The Role of the Poisson Distribution in Approximating System Reliability of k-Out-Of-n Structures

by

R. J. Serfling

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The Florida State University
Department of Statistics
Tallahassee, Florida 32306

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ABSTRACT

The determination of the exact reliability of a complex system of "components" (or subsystems) is typically a formidable theoretical and computational problem. Often one resorts merely to bounds on system reliability. The present paper delineates, for $k$-out-of-$n$ structures, a simple and easily implemented approach based on the Poisson distribution. The components are allowed to be statistically dependent and to have unequal reliabilities. Special attention is given to the cases of independent and Markov dependent components, and some numerical illustration is provided. Comparisons are made to other reliability bounds in the literature. Reliability bounds for $k$-out-of-$n$ structures have wide application, including "series" systems, "parallel" systems, monitoring systems, and systems with "spares". The results are also applicable in the context of availability.
1. Introduction. The determination of the exact reliability of a complex system of "components" (or subsystems) is typically a formidable theoretical and computational problem. Often one resorts merely to bounds on system reliability. Broadly applicable lower and upper bounds are provided by Barlow and Proschan (1975). However, the general formulas for these bounds are somewhat cumbersome.

The present paper delineates, for \( k \)-out-of-\( n \) structures, a simple and easily implemented approach based on the Poisson distribution. For such structures, where the \( n \)-component system functions if and only if at least \( k \) components function, the system reliability \( R \) may be represented in the form

\[
(1.1) \quad R = P\left[ \sum_{i=1}^{n} Y_i \leq n - k \right],
\]

where \( Y_i = 1 \) or 0 according as the \( i \)-th component has failed or is still functioning. The random variable \( \sum_{i=1}^{n} Y_i \) denotes the total number of failed components, i.e., the number of occurrences among the collection of "rare" (hopefully) events \( \{Y_1 = 1\}, \cdots, \{Y_n = 1\} \), and thus its probability distribution is subject to approximation by a suitably chosen Poisson distribution. In this fashion one obtains a Poisson approximation for the system reliability \( R \), as well as related lower and upper bounds for \( R \).

A general theorem for Poisson approximation of system reliability of \( k \)-out-of-\( n \) structures is presented in Section 2. The components of the system are allowed to be statistically dependent and to have unequal reliabilities. Also, the special case of independent components is examined.
In Section 3 we consider k-out-of-n structures with Markov-dependent components. For convenience, attention is confined here to the case of equal component reliabilities. For the 2-out-of-3 structure, an exact analysis is given in order to illustrate quantitatively the consequences of erroneously assuming independence in computing system reliability. For the general k-out-of-n structure, the theorem of Section 2 is utilized to obtain approximations and bounds explicitly involving parameters of the Markov dependence. Numerical illustration for the 2-out-of-3 case is provided.

Comparisons between the bounds of the present paper and those of Barlow and Proschan (1975), specialized to k-out-of-n structures, are made in Section 4. For example, in the case of independent components with unequal reliabilities, the bounds based on Poisson approximation are considerably easier to compute. On the other hand, in the case of dependent associated components, the bounds of the present paper require explicit computation of dependence parameters, whereas the Barlow and Proschan bounds do not entail such parameters. However, such computational advantage is offset by a loss of sharpness. Various examples and numerical illustrations are presented.

Our treatment is restricted to k-out-of-n structures for a technical reason, to accommodate the utilization of the Poisson distribution as the basis for a convenient approximation to system reliability. Nevertheless the results have wide practical application. Familiar special cases are "series" (n-out-of-n) and "parallel" (1-out-of-n) structures. The role of k-out-of-n structure in designing monitoring systems is discussed by Barlow and Proschan (1975), p. 49. A k-series system with a "spares pool" of size m may be regarded as a k-out-of-(k+m) structure. A maintenance policy of calling a repairman or ordering a batch of replacements as soon as n - k - 1 units have been
exhausted corresponds to a k-out-of-n system. Further, the results of the paper may be interpreted in the context of availability. Shaw and Shooman (1975) study the availability of systems having k-out-of-n structure with respect to n subsystems with known availabilities. Our results provide an alternative approach for such studies. Finally, we note that the results of the paper could be extended to systems which are j-out-of-m compositions of k-out-of-n structures.

