Treatment Contrasts in Paired Comparisons

II. Convergence of a Basic Iterative Scheme For Estimation

by

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Treatment Contrasts in Paired Comparisons

II. Convergence of a Basic Iterative Scheme for Estimation

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SUMMARY

In an earlier paper, the authors have presented the methodology for consideration of specified treatment contrasts in paired comparisons. An iterative method of solution of derived likelihood equations was given. This paper considers the iterative method. It is proved that, subject to a mild condition, the iterative scheme converges to a solution of the likelihood equations that is unique and yields the maximum of the likelihood function on a parameter space constrained by the condition that certain treatment contrasts are null.

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1. INTRODUCTION

Bradley and El-Helbawy (1976) presented basic procedures for consideration of treatment contrasts in paired comparisons with some emphasis on factorial treatment contrasts. The model of Bradley and Terry (1952) was used with a change of convenience in the scale-determining constraint to (1.3) below. Likelihood estimation was presented with some generality and an iterative procedure for solution of estimation equations was outlined. The purpose of this paper is to show convergence of the iterative procedure.

In brief summary, \( t \) treatments, \( T_1, \ldots, T_t \), are compared with \( n_{ij} \) comparisons of \( T_i \) and \( T_j \), \( n_{ij} \geq 0, i < j, i, j = 1, \ldots, t, n_{ji} = n_{ij}, j < i \). Associated with \( T_i \) is a parameter \( \pi_i, \pi_i > 0 \), such that the probability of selection of \( T_i \) when compared with \( T_j, i \neq j \), is

\[
pr(T_i > T_j) = \frac{\pi_i}{\pi_i + \pi_j}.
\]

Let \( \pi \) be the \( t \)-element column vector with typical element \( \pi_i \) and let
\( \gamma(\pi) \) be similarly defined with

\[
\gamma_i(\pi) = \log \pi_i, i = 1, \ldots, t.
\]

The scale-determining constraint used on the elements of \( \pi \) is

\[
\sum_{i=1}^{t} \gamma_i(\pi) = 0.
\]
On the assumption of independence of selections in pairs, the likelihood function is

\[ L(\pi) = \prod_{i<j} \pi_{ij}^{a_{ij}} \pi_i^{a_i} (\pi_i + \pi_j)^{-n_{ij}} = \prod_i \pi_i^{a_i} \prod_{i<j} (\pi_i + \pi_j)^{-n_{ij}} \quad (1.4) \]

where \( a_i = \sum_j a_{ij}, \ i = 1, \ldots, t, \) and \( a_{ij} \) is the number of selections of \( T_i \) in comparison with \( T_j, \ i \neq j, \ a_{ii} = 0. \) It is assumed that the \( a_{ij} \) satisfy the following assumption of Ford (1957) and that there are sufficient \( n_{ij} > 0 \) to permit this.

**Ford Assumption:** "In every possible partition of the \( (t) \) objects (treatments) into two nonempty subsets, some object in the second set has been preferred (selected) at least once to some object in the first set."

Let the elements of \( \pi \) be constrained by the requirement that

\[ B_m \gamma(\pi) = 0_m, \ 0 \leq m \leq (t - 1), \quad (1.5) \]

where \( B_m \) is an \( m \times t \) orthonormal matrix with zero-sum rows and \( 0_m \) is an \( m \)-element column vector of zeros. The general problem posed by Bradley and El-Helbawy is to obtain the estimator \( \hat{p} \) of \( \pi \) as the value of \( \pi \) maximizing \( L(\pi) \) in (1.4) subject to (1.3) and (1.5). The resulting equations for solution were shown to reduce to
\[ \frac{a_i}{p_i} - \phi_i(p) = 0, \quad i = 1, \ldots, t, \] (1.6)

\[ B_m \gamma(p) = 0_m, \] (1.7)

and

\[ \sum_{i} \gamma_i(p) = 0 \] (1.8)

