf{E} \left( \sum_{k=1}^{n} c_k X_k \right)^v \leq (\alpha_n v! + D_v K_{v,n})^{v/2} \left( \sum_{k=1}^{n} c_k^2 \right)^{v/2},
\]

where

\[
(4.33) \quad \alpha_n = \sum_{j_1=1}^{n-v+1} \sum_{j_2=1}^{n-v+2-j_1} \cdots \sum_{j_{v-1}=1}^{n-1-j_1-\cdots-j_{v-2}} \alpha(j_1, j_2, \ldots, j_{v-1}).
\]

(ii) \( \beta \)-multiplicative case: if, alternatively,

\[
(4.34) \quad |\mathbf{E}(X_{i_1} \cdots X_{i_v})| \leq \min \{ \beta(i_1 - i_1), \beta(i_2 - i_3), \ldots, \beta(i_v - i_{v-1}) \} \quad (1 \leq i_1 < i_2 < \cdots < i_v \leq n),
\]

then

\[
(4.35) \quad \mathbf{E} \left( \sum_{k=1}^{n} c_k X_k \right)^v \leq \left[ \beta_n \frac{v!}{(v/2 - 1)!} + D_v K_{v,n} \right]^{v/2} \left( \sum_{k=1}^{n} c_k^2 \right)^{v/2},
\]

where

\[
(4.36) \quad \beta_n = \sum_{j=1}^{n} j^{v/2-1} \beta(j).
\]

PROOF. Part (i). Put \( S_1 = \sum_{k=1}^{n} c_k X_k \) and \( \Delta = \sum_{k=1}^{n} c_k^2 \). By Lemmas 4.2, 4.3, and 4.4, we have

\[
(4.37) \quad \mathbf{E}(S_1^v) \leq D_v K_{v,n}^{(v-h)/v} \left[ \mathbf{E}(S_1^v) \right]^{h/\Delta(v-h)/2} + \alpha_n v! \Delta^{v/2},
\]

where \( D_v \) is given by (4.11) and \( h \) is an integer satisfying \( 0 \leq h \leq v - 2 \).

If \( \mathbf{E}(S_1^v) \leq K_{v,n} \Delta^{v/2} \), then, since \( D_v \geq 1 \), we immediately have

\[
(4.38) \quad \mathbf{E}(S_1^v) \leq (D_v K_{v,n} + \alpha_n v!) \Delta^{v/2}.
\]
Now (1.0) implies $K_{v,n} \geq 1$, by Jensen's inequality; since also $D_v \geq 1$, we have

$$(D_v K_{v,n} + \alpha_n v!) \geq 1.$$  

Hence (4.38) implies (4.32). If, on the other hand,

$E(S^\gamma_1) \geq K_{v,n}^{\Delta/2}$, then (4.37) yields

$$E(S^\gamma_1) \leq D_v K_{v,n}^{(v-h)/\gamma}[E(S^\gamma_1)^{h/\gamma} \Delta^{(v-h)/2} + \alpha_n v! E(S^\gamma_1)^{h/\gamma} \Delta^{(v-h)/2}] K_{v,n}^{-}\gamma/h/\gamma$$

$$= (D_v K_{v,n} + \alpha_n v!) K_{v,n}^{-\gamma/h/\gamma}[E(S^\gamma_1)^{h/\gamma} \Delta^{(v-h)/2}$$

(4.39) \leq (D_v K_{v,n} + \alpha_n v!) K_{v,n}^{-\gamma/h/\gamma}[E(S^\gamma_1)^{h/\gamma} \Delta^{(v-h)/2}$$

the latter step since $K_{v,n} \geq 1$. But (4.39) implies

$$[E(S^\gamma_1)]^{(v-h)/\gamma} \leq (D_v K_{v,n} + \alpha_n v!)^{\Delta^{(v-h)/2}}$$

or

$$E(S^\gamma_1) \leq (D_v K_{v,n} + \alpha_n v!)^{\gamma/(v-h)\Delta^{\gamma/2}}$$

$$\leq (D_v K_{v,n} + \alpha_n v!)^{\gamma/2\Delta^{\gamma/2}}$$

the latter step since $(D_v K_{v,n} + \alpha_n v!) \geq 1$ and $v - h \geq 2$. Thus, again, we obtain (4.32).