A useful feature of the Poisson approximation for system reliability is that, rather than directly estimating reliability of a given system, it may alternatively be used to design a k-out-of-n system to achieve desired reliability specifications.


Our approximations to the reliability R defined by (1.1) will be expressed in terms of Poisson distributions. Let \( Q_\lambda \) denote the Poisson distribution with mean \( \lambda \) (where \( \lambda > 0 \)), and let \( Q_\lambda(m) \) denote the probability assigned by \( Q_\lambda \) to the set \( \{0, 1, \ldots, m\} \). Thus

\[
Q_\lambda(m) = \sum_{j=0}^{m} \frac{e^{-\lambda} \lambda^j}{j!}, \quad m = 0, 1, \ldots
\]

Random variables taking only the values 0 and 1 will be called Bernoulli variables. Thus the reliability R is expressed by (1.1) in terms of the distribution of a sum of Bernoulli variables. A general theorem on approximation of such distributions by Poisson distributions is given in Serfling (1976). An approximation to R follows as a direct corollary, which is now presented.
Specifically, consider a $k$-out-of-$n$ structure with components having states indicated by the Bernoulli variables $Y_1, \ldots, Y_n$ defined in Section 1. Put

$$\theta_1 = P[Y_1 = 1]$$

and, for $2 \leq i \leq n$,

$$\theta_i(y_1, \ldots, y_{i-1}) = P[Y_i = 1|Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}],$$

for $y_1, \ldots, y_{i-1}$ taking values 0 or 1. The $\theta_i$ denote (conditional) failure probabilities of the components. Restating (1.1), the system reliability is given by $R = P[\bigwedge_{i=1}^n y_i \leq n - k]$.

**THEOREM 2.1.** Consider a $k$-out-of-$n$ structure with conditional component failure probabilities $\theta_1, \ldots, \theta_n$ and system reliability $R$ as given above. Let $\lambda_1, \ldots, \lambda_n$ be any set of values such that $0 < \lambda_1, \ldots, \lambda_n < 1$ and put

$$\lambda = \sum_{i=1}^n \lambda_i,$$

$$L = \frac{1}{2} \sum_{i=1}^n \lambda_i^2,$$

$$D = \sum_{i=1}^n E[|\theta_i - \lambda_i|],$$
\[ \gamma_i = -\log(1-\lambda_i), \quad 1 \leq i \leq n, \]

\[ \gamma = \sum_{i=1}^{n} \gamma_i, \]

and

\[ G = \frac{1}{2} \sum_{i=1}^{n} \gamma_i^2. \]

Then

(2.1a) \[ Q_{\lambda}(n-k) - D - L \leq R \leq Q_{\lambda}(n-k) + D + L \]

and

(2.1b) \[ Q_{\gamma}(n-k) - D \leq R \leq Q_{\gamma}(n-k) + D + G. \]

**REMARKS.** (i) Here \( E|\theta_i - \lambda_i| \) denotes the quantity

\[ E|\theta_i(y_1, ..., y_{i-1}) - \lambda_i|. \]

For \( i = 1 \), this is simply \( |\theta_1 - \lambda_1| \).

(ii) It is important to note that the values \( \lambda_i \) in this theorem may be selected arbitrarily. For example, a natural and convenient choice for \( \lambda_i \) is the **mean** \( E(\theta_i) \). On the other hand, if it is desired to minimize the quantity \( E|\theta_i - \lambda_i| \), the appropriate choice is \( \lambda_i \) equal to a **median** of the distribution of \( \theta_i(y_1, ..., y_{i-1}) \).
(iii) For the case of independent components, the quantities \( \theta_i \) are no longer random, so that the choice \( \lambda_i = \theta_i, \ 1 \leq i \leq n \), is feasible. This completely eliminates the term D. See Corollary 2.1 below.

(iv) Conditions (2.1a) and (2.1b) provide alternate sets of bounds for R. For small \( \lambda_i \)'s, (2.1b) tends to give a tighter interval for R. See Sections 3 and 4 for numerical illustration. []

The following result focuses upon the case of independent components and expresses the conditions in terms of the parameters

\[ p_i = 1 - \theta_i = P[\text{i-th component is functioning}], \ 1 \leq i \leq n, \]

which are more conventional than the \( \theta_i \)'s.