where

\[ \phi_i(p) = \sum'_{j} \frac{n_{ij} p_i}{p_i + p_j} - \frac{1}{p_i} \sum'_{j} E_j(p) \frac{D_{i\bar{j}}}{D_{i\bar{i}}}, \] (1.9)

\[ E_i(p) = a_i - \sum'_{j} \frac{n_{ij} p_i}{p_i + p_j}, \quad i = 1, \ldots, t, \] (1.10)

and

\[ D = I - B'B_m \] (1.11)

with \( I_t \) being the \( t \times t \) identity matrix, \( D_{ij} \), the typical element of \( D \), and \( \sum' \), the sum over values \( j = 1, \ldots, t \) excluding \( j = i \). Note that \( D_{ii} > 0, \quad m = 1, \ldots, (t - 2) \).

Equations (1.6), (1.7) and (1.8) are solved through an iterative method.

Let \( p^{(0)} \) and \( p^{(r)} \) satisfying (1.7) and (1.8) be initial and \( r \)-th approximations to \( p \). (One acceptable choice of \( p^{(0)} \) has each element unity). The iterative method is described when it is shown how to obtain \( p^{(r+1)} \) from \( p^{(r)} \).
Let \( r = t(J - 1) + j - 1, J = 1, 2, 3, \ldots, j = 1, \ldots, t; J \) indicates a cycle of iterations and \( j \), the step in the \( J \)-th cycle. Calculate \( \phi_j(p^{(r)}) \) from (1.9). When \( a_j/p_j^{(r)} = \phi_j(p^{(r)}), p^{(r+1)} = p^{(r)}; \) if this occurs for \( t \) successive values of \( r \), \( p^{(r)} = p \). When \( a_j/p_j^{(r)} \neq \phi_j(p^{(r)}) \), compute \( p^{(r+1)}(k) \) and \( L[p^{(r+1)}(k)] \) for \( k = 0, 1, 2, \ldots \) until a value \( k^* \) of \( k \) is found for which \( L[p^{(r+1)}(k^*)] > L(p^{(r)}) \) and take \( p^{(r+1)} = p^{(r+1)}(k^*) \). The elements of \( p^{(r+1)}(k) \) are

\[
P_{i1}^{(r+1)}(k) = \exp[\log p_{i1}^{(r)} + (\frac{1}{2})\Lambda_j^{(r)}(p_{ij} - \frac{1}{n})], \quad i = 1, \ldots, t, \quad (1.12)
\]

where

\[
\Lambda_j^{(r)} = \begin{cases} 
\log [a_j/p_j^{(r)} \phi_j(p^{(r)})] & \text{if } \phi_j(p^{(r)}) > 0 \\
1 & \text{otherwise.} 
\end{cases} \quad (1.13)
\]

It has been shown [Bradley and El-Helbawy (1976)] that \( p^{(r+1)} \) satisfies conditions (1.7) and (1.8) if \( p^{(r)} \) does.

When \( m = 0 \) in (1.5), \( B_m \) does not exist, \( D_{ij} = 0, i \neq j \), equations (1.6) and (1.8) apply, the iterative method reduces to that proposed by Bradley and Terry (1952), and the Ford (1957) proof of convergence and maximization of \( L(p) \) covers this special case. When \( m = (t - 1) \), (1.3) and (1.5) determine that \( \gamma_1(m) = 0, \pi_1 = 1, i = 1, \ldots, t \). Attention is thus now restricted to consideration of the iterative process for \( 1 \leq m \leq (t - 2) \).
2. THE MAIN RESULTS

Let

$$\omega = \{ \pi: \pi > 0_t, \sum_{i} y_i(\pi) = 0, B_m y(\pi) = 0_m \} \quad (2.1)$$

for given \( m, 1 \leq m \leq (t - 2) \). The main theorem to be proved is:

Theorem 2.1: Subject to the Ford Assumption, the iterative procedure described above converges to a solution \( p \in \omega \) satisfying equations (1.6) such that \( L(p) \) is the unique maximum of \( L(\pi) \) over \( \omega \).

