NONPARAMETRIC ESTIMATION OF A VECTOR-VALUED BIVARIATE

FAILURE RATE II: LIMITING DISTRIBUTION

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ABSTRACT

In Part I of the study, a nonparametric estimate of the Johnson-Kotz
(J. Multivariate Anal. 5 (1975), 53-66) vector-valued bivariate failure rate
is proposed and important asymptotic properties of the estimate, such as weak,
first mean, strong, and strong uniform consistency, are investigated. In
the current paper, the joint asymptotic normality of the estimate evaluated at
q distinct continuity points of the failure rate is established.

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1. Introduction. This is a continuation of Part I (consistency), where a nonparametric estimate \( \hat{\xi} \) of the Johnson-Kotz (1975) vector-valued bivariate failure rate is proposed. In the earlier study, important asymptotic properties of the estimate, such as weak, first mean, and strong (pointwise) consistency are investigated. A set of necessary and sufficient conditions is also obtained for the strong uniform consistency of the estimate in a bounded region where \( \hat{F}(x) > 0 \). In the current study, the joint asymptotic normality of \( \hat{\xi}'(x_\ell), \ldots, \hat{\xi}'(x_q) \) is established where \( x_\ell, \ldots, x_q \) are q distinct continuity points of the vector-valued bivariate failure rate \( \xi \). The definitions and notation used here are the same as those of Part I.

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2. **Limiting Distribution of the Estimate.** Let \( \mathbf{x}_a' = (x_{1a}, x_{2a}) \), \( a = 1, \ldots, q \), be \( q \) distinct continuity points of \( f \) such that \( \sum_{i=1}^{2} \sum_{a=1}^{q} G_i(x_a) > 0 \), and

\[
F(x_a) > 0, \ \alpha = 1, \ldots, q.
\]

The limiting distribution of \( \mathbf{\hat{x}}'(\mathbf{x}_1), \ldots, \mathbf{\hat{x}}'(\mathbf{x}_q) \) is derived in the following theorem.

**THEOREM 2.1.** Assume the following conditions:

(i) \( \int u^2 k_i(u) du = 0 \) and \( \int u^2 k_i(u) du < \infty, \ (i=1,2) \);

(ii) \( na_n \to \infty \) and \( na_n^{1/2} \to 0 \), as \( n \to \infty \); and

(iii) \( G_i(x) \) has bounded partial derivatives of first and second orders \( (i=1,2) \).

If \( \prod_{i=1}^{2} \prod_{\alpha \neq \beta = 1}^{q} (x_{i\alpha} - x_{i\beta}) \neq 0 \), then the limiting distribution of

\[
\mathbf{\hat{x}}'(\mathbf{x}_1), \ldots, \mathbf{\hat{x}}'(\mathbf{x}_q)
\]

is a \( 2q \)-variate normal distribution with mean vector \( \mathbf{y}' = (\mathbf{x}'(\mathbf{x}_1), \ldots, \mathbf{x}'(\mathbf{x}_q)) \) and covariance matrix \( (na_n)^{-1/2} \Sigma \) with

\[
\Sigma = \begin{bmatrix}
\sigma_{11} & \sigma_{11} & \sigma_{11} & \sigma_{11} & \cdots & \sigma_{11} & \sigma_{11} & \sigma_{11} & \sigma_{11} & 2q \\
\sigma_{21} & \sigma_{21} & \sigma_{21} & \sigma_{21} & \cdots & \sigma_{21} & \sigma_{21} & \sigma_{21} & \sigma_{21} & 2q \\
\sigma_{2q} & \sigma_{2q} & \sigma_{2q} & \sigma_{2q} & \cdots & \sigma_{2q} & \sigma_{2q} & \sigma_{2q} & \sigma_{2q} & 2q \\
\end{bmatrix}
\]

(2.1)
where

\[
\sigma_{i\alpha j\beta} = \lim_{n \to \infty} n \text{cov}[\hat{r}_i(x_{\alpha}), \hat{r}_j(x_{\beta})]
\]

