Positive Dependence of the Bivariate and Trivariate Absolute Normal, t, $\chi^2$ and F Distributions

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ABSTRACT

It is shown that the bivariate density of the absolute normal distribution is totally positive of order 2. Necessary and sufficient conditions are given for the trivariate density of the absolute normal distribution to be totally positive of order 2 in pairs of arguments. These results are then used to show that certain generalized bivariate and trivariate t, $\chi^2$ and F random variables are associated.

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Key Words: Total Positivity, positive quadrant dependence, conditionally increasing in sequence, association, multivariate t distribution, multivariate F distribution, multivariate normal distribution.
1. Introduction. Motivated by needs in simultaneous inference, numerous authors have established inequalities for joint probabilities in terms of marginal probabilities. Typically in these inequalities the underlying random variables are jointly normal and most of the proofs are of an analytic nature. In this paper we obtain stronger dependence results in the bivariate and trivariate cases by using certain notions of multivariate dependence.

Suppose \((X_1, \ldots, X_p)' \sim N_p(\mu, \Sigma)\), where \(N_p(\mu, \Sigma)\) denotes the law of a \(p\)-variate normal random vector with mean \(\mu\) and nonsingular covariance matrix \(\Sigma = \{\rho_{ij}^\sigma\}_{ij}\). For \(i = 1, \ldots, n\), let \(Z_i = (Z_{1i}, \ldots, Z_{pi})' \sim N_p(\mu, \Sigma_i)\), where \(Z_1, \ldots, Z_n\) are independent random variables. Further, for \(i = 1, \ldots, p\), let \(T_{i1}, \ldots, T_{iq_i}^1\) be independently and identically distributed according to \(N(0, \sigma_{T_i}^2)\).

Now assume \((X_1, \ldots, X_p)', \{Z_i\}, \{T_{1k}\}_{k=1}^{q_1}, \ldots, \{T_{pk}\}_{k=1}^{q_p}\) are all mutually independent sets of random variables. Define

\[
S_k^2 = \sum_{\ell=1}^n Z_{k\ell}^2, \quad k = 1, \ldots, p,
\]

and

\[
S_k^* = \sum_{\ell=1}^{q_k} T_{k\ell}^2, \quad k = 1, \ldots, p.
\]

**DEFINITION.** (Lehmann [1966]). The random variables \(X_1, \ldots, X_p\) are positively quadrant dependent (PQD) if

\[
P[ \cap_{i=1}^n (X_i \leq x_i) ] \geq \mathbb{P}[X_1 \leq x_1], \text{ for all real numbers } x_1, \ldots, x_n.
\]
In the case $p = 2$, i.e., the bivariate case, Khatri [1967] showed that $|X_1|$, $|X_2|$ are PQD and that $S_1^2$, $S_2^2$ are PQD. Šidák [1967], [1971] proved that $|X_1|/S_1$, $|X_2|/S_2$ are PQD. Halperin [1967] obtained the slightly stronger result that $|X_1|/(S_1^2 + S_1^2)^{1/2}$, $|X_2|/(S_2^2 + S_2^2)^{1/2}$ are PQD. Dunn [1958] had previously obtained similar results.

For the $p = 3$ case, similar results hold. Khatri showed that $|X_1|$, $|X_2|$, $|X_3|$ are PQD if $E$ is of the form $\{\beta_i \beta_j\}$. Khatri also showed under this condition that $S_1^2$, $S_2^2$, $S_3^2$ are PQD. Šidák [1971] proved that $|X_1|/S_1$, ..., $|X_3|/S_3$ are PQD if the correlation between $X_i$ and $X_j$ is of the form $\lambda_i \lambda_j \rho_{ij}$ $(i, j = 1, \ldots, 3; i \neq j)$, $|\lambda_i| \leq 1$ $(i = 1, \ldots, 3)$, $\{\rho_{ij}\}$ is any fixed correlation matrix; and if the correlation between $Z_{\ell i}$ and $Z_{ki}$ is of the form $\tau_{\ell i} \tau_{ki}$ $(\ell, k = 1, \ldots, 3; \ell \neq k$, $i = 1, \ldots, n)$ where $|\tau_{\ell i}| < 1$ $(\ell = 1, \ldots, 3; i = 1, \ldots, n)$.

Some results have been obtained for higher dimensions by the above authors.