Part (ii). Exactly as (i), using Lemma 4.5 in place of Lemma 4.4.[]

The preceding theorem applies to a finite sequence $X_1, \ldots, X_n$. For an infinite sequence the following corollary is useful.

**COROLLARY 4.1.** Let the sequence $\{X_i\}$ satisfy (1.0) and, for an even integer $v$, $E(X^\gamma_1) \leq K_v < \infty$ (all $i$). Suppose that $\{X_i\}$ is either $\alpha$- or $\beta$-multiplicative of order $v$, and denote by $\Delta_v$ the quantity $\alpha v!$ or $\beta v!/(v/2-1)!$, with $\alpha$ and $\beta$ given by (3.5b) and (3.6b), as the case may be. Then
(4.40) \[ \mathbb{E} \left( \sum_{k=1}^{n} c_k X_k \right)^\nu \leq \left( \Delta_\nu + D_\nu \right)^{\nu/2} \left( \sum_{k=1}^{n} c_k^2 \right)^{\nu/2}, \quad n = 1, 2, \ldots. \]

Theorem 4.1 serves as the basis of a useful maximal inequality, which will be obtained with the help of the following result of Billingsley (1968), page 94.

**Lemma 4.6.** Let \( Y_1, \ldots, Y_n \) satisfy \( \mathbb{E}(Y_i) = 0 \). Suppose that for constants \( \gamma \geq 0 \) and \( \alpha > 1 \) and nonnegative constants \( u_1, \ldots, u_n \),

\[ P \left( \left| \sum_{i=1}^{j} Y_i \right| \geq \lambda \right) \leq \lambda^{-\gamma} \left( \sum_{i=1}^{j} u_i \right)^{\alpha}, \quad 1 \leq i \leq j \leq n, \quad \text{all} \quad \lambda > 0. \]

Then

\[ P \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} Y_i \right| \geq \lambda \right) \leq \lambda^{-\gamma} A_{\gamma, \alpha} \left( \sum_{i=1}^{n} u_i \right)^{\alpha}, \quad \text{all} \quad \lambda > 0, \]

where \( A_{\gamma, \alpha} \) depends only on \( \gamma \) and \( \alpha \).

In fact, we may take

\[ A_{\gamma, \alpha} = 2^\gamma \left[ 1 + \left( 2^{-1/\gamma} - 2^{-\alpha/\gamma} \right) \right]^{\gamma+(\gamma+1)}. \]

A sufficient condition for (4.41) is, of course,

\[ \mathbb{E} \left( \sum_{i=1}^{j} Y_i \right)^\nu \leq \left( \sum_{i=1}^{j} u_i \right)^{\alpha}, \quad 1 \leq i \leq j \leq n. \]

**Theorem 4.2.** Let \( X_1, \ldots, X_n \) satisfy the conditions of Theorem 4.1 for an integer \( \nu \geq 4 \). Then, for \( 1 \leq m \leq n \),

\[ P \left( \max_{m \leq k \leq n} \left| \sum_{\ell=m}^{k} c_\ell X_\ell \right| \geq \lambda \right) \leq \lambda^{-\nu} A_{\nu, \alpha} \left( \Delta_{\nu, n} + D_\nu \Delta_{\nu, n} \right)^{\nu/2} \left( \sum_{k=m}^{n} c_k^2 \right)^{\nu/2}, \quad \text{all} \quad \lambda > 0, \]

where \( A_{\nu} \) depends only on \( \nu \) and \( \Delta_{\nu, n} \) denotes \( a_n \nu^! \) or \( b_n \nu^!/(\nu^2)! \) in the \( a- \) or \( b- \)multiplicative cases, respectively.
PROOF. Put \( \gamma = \nu, \alpha = \nu/2 > 1 \), and \( Y_k = c_kX_k, 1 \leq k \leq n \). Then (4.43) holds with

\[
u_k = (\Delta_{\nu,n} + D_{\nu,v,n})c_k^2, 1 \leq k \leq n,
\]

and (4.44) follows by Lemma 4.6.