(2.2)

\[
(i, j = 1, 2; \quad \alpha, \beta = 1, \ldots, q)
\]

\[
= g_i(x_{\alpha})[F(x_{\alpha})]^{-2} k_i^2(u) du + \frac{r_i^2(x_{\alpha})}{1 - F(x_{\alpha})}[\frac{1}{F(x_{\alpha})}]^{-1} \quad (2.2a)
\]

for \(i = j = 1, 2; \quad \alpha = \beta = 1, \ldots, q;\)

\[
= r_i(x_{\alpha}) r_j(x_{\alpha})[1 - F(x_{\alpha})][\frac{1}{F(x_{\alpha})}]^{-1} \quad (2.2b)
\]

for \(i \neq j = 1, 2; \quad \alpha = \beta = 1, \ldots, q;\)

\[
= r_j(x_{\beta}) \frac{r_j(x_{\beta})}{F(x_{\beta})[F(x_{\beta})]}^{-1} \quad (2.2c)
\]

for \(i, j = 1, 2; \quad \alpha \neq \beta = 1, \ldots, q.\)

**Remark.** For simplicity the theorem will be shown only for the case \(q = 2.\) It is clear that the method of proof applies equally well to the more general case. The following lemma will be useful in proving the theorem; it also may have an independent interest of its own right.

**Lemma 2.2.** Let \(x'_{\alpha} = (x_{1\alpha}, x_{2\alpha}), \alpha = 1, 2,\) be two distinct continuity points of \(G_i(x), i = 1, 2,\) such that \(\sum_{i=1}^{2} \sum_{\alpha=1}^{2} G_i(x_{\alpha}) > 0.\) Define

\[
Y_{\alpha \alpha} = a_n [\hat{G}_i(x_{\alpha}) - \hat{E}_i(x_{\alpha})],
\]
\[ Y_{a+n} = a_n \{ \hat{G}_2(x_a) - \hat{E}_2(x_a) \}, \quad (2.3) \]

\[ Y_{a+4+n} = \hat{F}(x_a) - \hat{E}_2(x_a), \quad a = 1, 2. \]

If assumption (ii) of Theorem 2.1 holds, then

\[ n^4 (y_1, \ldots, y_6) \sim AN(\Omega, \Gamma), \quad (2.4) \]

where

\[
\Gamma = \begin{pmatrix}
\Gamma_1 & 0 & 0 \\
0 & \Gamma_2 & 0 \\
0 & 0 & \Gamma_3
\end{pmatrix}
\]

\[
\Gamma_i = \int k_i^2(u) du \begin{pmatrix}
G_1(x_1) & 0 \\
0 & G_1(x_2)
\end{pmatrix}, \quad (i = 1, 2) \quad (2.5)
\]

\[
\gamma = \hat{F}(x_{\text{max}}) - \hat{F}(x_1) \hat{F}(x_2), \quad (2.7)
\]

where

\[
x_{\text{max}}' = (x_{1\text{max}}, x_{2\text{max}}) = (\max(x_{11}, x_{12}), \max(x_{21}, x_{22})).
\]
PROOF. Define, for $j = 1, \ldots, n$,

\[
V_{nj}(x) = \frac{1}{a_n^{3/2}} \left[ k_1 \left( \frac{x_1 - x_{1j}}{a_n} \right) k_2 \left( \frac{x_2 - x_{2j}}{a_n} \right) - \bar{E} k_1 \left( \frac{x_1 - x_{1j}}{a_n} \right) k_2 \left( \frac{x_2 - x_{2j}}{a_n} \right) \right],
\]

\[
W_{nj}(x) = \frac{1}{a_n^{3/2}} \left[ k_1 \left( \frac{x_1 - x_{1j}}{a_n} \right) k_2 \left( \frac{x_2 - x_{2j}}{a_n} \right) - \bar{E} k_1 \left( \frac{x_1 - x_{1j}}{a_n} \right) k_2 \left( \frac{x_2 - x_{2j}}{a_n} \right) \right],
\]

and

\[
Z_{nj}(x) = \frac{1}{a_n^{5/2}} \left[ k_1 \left( \frac{x_1 - x_{1j}}{a_n} \right) k_2 \left( \frac{x_2 - x_{2j}}{a_n} \right) - \bar{E} k_1 \left( \frac{x_1 - x_{1j}}{a_n} \right) k_2 \left( \frac{x_2 - x_{2j}}{a_n} \right) \right].
\]

