THE ESSENTIAL COMPLETENESS OF MONOTONE DECISION PROCEDURES FOR STRICT MONOTONE LIKELIHOOD RATIO

BY

Glen Meeden

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The Florida State University
Department of Statistics
Tallahassee, Florida 32306
ABSTRACT

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Suppose a random variable which has a family of densities indexed by a real parameter has strict monotone likelihood ratio. Let the action space $A$ and the loss function $L(\theta, a)$ specify a decision problem based on the random variable. For each pair of actions $a_1 \in A$ and $a_2 \in A$ with $a_1 < a_2$ assume that there exists a real number $\theta^* = \theta^*(a_1, a_2)$ such that $L(\theta, a_1) - L(\theta, a_2) < 0$ or $> 0$ as $\theta < \theta^*$ or $\theta > \theta^*$. Under an additional assumption it is shown that the class of monotone decision procedures forms an essentially complete class for the decision problem.


Key words and phrases: essentially complete class, monotone likelihood ratio, monotone procedure, Bayes procedure.
1. INTRODUCTION

Let $X$ be a real valued random variable with $\{f_\theta : \theta \in \Theta\}$ a family of possible densities for $X$ with respect to the $\sigma$-finite measure $\mu$, where $\Theta$ is a subset of the real line. Assume $f_\theta(x)$ has the strict monotone likelihood ratio property and for each $\theta$, $f_\theta(x) > 0$ for $x$ in the spectrum of $\mu$. Consider the decision problem specified by the action space $A$, a subset of the real line which contains at least two points and the loss function $L(\theta, a)$ for $\theta \in \Theta$ and $a \in A$. Under some additional assumptions on the loss function it was shown in Karlin and Rubbn (1956) that the class of monotone decision procedures is essentially complete. For example in the case when $A$ is finite, that is $A = \{a_1, \ldots, a_k\}$ where $a_i < a_j$ when $i < j$. Karlin and Rubin assumed that there exist numbers $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{k-1}$ such that the loss functions satisfies $L(\theta, a_1) - L(\theta, a_{i+1}) \leq 0$ or $\geq 0$ as $\theta < \theta_i$ or $\theta > \theta_i$ for $i = 1, 2, \ldots, k-1$. When $A$ is infinite they assumed that the loss function satisfied the following properties.

i) For each $\theta$, $L(\theta, a)$ attains its minimum as a function of $a$ at a point $q(\theta)$ which is a monotone increasing function of $\theta$.

ii) For each $\theta$, $L(\theta, a)$ as a function of $a$ increases away from that minimum.

Recently Brown, Cohen and Strawderman (1976) showed that the class of monotone procedures is a complete class under the assumption that the loss function $L(\theta, \cdot)$ is bowl shaped for each $\theta$.

The purpose of this note is to give a simple proof which shows for many loss functions not included in the above papers that the class of monotone procedures is essentially complete. Here it is assumed that for each pair of actions $a_1 \in A$ and $a_2 \in A$ with $a_1 < a_2$ there exists a real number $\theta^* = \theta^* (a_1, a_2)$ (depending on $a_1$ and $a_2$) such that
\[ L(\theta, a_1) - L(\theta, a_2) < 0 \text{ for } \theta < \theta^* \]
\[ > 0 \text{ for } \theta > \theta^*. \]  

(1)

There exist many loss functions which satisfy (1) but not the assumptions of the previous theory. See example 2.

2. The result. Before statement of the theorem some additional notation needs to be introduced. Let \( G \) denote a prior distribution over the parameter space \( \Theta \). For each such \( G \) it is assumed that a corresponding Bayes decision procedure exists. Let \( B \) be the class of decision procedures which are Bayes against a prior distribution which does not concentrate all its mass at one point.

Theorem Suppose for the decision problem defined above with loss function \( L \) satisfying (1) that the closure of \( B \), under regular convergence, is an essentially complete class of decision procedures; then the class of monotone decision, procedures is essentially complete.

Proof. Since under regular convergence the limit of a sequence of monotone procedures is monotone it suffices to show that any procedure which is Bayes against a prior distribution which does not concentrate all its mass at one point is a monotone procedure.