COROLLARY 4.2. Let the sequence \( \{X_i\} \) satisfy the conditions of Corollary 4.1 for an even integer \( \nu \geq 4 \). Then

(4.45) \( P(\max_{m \leq k \leq n} \left| \sum_{k=m}^{n} c_kX_k \right| \geq \lambda) \leq \lambda^{-\nu} \Lambda_{\nu}(\Delta_{\nu} + D_{\nu,v,n})^{\nu/2}\left(\sum_{k=m}^{n} c_k^2\right)^{\nu/2}, \text{ all } \lambda > 0 \), for \( 1 \leq m \leq n, n = 1,2, ..., \) with \( \Lambda_{\nu} \) as in Theorem 4.2 and \( \Delta_{\nu} \) and \( K_{\nu} \) as in Corollary 4.1.

(The proof is immediate.)

The preceding corollary has many applications. In particular, we treat now the question of a.s. convergence of \( L_{c_kX_k}^\infty \) under \( L_{c_kX_k}^1 < \infty \). The following result broadly generalizes Theorem 2.1.

COROLLARY 4.3. Let the sequence \( \{X_i\} \) satisfy (1.0), and, for an even integer \( \nu \geq 4 \), \( E(X_i^\nu) \leq K_{\nu} < \infty \) (all \( i \)). Suppose that \( \{X_i\} \) is either \( \alpha \)- or \( \beta \)-multiplicative of order \( \nu \). Then the condition \( \sum_{k=1}^{\infty} c_k^2 < \infty \) implies a.s. convergence of \( L_{c_kX_k}^\infty \).

PROOF. Assume \( \sum_{k=1}^{\infty} c_k^2 < \infty \). Putting \( \xi_n = L_{c_kX_k}^n \), we shall show that \( L_{c_kX_k}^\infty \) converges a.s. by showing that the sequence \( \{\xi_n\} \) is a.s. Cauchy, i.e., satisfies

\[
P(\left| \xi_n - \xi_m \right| \to 0 \text{ as } m,n \to \infty) = 1,
\]
or, equivalently,

\[(4.46) \quad P(\max_{n \geq m} |\xi_n - \xi_m| > \epsilon) \to 0, \ m \to \infty, \ \text{each} \ \epsilon > 0.\]

Applying Corollary 4.2, we obtain from (4.45) that

\[P(\max_{m \leq n \leq M} |\xi_n - \xi_m| > \epsilon) \leq \epsilon^{-\alpha(2)} \left( \sum_{k=m}^{M} c_k^{2} \right)^{\nu/2},\]

where \(\alpha(2)\) does not depend on \(m\) and \(M\). Letting \(M \to \infty\), we have

\[(4.47) \quad P(\max_{n \geq m} |\xi_n - \xi_m| > \epsilon) \leq \epsilon^{-\alpha(2)} \left( \sum_{k=m}^{\infty} c_k^{2} \right)^{\nu/2}.\]

Since \(\sum_{k=1}^{\infty} c_k^{2} < \infty\), the right hand side of (4.47) tends to 0 as \(m \to \infty\), establishing (4.46).

We may also obtain from Theorem 4.1 a convergence result for \(\sum_{k=1}^{\infty} c_k^{2} X_k\) in the case of a sequence \(\{X_i\}\) which is \(\alpha\)- or \(\beta\)-multiplicative merely of order 2 (in which case they are the same). The following result broadly generalizes the Mensov-Rademacher result discussed in Section 2.

**COROLLARY 4.4.** Let the sequence \(\{X_i\}\) satisfy (1.0) and be \(\alpha\)- (or \(\beta\)-) multiplicative of order 2. Then the condition \(\sum_{k=1}^{\infty} c_k^{2} \log^2 k < \infty\) implies a.s. convergence of \(\sum_{k=1}^{\infty} c_k^{2} X_k\).