Then it follows from (1.8) and (1.9) of Ahmad and Lin (1975) and (2.3)

that, for $\alpha = 1, 2$,

\[
y_{j} = n^{-1/2} \sum_{j=1}^{n} V_{nj}(x_{1j}), \quad y_{j+2} = n^{-1} \sum_{j=1}^{n} W_{nj}(x_{1j}), \quad y_{j+4} = n^{-1} \sum_{j=1}^{n} Z_{nj}(x_{1j}).
\]

The lemma will be proved if we can show that, for any real constants

c_i, \quad i = 1, \ldots, 6,

the linear combination

\[
S_{nn} = n^{1/2} \sum_{i=1}^{6} c_i y_{in}
\]

has an asymptotic normal distribution with mean 0 and variance $\zeta' \Gamma \zeta$ where

$\zeta' = (c_1, \ldots, c_6)$. Making use of the representation (2.9) it is clear that the linear combination (2.10) may be rewritten as

\[
S_{nn} = n^{-1/2} \sum_{j=1}^{n} T_{nj},
\]

where

\[
T_{nj} = \sum_{\alpha=1}^{2} \left[ c_\alpha V_{nj}(x_{1j}) + c_{\alpha+2} W_{nj}(x_{1j}) + c_{\alpha+4} Z_{nj}(x_{1j}) \right], \quad j = 1, \ldots, n.
\]

Since, for fixed $x_1$, $x_2$, and $n$, the random variables $T_{n1}, \ldots, T_{nn}$ are i.i.d., a sufficient condition under which $S_{nn}$ converges to a normal variate, in distribution, is given by (see, e.g., Loève (1963) page 277).
\[
\frac{n^{-2}E|T_{n1}|^3}{(\text{Var } S_{mn})^{3/2}} \to 0, \text{ as } n \to \infty.
\] (2.13)

To evaluate the asymptotic variance of \( S_{mn} \) using (2.10) it is necessary to obtain the asymptotic variances and covariances of \( Y_{in} \)'s. They turn out to be as follows:

(I) \( n \text{ Var } Y_{an} = G_1(x_a) \int k_1^2(u)du + O(a_n^2), \quad (\alpha=1,2); \)

\[ n \text{ Var } Y_{\alpha+2n} = G_2(x_a) \int k_2^2(u)du + O(a_n^2), \quad (\alpha=1,2); \]

(II) \( n \text{ Var } Y_{\alpha+4n} = \bar{f}(x_a)[1-\bar{F}(x_a)], \quad (\alpha=1,2); \)

(III) \( n \text{ cov}(Y_{1n}, Y_{2n}) = O(a_n), \quad n \text{ cov}(Y_{3n}, Y_{4n}) = O(a_n); \)

(IV) \( n \text{ cov}(Y_{5n}, Y_{6n}) = \bar{F}(x_{\max}) - \bar{F}(x_1) \bar{F}(x_2); \)

(V) \( n \text{ cov}(Y_{an}, Y_{\alpha+2n}) = O(a_n), \quad (\alpha=1,2); \)

\[ n \text{ cov}(Y_{1n}, Y_{4n}) = O(a_n); \quad n \text{ cov}(Y_{2n}, Y_{3n}) = O(a_n); \]

(VI) \( n \text{ cov}(Y_{an}, Y_{\beta n}) = O(a_n^2), \quad (\alpha=1,2,3,4; \quad \beta=5,6). \)