But before proof of the theorem is given some necessary results are given in lemmas below. Auxiliary lemmas are given in Section 3. The development of this section parallels that of Ford (1957) and constitutes a generalization of his work.

We establish the existence of a maximum for \( L(\pi) \) on \( \omega \).

Lemma 2.1: The Ford Assumption is sufficient to assure the existence of a maximum for \( L(\pi), \pi \in \omega \).

Proof: \( L(\pi) \) is positive and continuous for \( \pi \in \omega \). Consider

$$\omega^* = \{ \pi: \pi > 0_t, \sum_{i} y_i(\pi) = 0, B_m \pi = 0_m \}, \quad (2.2)$$

a closed and bounded set extending \( \omega \) to include its boundaries. Let \( \pi^* \) be any value of \( \pi \) on the boundary of \( \omega^* \) and define \( L(\pi^*) = 0 \). The lemma
is proved when it is shown that the extended definition of $L(\pi)$ provides continuity on $\omega^*$.

For any $\pi^*$, the $t$ treatments may be divided into two sets, one of which has positive corresponding elements in $\pi^*$ while the other has zero elements. Therefore two treatments exist, say $T_i$ and $T_j$, such that $\pi_i = 0$, $\pi_j > 0$ and, by the Ford Assumption, $a_{ij} > 1$. From (1.4), we may write

$$L(\pi) = \psi(\pi)[\pi_i / (\pi_i + \pi_j)]^{a_{ij}}$$

(2.3)

where $0 < \psi(\pi) \leq 1$, $\pi \in \omega$. Hence the right-hand side of (2.3) approaches zero as $\pi$ approaches $\pi^*$ and the continuity of $L(\pi)$ on $\omega^*$ follows as does the lemma.

With the existence of a maximum for $L(\pi)$ on $\omega$ established, it is shown that a maximum exists satisfying the estimation equations.

**Lemma 2.2:** Subject to the Ford Assumption, if $\pi \in \omega$ is such that $L(\pi)$ is a maximum of $L(\pi)$, $\pi \in \omega$, $\pi$ is a solution of equations (1.6).

**Proof:** The existence of $\pi$ is provided by Lemma 2.1. That $\pi$ satisfies the specified equations will follow from Apostol (1957, Theorem 7-10). The proof of the lemma follows when conditions of that theorem are verified.

The basic open set of the theorem is associated with

$$\omega^* = \{\pi: \pi_i > 0, i = 1, \ldots, t\}.$$  

(2.4)
The constraint functions of the theorem are

\[ g_0(\pi) = \sum_j \gamma_j(\pi), \]

\[ g_i(\pi) = \sum_j B_{ij} \gamma_j(\pi), \quad i = 1, \ldots, m, \]

from (1.7) and (1.8) and these functions vanish on \( \omega \subseteq \omega^+ \). The Jacobian of (2.5) has rank \((m + 1)\) from the definition of \( B_m \). Let

\[ \ell(\pi) = \log L(\pi). \]

When it is observed that \( \ell \) and each of the \((m + 1)\) functions in (2.5) have continuous partial derivatives on \( \omega^+ \), conditions for the theorem are met.

Maximization of \( L(\pi) \) is equivalent to maximization of \( \ell(\pi) \). Let

\[ Q(\pi) = \ell(\pi) + \sum_{i=0}^m \Gamma_i g_i(\pi) \]

be the Lagrangian function used for the maximization of \( \ell(\pi) \) subject to constraints (1.7) and (1.8). Then the Apostol theorem states that, given \( \pi \in \omega \), values of \( \Gamma_0, \Gamma_1, \ldots, \Gamma_m \) exist satisfying the equations,

\[ \frac{\partial Q(\pi)}{\partial \pi_i} \bigg|_{\pi = \pi_0} = \frac{a_i}{p_i} - \sum_j \frac{\nu_{ij}}{p_i + p_j} + \frac{1}{p_i} \sum_{j=0}^m \Gamma_j B_{ji} = 0, \quad i = 1, \ldots, t. \]
Bradley and El-Helbawy (1976) have shown, through elimination of the Lagrange multipliers, that (2.7) reduces to (1.6). Hence \( \bar{p} \) is a solution of equations (1.6), (1.7) and (1.8), maximizing \( L(\pi) \) on \( \omega \).

