Bounds for the Average Sample Number Function for a Binomial Sequential Test

by

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FSU Statistics Report M381

June, 1976

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ABSTRACT

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Recently, Billard (1977) developed a partial sequential procedure for comparing a null hypothesis against a two-sided alternative hypothesis when the parameter under test is that of the binomial distribution. In that paper, approximations to the operating characteristic and average sample number function were derived. In this note, bounds to the average sample number function are derived. Using numerical results a comparison of the approximations, bounds and empirical values is made.
Introduction.

Recently, Billard (1977) described a partial sequential procedure for testing a simple null hypothesis concerning the parameter of a binomial distribution against a two-sided alternative hypothesis. The test procedure was defined in terms of seven geometrical parameters. Further, expressions for the operating characteristic (O.C.) function and the average sample number (A.S.N.) function were derived. Values of the geometrical parameters were those which minimized the A.S.N. function at a specified value of the parameter subject to certain constraints on the O.C. function. The subsequent values for the A.S.N. function showed that for parameter values midway between the hypothesised values, this procedure has lower values than does the corresponding Sobel and Wald (1949) procedure.

Unfortunately, in the derivation of the formulae for the A.S.N. function, it was implicitly assumed that sampling ceased when the sample path fell exactly on a boundary line; whereas, in general, there could be overshooting of the boundaries. Hence, the formulae are only approximations. In view of the wide applicability of the procedure, it is of interest to know just how good these approximations are. Therefore, the purpose of this note is to remove this deficiency of the earlier work by providing bounds for the A.S.N. function. These bounds are derived in Section 3. We begin with a brief description of the procedure in Section 2.

2. The partial sequential procedure.

Suppose the Bernoulli distributed random variable takes values 1, or 0 with probability p, or q = 1-p. Then, X the number of 1's in a random sample of
size \( n \) follows a binomial distribution with parameters \( n \) and \( p \). Suppose we wish to test the simple null hypothesis \( H_0: p = p_0 \) against the two-sided alternative hypothesis \( H_a: p \neq p_0 \). In practice, this composite hypothesis is replaced by the two simple hypotheses \( H_i: p = p_i, i = \pm 1 \) with \( p_{-1} < p_0 < p_1 \).

It is convenient to work with the random variable \( Z \) defined as the excess of 1's over 0's, that is,

\[
Z = 2X - n.
\]

Hence, the probability function of \( Z \) is

\[
f(z) = f(z; p, n) = \begin{cases} 
\binom{n}{(n+z)/2} \frac{n+z}{p^z (1-p)^{n-z}} & 
\text{for } z = \begin{cases} 
\pm 1, \pm 3, \ldots, \pm n, n \text{ odd} \\
0, \pm 2, \ldots, \pm n, n \text{ even}
\end{cases}
\end{cases}
\]

The test procedure is described as follows. Assume the geometrical parameters \((n_0, a, b, \tan \psi, a', b', \tan \psi')\) are determined. First take \( n_0 \) observations. If \( z \geq a \), accept \( H_1 \), if \( b' \leq z \leq b \), accept \( H_0 \) and if \( z \leq a' \), accept \( H_{-1} \). In these cases, no further observations are necessary. If however \( b < z < a \), further observations are taken until one of the boundaries

\[
z = a + (n - n_0) \tan \psi, \quad \text{(1a)}
\]

or

\[
z = b + (n - n_0) \tan \psi, \quad \text{(1b)}
\]
is crossed at which time the hypothesis \( H_1 \), or \( H_0 \), respectively, is accepted. Similarly, if \( a' < z < b' \), further observations are taken until one of the boundaries

\[
z = a' + (n - n_0) \tan \psi',
\]

(2a)

or

\[
z = b' + (n - n_0) \tan \psi'
\]

(2b)
is crossed with the acceptance of the hypothesis \( H_{-1} \), or \( H_0 \), respectively.