**PROOF.** Assume \(\sum_{k=1}^{\infty} c_k^{2} \log^2 k < \infty\). Put \(\xi_n = \sum_{k=1}^{n} c_k X_k\). By Corollary 4.1 we have, for \(n \geq m\),

\[(4.48) \quad E(\xi_n - \xi_m)^2 \leq (\Delta_2 + D_2) \sum_{k=m+1}^{n} c_k^{2} \leq (\Delta_2 + D_2) \frac{\sum_{k=m+1}^{n} c_k^{2} \log^2 k}{\log^2 m}.\]
Thus \( E(\xi_n - \xi_m)^2 \to 0 \) as \( m,n \to \infty \), showing that the sequence \( \{T_n\} \) is Cauchy in mean square, so that \( \xi_\infty = \sum_1^\infty c_k X_k \) exists as a limit in mean square. It is also seen, by letting \( n \to \infty \) in (4.48), that

(4.49) \[ E(\xi_\infty - \xi_m)^2 \leq \frac{A}{\log^2 m}, \]

where \( A = (\Delta_2 + D_2)^2 \sum_1^\infty c_k^2 \log^2 k < \infty \). Put \( n_k = 2^k \). Then (4.49) implies that

\[ \sum_1^\infty E(\xi_\infty - \xi_{n_k})^2 < \infty, \]

which yields, by a standard result,

(4.50) \[ P(\xi_\infty - \xi_{n_k} \to 0, \ k \to \infty) = 1. \]

We now show that

(4.51) \[ P(\max_{n_k \leq n \leq n_{k+1}} |\xi_n - \xi_{n_k}| \to 0, \ k \to \infty) = 1. \]

For this, we shall use a maximal inequality of Billingsley (1968), page 102 (or see Theorem A of Serfling (1970a)), which asserts that if, for nonnegative constants \( u_1, \ldots, u_n \),

(4.52) \[ E\left[ \sum_1^n Y_i \right]^2 \leq \sum_1^n u_i, \ 1 \leq i \leq j \leq n, \]

then

(4.53) \[ E[\max_{1 \leq k \leq n} \left( \sum_1^k Y_i \right)^2] \leq \left( \log_{2n} \right)^2 \sum_1^n u_i. \]

Now, by (4.48), we have (4.52) with \( Y_k = c_k X_k \) and \( u_k = (\Delta_2 + D_2)c_k^2 \). Hence, since \( n_{k+1} - n_k = n_k \),
\[ E[ \max_{n_k \leq n \leq n_{k+1}} (\xi_n - \xi_{n_k})^2] \leq (\log_2 4n_k)^2 (\Delta_2 + D_2) \sum_{j=n_k}^{n_{k+1}} c_j^2 \leq 2^n \sum_{j=n_k}^{n_{k+1}} c_j^2 \log j, \]

which implies

\[ \sum_{k=1}^{\infty} E[ \max_{n_k \leq n \leq n_{k+1}} (\xi_n - \xi_{n_k})^2] < \infty, \]

which yields (4.51). Now, for any integer \( n, k \) such that \( n_k \leq n \leq n_{k+1} \), so that

\[ |\xi_n - \xi_{n_k}| \leq |\xi_{\infty} - \xi_{n_k}| + \max_{n_k \leq j \leq n_{k+1}} |\xi_j - \xi_{n_k}|. \]

Therefore, by (4.50) and (4.51),

\[ P(\xi_n \to \xi_{\infty}, \ n \to \infty) = 1, \]

completing the proof.[]

5. Specific applications. The dependence restrictions considered in this paper, with special attention to \( \alpha \)- and \( \beta \)-multiplicative dependence, will be illustrated for several kinds of sequence \( \{X_i\} \). First we shall examine in some detail the "random telegraph signal" and its applications, and it will be seen that this stochastic process is both \( \beta \)- and \( \gamma \)-multiplicative but not \( \alpha \)-multiplicative. Secondly, we shall obtain the same conclusion regarding the stochastic process given by the square of a Gaussian time series. Thirdly, we
shall discuss moving averages and linear systems and exhibit a moving average process which is both $\alpha$- and $\beta$-multiplicative. Finally, we shall consider some Markov chain and related sequences and find examples of multiplicative dependence and $\beta$-multiplicative dependence. In particular, it will be seen that Markov-dependent Bernoulli trials are $\beta$-multiplicative.

5.1. The random telegraph signal. Let $X(t)$ be a stochastic process taking values $+1$ or $-1$ in continuous time $t$, so that each realization of the process is of the form of a "flat top wave" (see Figure 1). The function shifts from $+1$ to $-1$ alternately at each of an infinite series of timepoints. Let the intervals between changes of sign be distributed exponentially, i.e., let the changes of sign occur according to a Poisson process and denote by $\mu$ the expected number of changes of sign per unit time.