Lemma 2.3: Subject to the Ford Assumption, if \( p \in \omega \) is a solution of equations (1.6),

\[
L(\pi) = \prod_{i<j} \left\{ \frac{\pi_i}{\pi_i + \pi_j} \right\}^{n_{ij}p_i/(\pi_i + \pi_j)} \left\{ \frac{\pi_j}{\pi_i + \pi_j} \right\}^{n_{ij}p_j/(\pi_i + \pi_j)},
\]

(2.8)

\( L(p) \) is the unique maximum of \( L(\pi), \pi \in \omega \), and, for any \( \pi \in \omega^* \) in (2.2) \( L(\pi) < L(p) \) if \( \pi \neq p \).

Proof: The existence of \( p \) is provided by Lemma 2.2. Since each \( p_i > 0 \), equations (1.6) may be rewritten

\[
\mathbf{D} E(p) = 0_t,
\]

(2.9)

where \( E(p) \) is the column vector with \( t \) elements given by (1.10). This result is the key to the proof.

Replace \( a_i \) in \( L(\pi) \) in (1.4) by \( p_i \phi_i(p) \) from (1.6), \( i = 1, \ldots, t \), since \( p \) is a solution of (1.6). \( L(\pi) \) has two parts, one given by the right-hand side of (2.8) and one that is

\[
\exp\left\{ \sum_i \log \pi_i \right\} \sum_j E_j(p)D_{ij} + \sum_i E_i(p) \log \pi_i \}
\]

Any \( \pi \in \omega \) must satisfy (1.5) and \( \gamma(\pi) = \mathbf{D} \gamma(\pi) \) by Part (e) of Lemma 3.1 below; thus \( \log \pi_i = \sum_j D_{ij} \log \pi_j \), \( i = 1, \ldots, t \). Substitution in the second
term of the exponential followed by interchange of the indices of summation and
use of (2.9) shows that this exponential part of \( L(\pi) \) is unity and (2.8) follows.

Notice that the right-hand side of (2.8) is of the form \( \theta^{nv(1 - \theta)^{n(1-w)}} \)
and that, for \( w > 0 \), this function of \( \theta \) achieves its unique maximum when \( \theta = w \).
Thus, as in Ford, each term in the right-hand side of (2.8), and hence \( L(\pi) \),
is maximized when \( \pi_i/(\pi_i + \pi_j) = p_i/(\pi_i + \pi_j) \), \( i \neq j \), \( i, j = 1, \ldots, t \), and
thus \( \pi = p \) in view of (1.2) and (1.3). \( L(\pi) \) has a single maximum for \( \pi \in \omega^* \)
and the solution \( p \in \omega \) is unique.

This section concludes with two final lemmas. The first is used in
the second to show that the iterative process converges to a solution of the estimation equations.

**Lemma 2.4:** If \( \pi^* \in \omega \) is such that

\[
\frac{a_i}{\pi^*_i} = \phi_i(\pi^*)
\]

then \( \eta > 0 \) exists such that, if

\[
\Delta_i = \begin{cases} 
\log \frac{a_i}{\pi^*_i} \phi_i(\pi^*) & \text{if } \phi_i(\pi^*) > 0 \\
1 & \text{otherwise}
\end{cases}
\]  

(2.10)

and, for any \( \varepsilon > 0 \),

\[
\pi_j(\varepsilon) = \exp\{\log \pi^*_j + \varepsilon \Delta_i(D_{ij} - \frac{1}{t})\}, \ j = 1, \ldots, t,
\]