3. **The Average Sample Number Function.**

The average sample number (A.S.N.) function, \( E_n(p) \) is defined as the expected number of observations required to reach a decision when \( p \) is the true parameter value. Clearly, we take at least \( n_0 \) observations. If after \( n_0 \) observations have been taken, \( b < z < a \), then let \( N_1(p) \) be the expected number of additional observations required to reach a decision; whilst if after \( n_0 \) observations, \( a' < z < b' \), then let \( N_{-1}(p) \) be the expected number of additional observations required to reach a decision; otherwise, a terminating decision is made immediately with no further observations being required. Hence, we have

\[
E_n(p) = n_0 + N_1(p) + N_{-1}(p)
\]

(3)

From Billard (1977) we see that

\[
N_1(p) = \sum_{z \in A} f(z; p, n) \eta(z), A \neq \phi
\]

(4a)

\[
= 0, \quad A = \phi
\]

(4b)
where \( A \) is the set \( A = [-n_0, n_0] \cap [b, a] \) and where \( \eta(z) \) is the expected number of observations required for a random walk starting at \( z \) and operating between the boundaries (1a) and (1b) to be absorbed by one of these boundaries, that is,

\[
\eta(z) = \left\{ \frac{(a-b) \exp(-hz) - a \exp(-hb) + b \exp(-ha) - z}{\exp(-ha) - \exp(-hb)} \right\} / (2p-1 - \tan \psi)
\]

, \( h \neq 0 \) \hspace{1cm} (5a)

\[
= (a - z)(z - b)/(4pq),
\]

, \( h = 0 \) \hspace{1cm} (5b)

where \( h \) is the non-zero solution of the equation

\[
\exp(h \tan \psi)(p \exp(-h) + q \exp(h)) = 1.
\]

The term \( N_{-1}(p) \) is obtained similarly.

In the derivation of the above expressions, it was assumed that there was no excess over the boundaries. However, in practice there is some overshooting of the boundaries so that these expressions are only approximations. In particular, let \( \eta^*(z) \) be the "true value" of \( \eta(z) \). We then write for the true value of \( E_n(p) \)

\[
E_n^*(p) = n_0 + N_1^*(p) + N_{-1}^*(p)
\]

(6)

where \( N_i^*(p), i = \pm 1 \) are the true values of \( N_i(p) \). Clearly, the exact equation corresponding to equation (4) is

\[
N_1^*(p) = \sum_{z \in A} f(z; p, n) \eta^*(z), \ A \neq \phi
\]

(7a)

\[
= 0, \hspace{1cm} A = \phi
\]

(7b)
Therefore, if bounds for $\eta^*(z)$ can be found, the bounds for $N^*_1(p)$ will follow from equation (7).

The bounds for $\eta^*(z)$ may be determined by a method similar to that used by Wald (1947, Appendix A3) when he derived bounds for his sequential probability ratio test. After making the necessary transformation required to transform the sequential probability ratio test procedure to that of the present case, these bounds are seen to be, when $2p - 1 > \tan \psi$,

$$\eta(z) + \pi(z)\xi_1 / (2p-1 - \tan \psi) \leq \eta^*(z) \leq \eta(z) + (1-\pi(z))\xi_1 / (2p-1 - \tan \psi); \quad (8a)$$

when $2p - 1 < \tan \psi$,

$$\eta(z) + (1-\pi(z))\xi_1 / (2p-1-\tan\psi) \leq \eta^*(z) \leq \eta(z) + \pi(z)\xi_1 / (2p-1-\tan\psi); \quad (8b)$$

and when $2p - 1 = \tan \psi$,

$$\eta(z) \leq \eta^*(z) \leq \eta(z) + [\pi(z)(2b-z)\xi_1 + \xi_1] + (1-\pi(z))[2(a-z)\xi + \xi_1] / \{4p(1-p)\}, \quad (8c)$$

where $\pi(z)$ is the probability that a random walk starting at the point $z$ and operating between the boundaries (1a) and (1b), is ultimately absorbed by the boundary (1b), that is,