Results (I) and (II) may be obtained directly from Lemma 2.1 (ii) and Lemma 2.2 (ii) of Ahmad and Lin (1975). We will only sketch the proof of (III) here since the others may be established similarly. For (III) note that
\[
n \text{cov}(Y_{1n}, Y_{2n}) = n a_n \text{cov} [ \hat{G}_1(x_1), \hat{G}_1(x_2)] \\
= \frac{1}{a_n^3} \int \int k_1 (x_{11} - \frac{u_1}{a_n}) \int \int k_2 (t - \frac{u_2}{a_n}) \, dt \, k_1 (x_{12} - \frac{u_1}{a_n}) \\
\times \left[ \int \int k_2 \left( \frac{s - u_2}{a_n} \right) ds \right] f(u_1, u_2) \, du_1 \, du_2 - a_n E_1(x_1) E_1(x_2) \\
\int \int k_1 (y_1) \left[ \frac{x_{11} - \frac{x_{11}}{y_1}}{a_n} \right] k_2 (y_2) \, dy_2 \int \int k_1 \left( \frac{x_{12} - \frac{x_{11}}{y_1}}{a_n} \right) \left[ \frac{x_{12} - \frac{x_{11}}{y_1}}{a_n} \right] f(x_{11} - a_n y_1, u_2) \, dy_1 \, du_2 + O(a_n) \\
\approx \int \int k_1 (y_1) k_1 \left( \frac{x_{12} - \frac{x_{11}}{y_1}}{a_n} \right) f(x_{11} - a_n y_1, u_2) \, dy_1 \, du_2 + O(a_n) \\
= a_n \lim_{n \to \infty} \sup \left[ \frac{1}{a_n} k_1 \left( \frac{x_{12} - \frac{x_{11}}{y_1}}{a_n} \right) \right] \int \int k_1 (y_1) f(x_{11} - a_n y_1, u_2) \, dy_1 \, du_2 \\
+ O(a_n).
\] (2.14)

The third equality in (2.14) is obtained by the change of variables
\[y_1 = (x_{11} - u_1)/a_n, \ y_2 = (t - u_2)/a_n, \ z = (s - u_2)/a_n\] and \(u_2 = u_2\) with the
Jacobian of the transformation being \(-a_3\). The fourth expression is obtained
since \(\int \int k_2(y) \, dy + 1 \) or \(0\) according to whether \(u_2 > x_{2a}\) or \(u_2 < x_{2a}\),
respectively, \((a = 1, 2)\), as \(n \to \infty\). The integral in the final expression is
finite while \(\lim \sup a_n^{-1} k_1 [(x_{12} - x_{11})/a_n + y_1] = 0\) due to (1.5) of Ahmad
and Lin (1975). This establishes the first part of (III). The second part can
be proved similarly.
Now, summarizing results (I) through (VI), it is clear that

\[
\text{Var} S_{nn} = n\sum_{j=1}^{6} c_i^2 \text{Var} Y_{i1} + 2n\sum_{i<j} c_i c_j \text{cov}(Y_{i1}, Y_{j1}) + \gamma_2^2 \left[ c_{a+1}^2 G_2(x_a) \int k_2(u) du + c_{a+2}^2 G_2(x_a) \int k_2(u) du + c_{a+4}^2 F(x_a) [1 - \bar{F}(x_a)] \right] + 2c_2 c_6 \left[ \bar{F}(x_{\text{max}}) - \bar{F}(x_1) \bar{F}(x_2) \right],
\]

as \( n \to \infty \). Hence

\[
\lim_{n \to \infty} \text{Var} S_{nn} < \infty. \quad (2.15)
\]

It remains to show that

\[
n^{-1/2} \mathbb{E}|T_{nl}|^3 \to 0, \quad \text{as} \quad n \to \infty. \quad (2.16)
\]

Using (2.12) and the \( c_r \)-inequality of Loève (1963, page 155) repeatedly, it follows that

\[
\mathbb{E}|T_{nl}|^3 \leq 16 \sum_{a=1}^{2} \left| c_a \right|^3 \mathbb{E}|V_{nl}(x_a)|^3
\]

\[
+ 4 \left| c_{a+2} \right|^3 \mathbb{E}|V_{nl}(x_a)|^3 + 4 \left| c_{a+4} \right|^3 \mathbb{E}|Z_{nl}(x_a)|^3.
\]

Thus (2.16) will be satisfied if we can show that

\[
n^{-1/2} \mathbb{E}|V_{nl}(x_a)|^3 \to 0, \quad \text{as} \quad n \to \infty, \quad (2.17)
\]

\[
n^{-1/2} \mathbb{E}|W_{nl}(x_a)|^3 \to 0, \quad \text{as} \quad n \to \infty, \quad (2.18)
\]

and

\[
n^{-1/2} \mathbb{E}|Z_{nl}(x_a)|^3 \to 0, \quad \text{as} \quad n \to \infty, \quad (2.19)
\]

for \( a = 1, 2 \). Since \( |Z_{nl}(x_a)|^3 \leq 8 \), with probability one, (2.19) follows immediately. The proofs for (2.17) and (2.18) are alike. We will only show (2.17) for \( a = 1, 2 \). Using the \( c_r \)-inequality again, with the same arguments as those employed in deriving (2.14), it follows that