(2.11)

then \( \pi(\varepsilon) \in \omega, 0 < \varepsilon < \eta \), and \( L(\pi(\varepsilon)) > L(\pi^*) \).
Proof: For simplicity, we refer to $\ell(\gamma)$ when $\ell$ is written as a function of $\gamma$ in (1.2) and use $\gamma^*$ corresponding to $\gamma(\pi^*)$ and $\gamma(\epsilon)$ to $\gamma(\pi(\epsilon))$. It is easy to check that

$$\frac{\partial \ell(\gamma)}{\partial \gamma_i} = a_i - \sum_{j} n_{ij} \frac{\gamma_i}{\gamma_j + e_j}, \quad i = 1, \ldots, t. \quad (2.12)$$

Let

$$\omega^*(\eta_1) = \{\gamma: \frac{1}{t} (\sum (\gamma_j - \gamma_j^*))^2 < \eta_1\}, \quad \eta_1 > 0, \quad (2.13)$$

and note that

$$\gamma_j(\epsilon) = \gamma_j^* + \epsilon \Delta_1 (D_{ij} - \frac{1}{t}), \quad j = 1, \ldots, t, \quad (2.14)$$

from (2.11). Let $\eta_2 = \frac{1}{2} \min(\eta_1, \eta_1/|\Delta_1|)$. If we take $\epsilon$ such that $0 < \epsilon < \eta_2$, $\gamma(\epsilon) \in \omega^*(\eta_1)$ since

$$\frac{1}{t} (\sum (\gamma_j(\epsilon) - \gamma_j^*)^2)^2 = \epsilon |\Delta_1| (\sum (D_{ij} - \frac{1}{t})^2)^2$$

$$= \epsilon |\Delta_1| (\sum (D_{ij}^2 - \frac{1}{t})^2)^2 < \epsilon |\Delta_1| < |\Delta_1| \eta_2 < \eta_1.$$ 

Parts (c) and (d) of Lemma 3.1 and the definition of $\eta_2$ are used in establishing the inequality above. Then $\gamma(\epsilon) \in \omega^*(\eta_1), 0 < \epsilon < \eta_2$. 


From (2.12), note that \( \mathcal{L}(\gamma) \) is defined on \( w^*(\eta_1) \) and that all of its partial derivatives exist and are continuous at \( \gamma^* \). From Apostol (1957, Theorem 6-18), it follows that \( \mathcal{L} \) has a differential at \( \gamma^* \) and that

\[
\lim_{\varepsilon \to 0} \left[ \frac{\mathcal{L}(\gamma(\varepsilon)) - \mathcal{L}(\gamma^*)}{\varepsilon} \right] = \Delta_i \sum_j E_j(p_i^*) D_{i}D_{j}
\]

(2.15)

in view of (2.12), (2.14), (1.10) and the fact that \( \sum_j E_j(p_i^*) = 0 \).

Let the right-hand side of (2.15) be \( \psi_i(p_i^*) \). We show that \( \psi_i(p_i^*) > 0 \) and therefore that there exists \( \eta > 0 \), \( 0 < \eta < \eta_2 \) such that \( \mathcal{L}(\pi(\varepsilon)) > \mathcal{L}(p_i^*) \), \( 0 < \varepsilon < \eta \). From (2.8) and (1.9),

\[
\psi_i(p_i^*) = D_{i} \Delta_{i} \frac{a_i}{\pi_i} \left[ \frac{a_i}{\pi_i} - \phi_i(p_i^*) \right].
\]

(2.16)

If \( \phi_i(p_i^*) \leq 0 \), \( \Delta_i = 1 \) and \( \psi_i(p_i^*) > 0 \). If \( \phi_i(p_i^*) > 0 \), from (2.10),

\[
a_i/\pi_i = \phi_i(p_i^*) \exp \Delta_i \text{ and}
\]

\[
\psi_i(p_i^*) = D_{i} \Delta_{i} \phi_i(p_i^*) \exp \Delta_i (\varepsilon - 1) > 0
\]

since \( \Delta_i > 0 \).