$$\pi(z) = [\exp(-ha) - \exp(-hz)] / [\exp(-ha) - \exp(-hb)], \quad h \neq 0 \quad (9a)$$

$$= (a - z) / (a - b), \quad h = 0; \quad (9b)$$

and where
\[ \xi = \xi(p, \psi) = \max_r E\{z \cdot \tan\psi - r \mid z \cdot \tan\psi \geq r\} \] (10a)

\[ \xi_1 = \xi_1(p, \psi) = \min_r E\{z \cdot \tan\psi + r \mid z \cdot \tan\psi \leq 0\} \] (10b)

\[ \zeta = \zeta(p, \psi) = \max_r E\{(z \cdot \tan\psi - r)^2 \mid z \cdot \tan\psi \geq r\} \] (10c)

\[ \zeta_1 = \zeta_1(p, \psi) = \max_r E\{(z \cdot \tan\psi + r)^2 \mid z \cdot \tan\psi + r \leq 0\} \] (10d)

with \( r \geq 0 \).

These terms of the equation (10) are readily determined as follows. Consider \( \xi \). The variable \( z \) can take the values 1 and \(-1\) only with probabilities \( p \) and \( 1-p \), respectively. We have

\[ E(z \cdot \tan\psi - r \mid z \cdot \tan\psi \geq r) = \sum_{z \cdot \tan\psi \geq r} (z \cdot \tan\psi - r)f(z)/\sum_{z \cdot \tan\psi \geq r} f(z) . \]

When \( z \) takes the value \(-1\), \( z \cdot \tan\psi \leq 0 \) and thus the conditions \( z \cdot \tan\psi \geq r \) and \( r \geq 0 \) are contradictory. There is no such contradiction when \( z \) assumes the value 1. Therefore, \( z \) can equal 1 only. Thus, we have

\[ E(z \cdot \tan\psi - r \mid z \cdot \tan\psi \geq r) = 1 \cdot \tan\psi - r . \]

Since the maximum is reached when \( r = 0 \), this gives

\[ \xi = 1 \cdot \tan\psi \] (11a)

In order to calculate \( \xi_1 \), it is remembered that

\[ \min_r E\{z \cdot \tan\psi + r \mid z \cdot \tan\psi + r \leq 0\} = -\max_r \{-z \cdot \tan\psi - r \mid -z \cdot \tan\psi - r \geq 0\} \]
As \( z \) can take only the value \(-1\), we have

\[
E\{\max\{-z+\tan\psi-r|-z+\tan\psi-r \geq 0\}\} = 1 + \tan\psi + r
\]

Hence, on setting \( r = 0 \) to obtain the maximum, we find that

\[
\xi_1 = -(1+\tan\psi) \ .
\]  \hspace{1cm} (11b)

Similarly, the remaining terms \( \zeta \) and \( \xi_1 \) are given as

\[
\zeta = (1-\tan\psi)^2
\]  \hspace{1cm} (11c)

\[
\xi_1 = (1+\tan\psi)^2
\]  \hspace{1cm} (11d)

Since the bounds for \( \eta_\ast(z) \) given in the equation (8) are now completely determined, it is possible to find the bounds for \( N_1^\ast(p) \). Consider the case where \( 2p-1 > \tan \psi \). The lower bound for \( N_1^\ast(p) \) is given by

\[
1.b. \ N_1^\ast(p) = \sum_{z \in A} f(z;p,n_0)[\eta(z)+\pi(z)\xi_1/(2p-1-\tan\psi)]
\]