![Figure 1](image)

The process $X(t)$ has been considered in the literature in several contexts. It was first examined analytically by Kenrick (1929), as a model for treating a hand-sent telegraph message as a sequence of dots, dashes and spaces of random lengths. It was studied by Magness (1954) as an example of non-Gaussian noise for the purpose of illustrating a quantitative measure of non-Gaussianity of
a stochastic process (as we have considered in Section 3). Further, Wonham and Fuller (1958), noting that there are standard electronic methods for generating the process $X(t)$, have shown that low frequency random signals having specified Gaussian, rectangular, parabolic or elliptical probability density functions may be generated experimentally conveniently as the output of a simple r-c (resistance-capacity) smoothing network with $X(t)$ as input.

We now consider the relevant quantitative features of $X(t)$. Clearly, $E[X(t)] = 0$ and $E[X^2(t)] = 1$. The covariance function,

$$(5.1) \quad \tau(\tau) = E[X(t)X(t + \tau)] = e^{-2\mu|\tau|}, \quad -\infty < \tau < \infty,$$

was derived by Kenrick (1929). A more thorough treatment was provided by Rice (1944, 1945). Consideration of 4-th order product moments $E[X(t)X(t + \tau_1)X(t + \tau_2)X(t + \tau_3)]$ was initiated by Magness (1954), who gave an explicit formula without proof. Extending to the case of higher order moments, Wonham and Fuller (1958) established the formula, for any even integer $v$,

$$(5.2) E[X(t)X(t + \tau_1) \cdots X(t + \tau_{v-1})] = e^{-2\mu(|\tau_1|+|\tau_2|+|\tau_3|+\cdots+|\tau_{v-1}|)},$$

for $-\infty < \tau_1 < \tau_2 < \cdots < \tau_{v-1} < \infty$. (The odd-order product moments are 0.)

For the purposes of Kenrick and Rice, who sought only a spectral analysis of $X(t)$, the covariance function (5.1) was sufficient. Extension to the 4-th order became relevant in the study, by Magness, of the non-Gaussianity of the process. The higher-order moments (5.2) were needed, by Wonham and Fuller, for the purpose of expressing the moments $E[Y(t)^v]$ of an output process from r-c smoothing of $X(t)$. Furthermore, we have seen in Section 3 that the higher-order moments are relevant as a fruitful way of expressing dependence restrictions on the process $X(t)$. 
We now apply the considerations of Sections 3 and 4 in the present setting. Consider the discrete-time sequence \( \{X_k\} \) given by

\[
X_1 = X(0), \quad X_2 = X(-1), \ldots
\]

(5.3)

Note that \( \{X_i\} \) satisfies (1.0), and also \( E(X_i^v) = 1 \) for all even \( v \). The output \( Y(t) \) of the r-c smoother considered by Wonham and Fuller (1958) is given by \( Y(t) = \int_0^\infty X(t - u)h(u)du \), where \( h(u) = T^{-1} \exp(-uT^{-1}) \), \( u \geq 0 \), and \( h(u) = 0 \), \( u < 0 \), with \( T \) being a parameter of the smoother. Thus, by analogy, we are interested in convergence properties of \( \sum_{k=1}^\infty c_k X_k \) for \( \{X_i\} \) given by (5.3) and \( \{c_i\} \) given by \( c_k = h(k) \). It is noted that \( \sum_{k=1}^\infty c_k^2 \log^2 k < \infty \); in fact

\[
\sum_{k=1}^\infty c_k^2 \log^2 k < \infty
\]