\[
n^{-1/2} \mathbb{E}|V_{nl}(x_a)|^3
\]
\[
\leq \frac{h}{(n \alpha_n)^{1/2}} \left\{ E_k \left\{ \frac{x_{1a} - X_{11}}{a_n} \right\} - \frac{x_{2a} - X_{21}}{a_n} \right\} + E_k \left\{ \frac{x_{1a} - X_{11}}{a_n} \right\} - \frac{x_{2a} - X_{21}}{a_n} \right\} \right\}^{3/2}
\]

\[
= \frac{h}{(n \alpha_n)^{1/2}} \left\{ \int_{\mathbb{R}^2} k_1(y_1) f(x_{1a} - a_n y_1, u_2) dy_1 du_2 + a_n^2 E \left( \frac{X_{1a}}{a_n} \right)^3 \right\}
\]

\[
= o((n \alpha_n)^{-1/2}),
\]

which converges to 0 as \( n \to \infty \), by the first part of assumption (ii) in Theorem 2.1. This establishes (2.17). Hence (2.16) is satisfied which, together with (2.15), implies (2.13).

**Lemma 2.3.** Assume that the conditions of Theorem 2.1 hold and \( q = 2 \). Define

\[
Z_{an} = a_n^{1/2} [ \hat{\mathcal{G}}_1(x_a) - \mathcal{G}_1(x_a) ],
\]

\[
Z_{a+n^2} = a_n^{1/2} [ \hat{\mathcal{G}}_2(x_a) - \mathcal{G}_2(x_a) ],
\]

\[
Z_{a+n^4} = \hat{\mathcal{F}}(x_a) - \mathcal{F}(x_a), \quad a=1, 2.
\]

Then

\[
n^{1/2} (Z_{1n}, ..., Z_{6n})' \sim AN(q, \Gamma),
\]

where \( \Gamma \) is given by (2.5).

**Proof.** In view of Lemma 2.2, it suffices to show that, for \( i = 1, ..., 6 \),

\[
n^{1/2} |Y_{in} - Z_{in}| \to 0, \text{ in probability, as } n \to \infty.
\]

Recalling (1.7) - (1.9) of Ahmad and Lin (1975) and after changing the order of integration, it follows that
\[
\hat{F}(\tilde{x}) - F(\tilde{x}) \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tilde{x} - a_n y) \, dy \, d\tilde{x} - \tilde{F}(\tilde{x}) \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_1(y_1)k_2(y_2) f(\tilde{x} - a_n y) - f(\tilde{x}) \, dy \, d\tilde{x} \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ f(\tilde{x} - a_n y) - f(\tilde{x}) \right] \, dy \, d\tilde{x} \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \hat{F}(\tilde{x} - a_n y) - \hat{F}(\tilde{x}) \right] k_1(y_1)k_2(y_2) \, dy \\
= \frac{a_n^2}{2} \frac{\delta^2 \hat{F}(\tilde{x})}{\delta \tilde{x}_i^2} \int_{-\infty}^{\infty} y_1^2 k_1(y_1) \, dy_i. \\
\]

(2.23)

The last expression of (2.23) is obtained by a Taylor expansion of \( \hat{F}(\tilde{x} - a_n y) \) about \( \tilde{x} \) and by assumptions (i) and (iii) of Theorem 2.1.

Similarly, it can be shown that, for \( i = 1, 2, \)

\[
\hat{G}_{i}(\tilde{x}) - G_{i}(\tilde{x}) = \frac{a_n^2}{2} \sum_{j=1}^{2} \frac{\delta^2 G_{i}(\tilde{x})}{\delta \tilde{x}_j^2} \int_{-\infty}^{\infty} y_j^2 k_j(y_j) \, dy_j.
\]

Thus

\[
n^{1/2}|y_i - z_i| = \begin{cases} 
0(\delta_n^{5/2}), & i = 1, 2, 3, 4, \\
0(\delta_n^{4/2}), & i = 5, 6,
\end{cases}
\]

which converges to 0, as \( n \to \infty \), by assumption (ii) of the theorem. This establishes (2.22). □

The following lemma is a trivial generalization of Theorem (iii) of Rao (1965), page 322. The proof is omitted.