**COROLLARY 2.1.** For a k-out-of-n structure with independent components having reliabilities \( p_i, \ 1 \leq i \leq n \), the system reliability \( R \) satisfies

\[ (2.2a) \quad Q_\lambda(n - k) - \frac{1}{2} \sum_{i=1}^{n} (1 - p_i)^2 \leq R \leq Q_\lambda(n - k) + \frac{1}{2} \sum_{i=1}^{n} (1 - p_i)^2 \]

and

\[ (2.2b) \quad Q_\gamma(n - k) \leq R \leq Q_\gamma(n - k) + \frac{1}{2} \sum_{i=1}^{n} \log^2(p_i), \]

where \( \lambda = \frac{r^n}{t_1} (1 - p_1) \) and \( \gamma = -\frac{r^n}{t_1} \log p_1 \).

Another special case, that of Markov-dependent components, is treated in Section 3. However, there attention is confined for convenience to the case of equal component reliabilities \( p_i \equiv p \).
3. Reliability of k-out-of-n structures with identical Markov-dependent components. Let us indicate the states of the components by $X_1, \ldots, X_n$, where $X_i = 1$ or 0 according as the $i$-th component is still functioning or has failed. Suppose that the sequence of Bernoulli variables $X_1, \ldots, X_n$ is Markov-dependent, with transition probabilities

$$\alpha = P[X_i = 1|X_{i-1} = 1]$$

and

$$\beta = P[X_i = 1|X_{i-1} = 0]$$

for $2 \leq i \leq n$. Assume that $0 < \alpha < 1$ and $0 < \beta < 1$ and put

$$\delta = \alpha - \beta, \quad p = \frac{\beta}{1 - \delta}.$$ 

Assume for convenience that the components have equal reliabilities, in which case it is found that

$$P[X_i = 1] = p, \quad 1 \leq i \leq n.$$ 

Suppose also that $\alpha > \beta$ (or $1 - \beta > 1 - \alpha$), reflecting the assumption that failure of the $(i-1)$-th component only enhances the chance of failure of the $i$-th component, $2 \leq i \leq n$. Thus $\delta > 0$ and $\alpha > p > \beta$.

The assumption that $\delta \geq 0$ makes $X_1, \ldots, X_n$ conditionally increasing in sequence in the sense of Barlow and Proschan (1975) and hence, by their
Theorem 4.7, p. 146, associated. This fact, namely that Markov-dependent Bernoulli variables $X_1, \ldots, X_n$ with $\delta \geq 0$ are associated, is apropos to the comparisons made in Section 4.

Let us further define

$$z = \frac{1 - \beta}{1 - \alpha}.$$ 

The conditions $\delta > 0$ and $\delta > 1$ are equivalent assertions, but the quantities $\delta$ and $\delta$ measure the dependence in different ways. The quantity $\delta$ measures the relative severity of the influence of failure of the $(i - 1)$-th component upon the chance of failure of the $i$-th component. In considering examples, it is intuitively appealing to specify values for $p$ and $d$ and then solve for $\delta$, $\alpha$ and $\beta$ via

$$\delta = \frac{(1 - p)(d - 1)}{1 + (1 - p)(d - 1)}.$$ 

In order to see how Markov dependence alters the system reliability from its value in the case of independence, let us examine the exact reliability function in the simple case of 2-out-of-3 structure.

EXAMPLE: 2-out-of-3 structure. Let $X_1$, $X_2$ and $X_3$ be Markov dependent as described above. Considering the system reliability as a function of the component reliability $p$, the reliability function is found to be
\[ R(p) = P[X_1 + X_2 + X_3 \geq 2] \]

\[ = P[X_1 = X_2 = X_3 = 1] + P[X_1 = X_2 = 1, X_3 = 0] + P[X_1 = X_2 = 0, X_3 = X_3 = 1] \]

\[ = P[X_3 = 1|X_2 = 1] P[X_2 = 1|X_1 = 1] P[X_1 = 1] + \cdots \]

\[ = \alpha^2 p + (1 - \alpha)\alpha p + \beta(1 - \alpha)p + \alpha\beta(1 - p) \]

\[ = \alpha p + \beta p + \alpha\beta - 2\alpha\beta p. \]