Moreover, by (5.1), it is seen that \( \{X_i\} \) is \( \alpha \)- and \( \beta \)-multiplicative of order 2. Hence, by Corollary 4.4, the series \( \sum_{k=1}^\infty c_k X_k \) converges a.s. for \( \{c_i\} \) as given. We also explore the possibility of a.s. convergence of

\[
\sum_{k=1}^\infty c_k X_k
\]

for choices of \( \{c_i\} \) satisfying \( \sum_{k=1}^\infty c_k^2 < \infty \) but not necessarily \( \sum_{k=1}^\infty c_k^2 \log^2 k < \infty \). For this purpose we investigate whether \( \{X_i\} \) is \( \alpha \)- or \( \beta \)-multiplicative of even order \( v \geq 4 \). Applying (5.2), with \( t = i_1 + 1 \), \( \tau_1 = i_1 - i_2 \), \( \tau_2 = i_1 - i_3 \), \ldots, \( \tau_{v-1} = i_1 - i_v \), where \( 1 \leq i_1 < \cdots < i_v \), we obtain

\[
E(X_{i_1} X_{i_2} \cdots X_{i_v}) = e^{-2\mu(|i_1-i_2| + |i_1-i_3| + \cdots + |i_1-i_v|)}
\]

(5.4)

\[
= e^{-2\mu[(i_2-i_1)+(i_4-i_3)+\cdots+(i_v-i_{v-1})]}
\]

(5.5)

\[
= \beta(i_2-i_1)\beta(i_4-i_3) \cdots \beta(i_v-i_{v-1})
\]

where

\[
\beta(j) = e^{-2\mu j}, \quad j \geq 1.
\]

(5.6)
It is immediate from (5.4) and (5.5) that for any even integer \( n \geq 4 \) the sequence \( \{X_1\} \) is \( \beta \)-multiplicative of order \( n \) but not \( \alpha \)-multiplicative. It is also readily seen, by checking criterion (3.8), making use of (5.1) and (5.4), that \( \{X_1\} \) is \( \gamma \)-multiplicative. Finally, apply the \( \beta \)-multiplicativity, we assert by Corollary 4.3 that convergence of \( \sum_{k=1}^{\infty} c_k^2 \) implies a.s. convergence of \( \sum_{k=1}^{\infty} c_k X_k \).

5.2. The square of a Gaussian time series. Another process considered by Magness (1954) for quantitative illustration of non-Gaussianity is

\[
X(t) = 2^{-\frac{1}{2}}[\xi(t)^2 - 1] ,
\]

where \( \xi(t) \) is a Gaussian process with \( E[\xi(t)] = 0, \ E[\xi(t)^2] = 1 \), and \( E[\xi(t)\xi(t+\tau)] = r(\tau) \). (See Magness (1954) for description of how this process could be physically realized.) Consider the associated discrete-time sequence \( \{X_1\} \), where

\[
X_k = X(k) , \quad k = 1, 2, \ldots
\]

It is readily seen that \( \{X_1\} \) satisfies (1.0) and that, for any even integer \( n \), \( E(X_1^n) \leq K_n < \infty \). We now investigate the dependence structure of such a sequence. It is readily found (or see Magness (1954)) that

\[
E(X_1 X_j) = r^2(j - 1)
\]

and
\[ E(X_{i_1} X_{i_2} X_{i_3} X_{i_4}) = 4r(i_2-i_1)r(i_3-i_2)r(i_4-i_3)r(i_4-i_1) \]
\[ + 4r(i_3-i_1)r(i_4-i_3)r(i_4-i_2)r(i_2-i_1) \]
\[ + 4r(i_4-i_1)r(i_4-i_2)r(i_3-i_2)r(i_3-i_1) \]
\[ + r^2(i_4-i_1)^2(i_3-i_2) + r^2(i_3-i_1)^2(i_4-i_2) \]
\[ + r^2(i_2-i_1)^2(i_4-i_3). \]
\[(5.10)\]

Let us suppose that \(|r(\tau)|\) is nonincreasing as \(|\tau|\) increases. Then (5.10) yields

\[(5.11) \quad |E(X_{i_1} X_{i_2} X_{i_3} X_{i_4})| \leq 15|r(i_2-i_1)r(i_4-i_3)| \ (i_1 < i_2 < i_3 < i_4),\]

which shows that \(\{X_i\}\) is \(\beta\)-multiplicative of order 4 with \(\beta(j) = 15|r(j)|\), provided that \(\sum_j |j| |r(j)| < \infty\). It is also readily seen that \(\{X_i\}\) is \(\gamma\)-multiplicative, with \(\gamma(j_1,j_2,j_3) = 15|r(j_1)r(j_2)r(j_3)|\), provided merely that \(\sum_j |r(j)| < \infty\). However, inspection of the final term in the right hand side of (5.10) shows that \(\{X_i\}\) cannot be \(\alpha\)-multiplicative unless \(r(j) \equiv 0\) (in which case the \(X_i's\) are independent).