Since \( \pi^* \in \omega \), Lemma 3.1, Part (e), and Lemma 3.2 are sufficient to show that \( \pi(\varepsilon) \in \omega \), \( 0 < \varepsilon < \eta \).

Lemma 2.5: The iterative scheme characterized by (1.12) converges to a solution \( p \in \omega \) satisfying equations (1.6).

Proof: Lemma 2.4 is used. When \( a_i/p_i^{(r)} = \phi_i(p_i^{(r)}) \), \( p_i^{(r)} \) is associated with \( p_i^* \), \( p_i^{(r+1)} \) with \( p(\varepsilon) \), and \( \varepsilon \) with \( (1/2)^k \). Clearly \( k^* \) exists such that \( 0 < (1/2)^k < \eta \) and \( p_i^{(r)} \), \( p_i^{(r+1)} \) \( \varepsilon \in \omega \) since \( p_i(0) \in \omega \). Then \( p_i^{(r+1)} \) is such that \( \lambda(p_i^{(r+1)}) > \lambda(p_i^{(r)}) \) and \( L(p_i^{(r+1)}) > L(p_i^{(r)}) \).
Let $p^{[j]}$, $j = 0, 1, 2, \ldots$, represent $p^{(r)}$ where $r$ completes a cycle of iterations, $p^{(0)} = p^{[0]}$. If $a_i/p_i^{[j]} = \phi_i(p^{[j]})$ for all $i = 1, \ldots, t$ and some $j$, $p = p^{[j]}$ and equations (1.6) are satisfied. If $a_i/p_i^{[j]} = \phi_i(p^{[j]})$ for some $i$, $L(p^{[j+1]}) > L(p^{[j]})$. Successive values of $j$ lead to a monotone increasing, bounded sequence $\{L(p^{[j]})\}$ with corresponding sequence of vectors $\{p^{[j]}\}$. The sequence converges and the sequence $\{p^{[j]}\}$ must converge to a vector $p$. It follows from the right-hand member of (2.15), given in the form (2.16), that $\psi_i(p) = 0$, $i = 1, \ldots, t$, and that equations (1.6) are satisfied. Since $L(p) = 0$ on the boundary of $\omega$ and is positive for $p \in \omega$, $p \in \omega$.

The lemmas developed in this section yield Theorem 2.1 directly.

3. SOME AUXILIARY RESULTS

Let $D = (I_t - B'B/m\cdot m)$ be a matrix of the form (1.11), $1 \leq m < t$, and let $D_j$ be the $j$-th column of $D$. Some properties of $D$ have been used above and are summarized in the two following lemmas.

**Lemma 3.1:** Given $D$ and $D_j$ described above, then (a) $D$ is symmetric, (b) $D$ is idempotent, (c) $\sum_{j} D_{ij} = \sum_{j} D_{ji} = 1$, $i = 1, \ldots, t$, (d) $0 < D_{ii} \leq 1$, $i = 1, \ldots, t$, (e) for any $t$-element column vector $x$, $x = Dx$ iff $Bx = 0_m$, (f) $D_j' D_j \leq 1$, and (g) $DD_j = D_j$. 
Lemma 3.2: If $\mathbf{Y}^*$ is a t-element column vector such that $|Y_i^*| < \infty$, $i = 1, \ldots, t$, $\sum_i Y_i^* = 0$, and $\mathbf{Y}^* = D \mathbf{Y}$ and if, for any real number $\Delta$,

$$\mathbf{Y} = \mathbf{Y}^* + \Delta \left( \frac{D}{t} - \frac{1}{t} \mathbf{1}_t \right)$$

where $\mathbf{1}_t$ is a column vector of unit elements, then (a) $|Y_i| < \infty$,
(b) $\sum_i Y_i = 0$, and (c) $\mathbf{Y} = D \mathbf{Y}$.

The proofs of the various parts of the lemmas are trivial or elementary.

Some parts of Lemma 3.1 assist in the proofs of Lemma 3.2.

REFERENCES


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