In terms of simply the parameters \( p \) and \( \delta \), we have

\[ R(p) = 3p^2 - 2p^3 - 2p(1 - p)(2p - 1)\delta + (3p^2 - 2p^3 - p)\delta^2. \]

The special case of independence \((\delta = 0)\) is given by

\[ R_o(p) = 3p^2 - 2p^3. \]

The disparity between \( R(p) \) and \( R_o(p) \) is illustrated in Table 3.1.
Table 3.1. The reliability function $R(p)$ for selected values of $p$ and $d$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$R_0(p)$</th>
<th>$d = 2$</th>
<th>$d = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.8</td>
<td>.896</td>
<td>.871</td>
<td>.850</td>
</tr>
<tr>
<td>.9</td>
<td>.972</td>
<td>.966</td>
<td>.950</td>
</tr>
<tr>
<td>.95</td>
<td>.993</td>
<td>.989</td>
<td>.986</td>
</tr>
<tr>
<td>.99</td>
<td>.9997</td>
<td>.9995</td>
<td>.9993</td>
</tr>
</tbody>
</table>

The change in $R(.99)$ in passing from independence ($d = 1$) to the dependence cases $d = 2$, $d = 3$ is rather substantial, if loss is measured in terms of the unreliability $1 - R$. □

Turning now to the general $k$-out-of-$n$ structure, we consider the implications of Theorem 2.1 for Markov-dependent components. For the random variables $Y_i = 1 - X_i$, we have $\Theta_i = P[Y_1 = 1] = 1 - p$ and, for $i \geq 2$,

$$\Theta_i(Y_1, \ldots, Y_{i-1}) = \begin{cases} 1 - \alpha \text{ if } Y_{i-1} = 0 \\ 1 - \beta \text{ if } Y_{i-1} = 1. \end{cases}$$

It follows, for $i \geq 2$, that $\Theta_i$ has mean $1 - p$ and (since $p > \frac{1}{2}$) median $1 - \alpha$. We find, for $i \geq 2$, that $E[\Theta_i - (1 - p)]^2 = 2\sigma (1 - p)$, whereas $E[\Theta_i - (1 - \alpha)] = (1 - p)\delta$, and thus that $E[\Theta_i - \lambda_i]$ is not only minimized at $\lambda_i = 1 - \alpha$ but is actually about half the value corresponding to $\lambda_i = 1 - p$. Moreover, since $1 - \alpha < 1 - p$, the term $L = \frac{\ln(p^2)}{\sum_i \lambda_i^2}$ is smaller for $\lambda_i = 1 - \alpha$ than for $\lambda_i = 1 - p$, $i \geq 2$. Consequently, in order to minimize the contribution $D = \sum_i E[\Theta_i - \lambda_i]$ in the bounds (2.1), we choose $\lambda_i = 1 - p$ and $\lambda_i = 1 - \alpha$, $i \geq 2$. We obtain
COROLLARY 3.1. Consider a k-out-of-n system of Markov-dependent components with parameters $p$ and $\delta$. Put

$$\lambda = (1 - p) + (n - 1)(1 - \alpha),$$

$$L = \frac{1}{2}[(1 - p)^2 + (n - 1)(1 - \alpha)^2],$$

$$D = (n - 1)(1 - p)\delta,$$

$$\gamma = -\log p - (n - 1) \log \alpha,$$

and

$$G = \frac{1}{2}[\log^2 p + (n - 1) \log^2 \alpha].$$

Then

$$Q_\lambda(n - k) - D - L \leq R \leq Q_\lambda(n - k) + D + L$$

and

$$Q_\gamma(n - k) - D \leq R \leq Q_\gamma(n - k) + D + G.$$
Table 3.2. Exact values and associated bounds for \( R(p) \) in the 2-out-of-3 structure, for selected values of \( p \) and \( d \).

\[
\begin{array}{cccccccccc}
\hline
p & R_o(p) & R(p), d=2 & \text{Lower} & \text{Upper} & \text{Lower} & \text{Upper} & \text{Lower} & \text{Upper} \\
0.8 & 0.896 & 0.871 & 0.789 & 1 & 0.814 & 1 & 0.64 & 0.96 \\
0.9 & 0.972 & 0.966 & 0.936 & 0.998 & 0.945 & 0.991 & 0.81 & 0.99 \\
0.95 & 0.993 & 0.989 & 0.981 & 0.998 & 0.985 & 0.998 & 0.903 & 0.9975 \\
0.99 & 0.9997 & 0.9995 & 0.9992 & 0.9999 & 0.99936 & 0.9999 & 0.980 & 0.9999 \\
\hline
\end{array}
\]