Let \( G \) be a fixed prior distribution which does not concentrate all its mass at one point. For each \( a \in A \) let \( \Lambda_a = \{ v : a \text{ is at least as good as every other } a' \text{ given that } X = x \text{ under } G \} \). Note \( \Lambda_a \) may be the empty set. Let \( a^* \in A \) and \( a_0 \in A \) be fixed and suppose \( a^* < a_0 \). By (1) and Corollary 2 of Karlin and Rubin (1956) there exists an \( x^* \) such that given \( X = x \).
\( a^* \) is preferred to \( a_0 \) \hspace{1cm} \text{when } x < x^* \\
\( a^* \) and \( a_0 \) are equivalent \hspace{1cm} \text{when } x = x^* \\
\( a_0 \) is preferred to \( a^* \) \hspace{1cm} \text{when } x > x^* \\

Hence when \( \Lambda_{a^*} \) and \( \Lambda_{a_0} \) are non-empty it follows that \( \Lambda_{a^*} \) lies to the left of \( \Lambda_{a_1} \) and they are disjoint or have only one point in common. Now let \\
\( A_x = \{ a: a \text{ is at least as good as every other } a' \text{ given } X = x \text{ under } G \} \). For \\
x_1 < x_2 \text{ it follows that } A_{x_1} \text{ lies to the left of } A_{x_2} \text{ with at most one point in } \\
A_{x_1} \cap A_{x_2} \text{. Since given } X = x \text{ a Bayes rule against } G \text{ must concentrate on the } \\
\text{set } A_x \text{ it follows that a Bayes rule against } G \text{ is monotone. This completes the } \\
\text{proof of the theorem.} \\

Now it will usually be the case that the closure of \( B \) under regular 
\text{convergence will be an essentially complete class. See Le Cam (1955) and Wald} 
(1950) for a discussion of this point. Using this result and the fact that a 
\text{Bayes procedure must be monotone, this note gives a simple proof of the} 
\text{essential completeness of the class of monotone procedures. This} 
\text{crucial property of the Bayes rules is not used in the other proofs. These} 
\text{other proofs are constructive in nature however and given a non-monotone} 
\text{procedure they exhibit a dominating monotone procedure which is independent} 
\text{of loss function over a large class of possible loss functions.} \\

It might be hoped that condition (1) with the strict inequalities replaced 
\text{by inequalities would be a necessary condition for the essential completeness} 
\text{of the class of monotone procedures. Example 1 shows that this is not the} 
\text{case.}
Example 1. Let $X$ be Bernoulli ($\theta$) with $\Theta = \{1/4, 1/2, 3/4\}$ and let $A = \{a_0, a_1, a_2\}$ where $a_0 < a_1 < a_2$. Let $L$ be given as follows: $L(\theta, a_0) = 0$ or 1 as $\theta = 1/4$ or $\theta \neq 1/4$, $L(\theta, a_1) = 0$ or 2 as $\theta = 1/2$ or $\theta \neq 1/2$ and $L(\theta, a_2) = 0$ or 1 as $\theta = 3/4$ or $\theta \neq 3/4$. Note that $L$ does not satisfy condition (1). It is easily checked that the class of monotone procedures is essentially complete for this problem.

It is easy to give many examples where the loss function satisfies condition (1) but not the assumptions of the earlier theory. The next example is one such case.

Example 2. Let $X$ be Binomial ($n, \theta$) where $n$ is known and $\Theta = [0, 1]$ and $A = [0, 1]$. Let $L(\theta, 1/2) = (1/2 - \theta)^2$ and for $a \neq 1/2$ let $L(\theta, a) = (a - \theta)^2 + c^2$ where $0 < c < 1/2$. We can imagine such a loss function arising in the following way. Suppose we are observing a Bernoulli process which is producing items of two types. If $\theta = 1/2$ we wish the process to continue. If $\theta \neq 1/2$ we wish to stop the process and adjust it so that each of the two types becomes equally likely. However when the process is stopped and re-adjusted there is a cost incurred. In the above loss function this cost is represented by $c^2$ which we incur when we estimate $\theta$ to be unequal to $1/2$. Note that although the loss function is not bowl shaped it does satisfy (1) and hence by the theorem the monotone procedures form an essentially complete class. Using the technique of Blyth (1951) it is easy to show that

$$
\delta(x) = \frac{x}{n} \quad \text{when} \quad |\frac{x}{n} - 1/2| > c
$$

$$
= 1/2 \quad \text{when} \quad |\frac{x}{n} - 1/2| \leq c
$$
is an admissible monotone decision procedure.

Recently Meeden and Arnold (1976) have considered the problem of estimating a normal mean when the loss function includes a complexity cost similar to that of example 2. Again the class of monotone procedures is essential complete even though the loss function is not bowl shaped.

REFERENCES