5.3. Moving averages; linear systems. Given a time series \(\{X_n\}\), a related series \(\{\xi_n\}\) generated by

\[(5.12) \quad \xi_n = \sum_{k=0}^{\infty} c_k X_{n-k}\]

is called a "moving average". As discussed in Section 2 (see (2.9), (2.10), and (2.11)), such series arise in connection with various types of sequences.
{X_i} of dependent r.v.'s, in problems of representation of time series with absolutely continuous spectral distribution and in related prediction problems. In other kinds of application, the sequence {X_i} represents some non-ideal input process and the constants {c_i} are "design" constants selected to make the output process {ξ_i} have desired statistical properties in terms of reliability and performance characteristics and in terms of smoothness characteristics. In this case, (5.12) is referred to as a "linear system". The spectral analysis of such systems, and application to the design of gyroscopic devices for inertial guidance systems, is discussed in Crámer and Leadbetter (1967), Chapter 15. Also, we recall the linear system considered by Wonham and Fuller (1958), which was discussed in Section 5.1 above, where the input {X_i} was selected for convenience to produce, through a linear operation, a desired output for experimental purposes. Further general discussion is available in Anderson (1971), §7.2.2.

When the sequence {X_i} is α- or β-multiplicative, the a.s. convergence of the sequence {ξ_i} given by (5.12) is assured for a wide class of constants {c_i}, by Corollaries 4.3 and 4.4. It is also of interest to inquire into the dependence structure of the output process {ξ_i}. For example, it is found that the familiar moving average model

\[(5.13) \xi_n = \sum_{k=0}^{\infty} a^k x_{n-k},\]

where |a| < 1 and {X_i} is a sequence of independent r.v.'s, is an example of a sequence {ξ_i} which is both α- and β-multiplicative but which does not satisfy typical stronger dependence restrictions such as (3.2). We see this by computing


(5.14) \[ E(\xi_1, \xi_1, \xi_2, \xi_4) \mid \xi_3 = \sum_{\xi = \infty} a_{i_1} i_1 + i_2 + i_3 + i_4 - 4\xi = a_{i_2 + i_3 + i_4 - 3i_1} \]

so that \(\alpha\)-multiplicativity holds with \(\alpha(j_1, j_2, j_3) = (1 - a^4)^{-1} |a|^{j_1 + j_2 + j_3}\)

and \(\beta\)-multiplicativity holds with \(\beta(j) = (1 - a^4) |a|^j\).

5.4. Markov-dependent sequences. First let us consider a very simple case, a stationary sequence \(\{X_i\}\) of Markov-dependent Bernoulli trials (taking values \(\pm 1\), however, rather than 0 and 1), characterized by

(5.15) \[ P(X_i = 1) = P(X_i = -1) = \frac{1}{2} \]

(5.16) \[ P(X_i = 1 | X_{i-1} = 1) = \frac{1}{2} + \delta, \quad P(X_i = -1 | X_{i-1} = 1) = \frac{1}{2} - \delta \]

and

(5.17) \[ P(X_i = 1 | X_{i-1} = -1) = \frac{1}{2} - \delta, \quad P(X_i = -1 | X_{i-1} = -1) = \frac{1}{2} + \delta \]

where \(0 \leq \delta < \frac{1}{2}\). Note that \(\{X_i\}\) satisfies (1.0). Also, it is readily seen

(5.18) \[ E(X_j | X_i, X_{i-1}, \ldots) = (2\delta)^{j-i} X_i, \quad i \leq j \]

and thus

(5.19) \[ E(X_{i_1} X_{i_2}) = (2\delta)^{i_2 - i_1} \quad (i_1 < i_2) \]

(5.20) \[ E(X_{i_1} X_{i_2} X_{i_3} X_{i_4}) = (2\delta)^{(i_2 - i_1) + (i_4 - i_3)} \]
Thus \( \{X_i\} \) is \( \beta \)-multiplicative of order 4 (and likewise of any even order \( \nu \)), with \( \beta(j) = (2\delta)^j \), \( j \geq 1 \). It is also clear from (5.19) that \( \{X_i\} \) is not orthogonal and from (5.20) that it is not \( \alpha \)-multiplicative.