Note that up to constants $(X_1/S_1, \ldots, X_p/S_1)'$ is a multivariate Student's t-random vector (considered in the bivariate case by Siddiqui [1967]); $X_1/S_1, \ldots, X_p/S_p$ is a generalized multivariate Student's t-random vector (Šidák [1971]); and $S_1^2, \ldots, S_p^2$ is a multivariate $\chi^2$ random vector (Krishnamoorthy and Parthasarathy [1951], Jensen [1970]).

The preceding results were derived basically independently of each other and each proof involved analytic techniques specific to that result. In this paper we obtain the
followig basic results: (a) the density of $|X_1|, |X_2|$ is totally positive of order 2 (b) a necessary and sufficient condition that $|X_1|, |X_2|, |X_3|$ be totally positive of order 2 in pairs of arguments is that $\sum_{j=1}^p \text{sgn}(\lambda_{ij}) \leq 0$, where $A = (\lambda_{ij}) = \mathbf{E}^{-1}$; (c) $S^2_1 + S^2_2, S^2_2 + S^2_3, S^2_3 + S^2_4$, are associated random variables and that $|X_1|/(S^2_1 + S^2_2), |X_2|/(S^2_2 + S^2_3), |X_3|/(S^2_3 + S^2_4)$ are associated random variables. (The same results hold for $p = 2$.)

2. Total Positivity of the Bivariate Absolute Normal. We employ the following definitions and implications.

**DEFINITION.** (Marlin [1960]). A function $f: \mathbb{R}^2 \to [0, \infty)$ is totally positive of order 2 (TP$_2$) if the second order determinant $\det(f(x_1, y_1))$ is nonnegative for each choice $x_1 < x_2, y_1 < y_2$.

**DEFINITION.** (Searle, Proschan and Walkup [1967]). The random variables $X_1, \ldots, X_n$ are associated if $\text{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) > 0$ for all nondecreasing functions $f, g$.

**DEFINITION.** (Barlow and Proschan [1974]). Let $a$ be an integer exceeding 2. A function $X: \mathbb{R}^a \to [0, \infty)$ is said to be totally positive of order 2 in pairs (TP$_2$ in pairs) if for any pair of arguments $x_a, x_b, f(x_1, \ldots, x_a, \ldots, x_b, \ldots, x_n)$, viewed as a function of $x_a, x_b$ with remaining arguments fixed, is TP$_2$.

**DEFINITION.** (Barlow and Proschan [1974]). The random variables $X_1, \ldots, X_n$ are conditionally increasing in sequence if for $i = 1, \ldots, n$ $F(X_i > x | X_{i-1} = x_{i-1}, \ldots, X_1 = x_1)$ is increasing in $x_1, \ldots, x_{i-1}$.
**Lemma 2.1.** Let the random variables $X_1, \ldots, X_a$ have joint density $f_{X_1, \ldots, X_a}(x_1, \ldots, x_a)$. Then the following implications hold: $f_{X_1, \ldots, X_a}(x_1, \ldots, x_a)$ is TP$_2$ in pairs $\Rightarrow X_1, \ldots, X_a$ are conditionally increasing in sequence $\Rightarrow X_1, \ldots, X_a$ are associated $\Rightarrow X_1, \ldots, X_a$ are PQR.

A proof of Lemma 2.1 can be found in Chapter 5 of Barlow and Proschan [1974], and a more detailed chain of implications can be found in Esary, Proschan and Walkup [1967].

In order to obtain our main bivariate result, we require the following lemma.

**Lemma 2.2.** Let $f(u, v) = k_1(u) k_2(v) g(uv)$ for $u \geq 0, v \geq 0$ and $f(u, v) = 0$, otherwise. Assume $k_1 \geq 0, k_2 \geq 0$, and $g \geq 0$. If $g$ is nondecreasing and $\ln g$ is convex, then $f$ is TP$_2$.