4. Comparisons with other reliability bounds. Barlow and Proschan (1975), p. 37, provide bounds on system reliability for an arbitrary coherent structure. From these general bounds they derive more explicit bounds for the case of associated components (this form of dependence includes independence as a special case and also includes the Markov dependence treated in Section 3). Specialized to the case of a \( k \)-out-of-\( n \) structure with components having reliabilities \( p_1, \ldots, p_n \) and with system reliability \( R \), the latter bounds are

\[
\begin{align*}
&(4.1) \quad \max_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{i_1} p_j \leq R \leq \min_{1 \leq i_1 < \cdots < i_{n-k+1} \leq n} \prod_{j=1}^{n-k+1} p_j,
\end{align*}
\]

where \( \prod_{i=1}^{b} y_i = 1 - \prod_{i=a}^{b} (1 - y_i) \). For the case of equal component reliabilities \( p_i \equiv p \), (4.1) reduces to
(4.2) \[ p^k \leq R \leq 1 - (1 - p)^{n-k+1}. \]

We have utilized (4.2) in Table 3.2.

Barlow and Proschan (1975), p. 34, also provide another set of bounds for the case of associated components. Specializing to the case of a k-out-of-n structure with independent components having reliabilities \( p_1, \ldots, p_n \), these bounds are:

\[
(4.3) \quad \prod_{1 \leq i_1 < \cdots < i_{n-k+1} \leq n} p_{i_1} \leq R \leq \prod_{1 \leq i_1 < \cdots < i_k < n} p_{i_1}.
\]

For the case of equal component reliabilities \( p_i = p \), (4.3) reduces to

\[
(4.4) \quad [1 - (1 - p)^{n-k+1} \binom{n}{n-k+1}] \leq R \leq 1 - [1 - p^k] \binom{n}{k}.
\]

Let us now focus upon the case of independent components with equal reliabilities \( p_i = p \). Corollary 2.1 yields

\[
(4.5) \quad Q_n(1-p)(n-k) - \frac{1}{2n}(1-p)^2 \leq R \leq Q_n(1-p)(n-k) + \frac{1}{2n}(1-p)^2
\]

and

\[
(4.6) \quad Q_n \log p (n-k) \leq R \leq Q_n \log p (n-k) + \frac{1}{2n} \log^2(p).
\]

We may compare (4.2), (4.4), (4.5) and (4.6). As regards computational ease, these are all relatively simple. For a numerical comparison, Table 4.1 provides
evaluations of the bounds for the 2-out-of-3 structure. It is seen that collectively the bounds (4.2), (4.4), (4.5) and (4.6) yield more information than any one of them individually. The composite information thus supplied by Table 4.1 is presented in Table 4.2. It is thus seen that the Barlow and Proschan bounds (4.2) and (4.4) and the Poisson approximation bounds (4.5) and (4.6) are competitive in the case of independent components with equal reliabilities.

**Table 4.1.** Various bounds on $R(p)$ for the 2-out-of-3 structure with independent components, for selected values of $p$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$R(p)$</th>
<th>Lower</th>
<th>Upper</th>
<th>Lower</th>
<th>Upper</th>
<th>Lower</th>
<th>Upper</th>
<th>Lower</th>
<th>Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>.8</td>
<td>.896</td>
<td>.64</td>
<td>.96</td>
<td>.885</td>
<td>.953</td>
<td>.818</td>
<td>.938</td>
<td>.854</td>
<td>.929</td>
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<tr>
<td>.9</td>
<td>.972</td>
<td>.81</td>
<td>.99</td>
<td>.970</td>
<td>.99</td>
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<td>.989</td>
<td>.993</td>
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<tr>
<td>.99</td>
<td>.9997</td>
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<td>.9999</td>
<td>.9997</td>
<td>.9997</td>
<td>.9994</td>
<td>.9997</td>
<td>.99956</td>
<td>.9997</td>
</tr>
</tbody>
</table>