What happens if we generate a new sequence \( \{Y_i\} \) from \( \{X_i\} \) through the mechanism

\[
Y_i = X_i X_{i-1}
\]

This is a sequence of \( \pm 1 \) values with \( E(Y_i) \equiv 2\delta \) and \( E(Y_i^2) \equiv 1 \). It is readily found that the sequence

\[
(5.21) \quad Y_i^* = (1 - 4\delta^2)^{-\frac{1}{2}}(Y_i - 2\delta), \quad i = 2, 3, \ldots
\]

which is normalized to satisfy (1.0), is a multiplicative dependent sequence.

What happens if instead we generate a moving average of the \( X_i \)'s, say

\[
Z_i = \frac{1}{2}(X_i + X_{i-1})
\]

This is a sequence of 1, 0, -1 values with \( E(Z_i) \equiv 0 \) and \( E(Z_i^2) \equiv \frac{1}{2} + \delta \). Now confining attention to the case \( \delta = 0 \), the normalized sequence \( Z_i^* = 2^{k_i}Z_i \) satisfies (1.0) and is almost orthogonal:

\[
E(Z_i^* Z_j^*) = \begin{cases} 
\frac{1}{4} & j = i + 1 \\
0 & j > i + 1
\end{cases}
\]

As an entertainment, the reader may check that \( \{Z_i^*\} \) is another case of a \( \beta \)-multiplicative sequence.

Finally, we consider a 3-state Markov chain of r.v.'s \( \{X_i\} \) taking values 1, 0, -1, as characterized by
\[(5.22) \quad P(X_i = \ell) = \frac{1}{3} \quad (\ell = 1, 0, -1) \]

\[(5.23) \quad P(X_i = 0 | X_{i-1} = \ell) = \frac{1}{3} \quad (\ell = 1, 0, -1) \]

\[(5.24) \quad P(X_i = 1 | X_{i-1} = 1) = \frac{1}{3} + \delta, \quad P(X_i = -1 | X_{i-1} = 1) = \frac{1}{3} - \delta \]

and

\[(5.25) \quad P(X_i = 1 | X_{i-1} = -1) = \frac{1}{3} - \delta, \quad P(X_i = -1 | X_{i-1} = -1) = \frac{1}{3} + \delta \]

where \(0 \leq \delta < \frac{1}{3}\). This sequence satisfies (1.0) and is \(\beta\)-multiplicative but not \(\alpha\)-multiplicative.

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**On Almost Sure Convergence of Infinite Series**

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20. ABSTRACT

This paper focuses upon the question of almost sure convergence of an infinite series \( \sum_{k=1}^{\infty} c_k X_k \), subject to mild restrictions on the growth of the constants \( c_i \) and weak dependence restrictions on the random variables \( X_i \). Two new forms of weak multiplicative dependence are introduced, which apply to a large class of orthogonal sequences and time series \( \{X_i\} \) not satisfying dependence restrictions presently considered in the literature. Examples are the "random telegraph signal", the square of a Gaussian time series, and various moving average models. The dependence restrictions involve product moments \( E(X_{i_1} \cdots X_{i_v}) \) rather than conditional expectations and the like, and are thus relatively easy to check in practice. Our main theorem provides a basic foundation consisting of useful upper bounds of the form \( C \cdot (\sum_{k=1}^{n} c_k^2)^{v/2} \) for \( E(\sum_{k=1}^{n} c_k X_k)^v \) under either of the two new dependence restrictions introduced. This result is utilized to obtain corresponding "maximal inequalities" which are of general interest. These various results are applied to establish the almost sure convergence of \( \sum_{k=1}^{\infty} c_k X_k \) under mild dependence restrictions and the condition \( \sum_{k=1}^{\infty} c_k^2 < \infty \), and under still milder dependence restrictions and the condition \( \sum_{k=1}^{\infty} c_k^2 \log^2 k < \infty \). Specific applications to time series are considered, with special attention to the "random telegraph signal".