**Lemma 2.4.** Let \( T_n = (T_{1n}, \ldots, T_{3q n})' \) be a \( 3q \)-dimensional statistic such that the asymptotic distribution of \( (n a_n^{1/2}(T_{1n} - \theta_1), \ldots, n a_n^{1/2}(T_{2q n} - \theta_2 q), n^{1/2}(T_{2q+1 n} - \theta_2 q+1), \ldots, n^{1/2}(T_{3q n} - \theta_3 q) \) is
3q-variate normal with mean vector \( \theta \) and covariance matrix \( H \), where 
\[
\theta_i \neq 0 \text{ for } i=2q+1, \ldots, 3q, \text{ and } n \to \infty \text{ as } n \to \infty. \quad \text{Let } h_{2i-1}(\theta_1, \ldots, \theta_{3q}) = \theta_i / \theta_{2q+i} \text{ and } h_{2i}(\theta_1, \ldots, \theta_{3q}) = \theta_{q+i} / \theta_{2q+i}, i=1, \ldots, q. \quad \text{Then the asymptotic distribution of } (n^{-1/2} h_1(T_n) - h_1(\theta), \ldots, h_{2q}(T_n) - h_{2q}(\theta))^t \text{ is 2q-variate normal with mean vector } \theta \text{ and covariance matrix } \Sigma H H^t, \text{ where } \theta = (\theta_1, \ldots, \theta_{3q}) \text{ and } H = (\partial h_i / \partial \theta_j). \text{ The rank of the distribution is that of } \Sigma H H^t. \]

**Proof of Theorem 2.1.** Using Lemmas 2.3 and 2.4 with \( q = 2 \), we have 
\[
\theta' = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6), \text{ where } \theta_\alpha = g_1(x_\alpha), \theta_{\alpha+2} = g_2(x_\alpha), \theta_{\alpha+4} = \bar{F}(x_\alpha) > 0, \alpha = 1, 2.
\]

Define, as in Lemma 2.4,
\[
h_1(\theta) = \theta_1 / \theta_5, \quad h_2(\theta) = \theta_3 / \theta_5, \quad h_3(\theta) = \theta_2 / \theta_6, \quad h_4(\theta) = \theta_4 / \theta_6.
\]

Then
\[
H = \begin{bmatrix}
\frac{\partial h_1}{\partial \theta_1} \\
\frac{\partial h_1}{\partial \theta_3} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[F(x_1)]^{-1} & 0 & 0 & 0 & -r_1(x_1)[F(x_1)]^{-1} & 0 \\
0 & [F(x_1)]^{-1} & 0 & -r_2(x_1)[F(x_1)]^{-1} & 0 \\
0 & [F(x_2)]^{-1} & 0 & 0 & -r_1(x_2)[F(x_2)]^{-1} \\
0 & 0 & [F(x_2)]^{-1} & 0 & -r_2(x_2)[F(x_2)]^{-1} \\
\end{bmatrix}, \quad (2.24)
\]
Now the theorem follows directly from Lemma 2.4 with $HH' = \Sigma$, as given by (2.1). □

It should be remarked that the assumption $\Pi_{i=1}^{2} \Pi_{\alpha \neq \beta = 1}^{q} (x_{1 \alpha} - x_{i \beta}) \neq 0$ seems strong and undesirable. However, the theorem still holds with $\Sigma$ slightly modified when the $i$th component of some $x_\alpha$'s is identical. For example, if $x_{11} = x_{12}$ and $x_{21} \neq x_{22}$ then the only changes needed to be made are entries in the submatrix $\Gamma_i (i = 1, 2)$ given by (2.6). The modified matrix is then

$$\Gamma_i^* = \int k_i^2(u) du \begin{pmatrix} G_i(x_1) & G_i(x_{11}, x_{2\text{max}}) \\ G_i(x_{11}, x_{2\text{max}}) & G_i(x_2) \end{pmatrix}.$$ 

Now, define $\Gamma^*$ as in (2.5) with $\Gamma_i^*$ replacing $\Gamma_i$ (i=1,2). Then Theorem 2.1 holds with covariance matrix $(n a_n)^{-1/2} \Sigma^*$, where $\Sigma^* = HH'$. 
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