**Table 4.2.** Composite bounds on $R(p)$, from Table 4.1.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$R(p)$</th>
<th>Lower</th>
<th>Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>.8</td>
<td>.896</td>
<td>.885</td>
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<tr>
<td>.99</td>
<td>.9997</td>
<td>.9997</td>
<td>.9997</td>
</tr>
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</table>
For the case of independent components with unequal reliabilities, the bounds given by (4.1) and (4.3) may be compared with those of Corollary 2.1. It is evident that (4.1) and (4.3) are considerably more cumbersome computationally than (4.2) and (4.4), whereas the general bounds of Corollary 2.1 are only slightly more cumbersome than (4.5) and (4.6). This difference in computational ease becomes increasingly pronounced for larger values of n and n - k. Consider the following example.

**EXAMPLE: k-out-of-20 structure.** Let us consider independent components with reliabilities \( p_1 = \cdots = p_5 = .99, \ p_6 = \cdots = p_{10} = .98, \ p_{11} = \cdots = p_{15} = .97, \) and \( p_{16} = \cdots = p_{20} = .96. \) We shall not bother to compute (4.1) and (4.3), which are tedious. In order to utilize Corollary 2.1, we readily compute

\[
\lambda = 5(.01 + .02 + .03 + .04) = .5,
\]

\[
\gamma = -5(\log .99 + \log .98 + \log .97 + \log .96) = .508,
\]

\[
L = \frac{1}{2} \cdot 5 \cdot \left[ (.01)^2 + \cdots \right] = .0075,
\]

and

\[
G = \frac{1}{2} \cdot 5 \cdot \left[ \log^2(.99) + \cdots \right] = .0078.
\]

Using a table of the Poisson distribution to obtain \( Q_{.5}(20 - k) \) and \( Q_{.508}(20 - k) \), and then evaluating the bounds (2.2a) and (2.2b) with the use of \( \lambda, \gamma, L \) and \( G \), we find that in the present example the bounds (2.2b) are more effective than (2.2a). On the other hand, the quantity \( Q_{.5}(20 - k) \) involved in (2.2a) may be interpreted as an approximation to \( R \), whereas \( Q_{.508}(20 - k) \)
arises as a lower bound rather than as an approximation. The relevant values are given in Table 4.3.

<table>
<thead>
<tr>
<th>k</th>
<th>(Q_{0.5(20-k)})</th>
<th>Lower</th>
<th>Upper</th>
<th>(2.2a)</th>
<th>Lower</th>
<th>Upper</th>
<th>(2.2b)</th>
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<tr>
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<td></td>
</tr>
<tr>
<td>16</td>
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<td>.99233</td>
<td>1</td>
<td>.99981</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that the second and third columns in Table 4.3 may be eliminated, being improved by the fourth and fifth columns, respectively.

Finally, we discuss the case of dependent components. Whereas the bounds given by Theorem 2.1 require explicit computation of dependence parameters, the Barlow and Proschan bounds (4.1) for associated components do not entail explicit dependence parameters. Whether or not (4.1) is actually easier to compute than (2.1a, b) depends on the nature of the actual dependence and the feasibility of computing the quantity \(D\). In Section 3 we have seen that for Markov-dependent components the computation of the bounds (2.1) is easy and yields substantial sharpening of the bounds (4.3). Indeed, this illustrates the potential gains from utilizing when possible any knowledge of the specific dependence structure. Of course, the bounds (2.1) are applicable also to other varieties of dependence.
5. Acknowledgments. I am indebted to Frank Proschan for correcting a
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<td>The Role of the Poisson Distribution in Approximating System Reliability of k-Out-Of-n Structures</td>
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The determination of the exact reliability of a complex system of "components" (or subsystems) is typically a formidable theoretical and computational problem. Often one resorts merely to bounds on system reliability. The present paper delineates, for $k$-out-of-$n$ structures, a simple and easily implemented approach based on the Poisson distribution. The components are allowed to be statistically dependent and to have unequal reliabilities. Special attention is given to the cases of independent and Markov dependent components, and some numerical illustration is provided. Comparisons are made to other reliability bounds in the literature. Reliability bounds for $k$-out-of-$n$ structures have wide application, including "series" systems, "parallel" systems, monitoring systems, and systems with "spares". The results are also applicable in the context of availability.