That is,

\[
1.b. \ N_1^\ast(p) = N_1(p) - L_1(p)(1+\tan\psi)/(2p-1-\tan\psi)
\]  \hspace{1cm} (12a)

where \( L_1(p) \) is the probability the boundary line first crossed is (1b), that is,

\[
L_1(p) = \sum_{z \in A} f(z;p,n_0)\pi(z) \ , \quad A \neq \emptyset
\]  \hspace{1cm} (13a)

\[
= 0 \ , \quad A = \emptyset
\]  \hspace{1cm} (13b)
Similarly, the upper bound for \( N_1^*(p) \), when \( 2p-1 > \tan \psi \), is

\[
\text{u.b. } N_1^*(p) = N_1(p) - \left( \frac{1-\tan \psi}{2p-1-\tan \psi} \right) \left\{ L_1(p) - \sum_{z \in A} f(z;p,n_o) \right\} \tag{14a}
\]

The bounds for \( N_1^*(p) \) obtained from the equations (8b) and (8c) are determined in a similar way to give, when \( 2p-1 < \tan \psi \),

\[
\text{1.b. } N_1^*(p) = N_1(p) - \left( \frac{1-\tan \psi}{2p-1-\tan \psi} \right) \left\{ L_1(p) - \sum_{z \in A} f(z;p,n_o) \right\} \tag{12b}
\]

\[
\text{u.b. } N_1^*(p) = N_1(p) - L_1(p)(1+\tan \psi)/(2p-1-\tan \psi) \tag{14b}
\]

and when \( 2p-1 = \tan \psi \),

\[
\text{1.b. } N_1^*(p) = N_1(p) \tag{12c}
\]

\[
\text{u.b. } N_1^*(p) = N_1(p) + \{ 4 \tan \psi L_1(p) + 2(a-b)\tan \psi - a-b \} L_1(p)
\]

\[
+ (2a+1-\tan \psi)(1-\tan \psi) \sum_{z \in A} f(z;p,n_o)
\]

\[
+ 2 \sum_{z \in A} \{ 2\pi(z)-1+\tan \psi \} f(z;p,n_o) \} /[4p(1-p)] \tag{14c}
\]

The bounds for the term \( N_{-1}^*(p) \) are derived analogously. Hence, by addition of the relevant parts, the lower bound for \( E_n^*(p) \) and the upper bound for \( E_n(p) \), respectively, can be obtained.

4. **An Example.**

Consider the symmetrical case in which \( p_o = .5, p_{-1} = 1-p_1 = .5 \), and the Type I and Type II errors are \( \alpha = \beta_1 = \beta_2 = .05 \). The geometrical parameters
(n₀, a, b, ψ, a', b', ψ') were determined in Billard (1977) to be those values
which minimized the A.S.N. function under the null hypothesis when the following
constraints on the O.C. function, L(p) were satisfied:

\[ L(p_o) \geq 1 - \alpha, \quad \text{and} \quad L(p_i) \leq \beta_i, \quad i = \pm 1. \]

Thus, in the present example, the geometrical parameters take the values n₀ = 12.019,
a = 9.203, b = .228, tan ψ = .332 with a' = -a, b' = -b and ψ' = -ψ.

Table 1 provides the values of the approximate A.S.N. function as given by
equation (4), and the upper and lower bounds for \( \frac{E^*(p)}{n} \) using the relevant terms
of Section 3. It is interesting to note that the approximate values are very
close to the lower bounds for the exact values. Also shown are the A.S.N. values
obtained empirically when for each \( p \) value 1000 trials were conducted. A
comparison of these results show that except for \( p > p_i \), the empirical value is
less than the corresponding theoretical upper bound value. The differences between
these values is quite large for values of \( p \) midway between the values specified
under the hypotheses. These results suggest that the upper bound may in fact be
quite a "wide" bound and that the lower bound is quite "close".

References.

Billard, L. (1977): Sequential procedures for testing a two-sided alternative
hypothesis concerning the parameter of the binomial distribution. J. Amer.

Table 1

A.S.N. Function values for $p_0 = .5$, $p_1 = .8$, $p_1 = .2$, $\alpha = \beta_1 = \beta_{-1} = .05$

<table>
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<th>Approximate</th>
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<th>Upper bound</th>
<th>Empirical</th>
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