**Proof:** Since $f = 0$ for $u < 0$ or $v < 0$, it suffices to consider $0 \leq u_1 < u_2, 0 \leq v_1 < v_2$, in showing $\det(f(u_i, v_j)) \geq 0$. Note that $\det(f(u_i, v_j)) = \prod_{i=1}^{2} (k_1(u_i)k_2(v_i)) \det(g(u_i v_j))$, and thus we need only to show that $\det(g(u_i v_j)) \geq 0$. Define $t_1 = u_1 v_1, t_2 = u_1 v_2, t_2 + \Lambda_2 = u_2 v_2$, so that $0 \leq \Lambda_1 < \Lambda_2$. Observe that

$$
\det(g(u_i v_j)) = g(t_1)g(t_2 + \Lambda_2) - g(t_1 + \Lambda_1)g(t_2)
$$

$$
\geq g(t_1)g(t_2 + \Lambda_1) - g(t_1 + \Lambda_1)g(t_2)
$$

$$
\geq 0,
$$
where the first inequality follows because $g > 0$ and nondecreasing and the second inequality because $g$ is logarithmically convex. \[ \square \]

**THEOREM 2.1.** Let $(X_1, X_2)' \sim N_2(\mu, \Sigma)$. Then the joint density function $f_{|X_1|, |X_2|}(x_1, x_2)$, of $|X_1|, |X_1|$ is TP$_2$.

**PROOF.** For $x_1 < 0$ or $x_2 < 0$, $f_{|X_1|, |X_2|}(x_1, x_2) = 0$, and for $x_1 > 0$, $x_2 > 0$ it is readily shown that

$$f_{|X_1|, |X_2|}(x_1, x_2) = k_1(x_1) k_2(x_2) g(x_1 x_2),$$

where

$$k_i(s) = \exp[-s^2/\theta_i^2], \ i = 1, 2,$$

$$g(s) = c \exp[-ps/(\theta_1 \theta_2)] + \exp[-ps/(\theta_1 \theta_2)]$$

and

$$\theta_i = (2 - 2\rho^2)^{1/2}, \ i = 1, 2; \ c^{-1} = 2\pi \sigma_1 \sigma_2 (1 - \rho^2)^{1/2}.$$

Straightforward calculations yield that $g$ is nondecreasing and logarithmically convex, so that Lemma 2 immediately yields that $f_{|X_1|, |X_2|}(x_1, x_2)$ is TP$_2$. \[ \square \]

**REMARK 2.1.** From Lemma 2.1, it follows that the random variables $|X_1|, |X_2|$ are conditionally increasing in sequence, associated, and PQD.

3. **Total Positivity of the Trivariate Absolute Normal.** In this section we give a necessary and sufficient condition for the density function of the trivariate absolute normal variable to be TP$_2$ in pairs. In section 4 we use this result to show that a trivariate $\chi^2$ and a trivariate t-distribution are associated and, hence, are PQD.
Let \((X_1, X_2, X_3)' \sim N_3(\mathbf{0}, \Sigma)\) have a trivariate normal distribution with mean \(\mathbf{0}\) and covariate matrix \(\Sigma\). Let \(A \equiv \{\lambda_{ij}\} = \Sigma^{-1}\). Then the joint p.d.f.,

\[
f(|X_1|, |X_2|, |X_3|(x_1, x_2, x_3), \text{ of } |X_1|, |X_2|, |X_3|, \text{ for } (x_1, x_2, x_3) \text{ in the positive octant is given by}
\]

\[
(3.1) \quad f(|X_1|, |X_2|, |X_3|(x_1, x_2, x_3) = K_A \exp[-\frac{1}{2}(\lambda_{11} x_1^2 + \lambda_{22} x_2^2 + \lambda_{33} x_3^2)] g(x_1, x_2, x_3),
\]

where \(K_A = 2(\sqrt{2\pi})^{-3} |A|^{3/2}\), and

\[
g(x_1, x_2, x_3) = \sum_{i=0}^{1} \sum_{j=0}^{1} \exp[(-1)^i \lambda_{12} x_1 x_2 + (-1)^j \lambda_{13} x_1 x_3 + (-1)^{i+j+1} \lambda_{23} x_2 x_3].
\]

The density is 0, otherwise.

Hence, to show that \(f(|X_1|, |X_2|, |X_3|(x_1, x_2, x_3)\) is TP_2 in pairs it suffices to show that \(g(x_1, x_2, x_3)\) is TP_2 in pairs. To do so we require the following two lemmas whose proofs are straightforward.

**Lemma 3.1.** Let \(A_0\) be a fixed 3 x 3 positive definite matrix and define \(D_e\) as a diagonal matrix with elements \(\pm 1\). Then the p.d.f. \(f(|X_1|, |X_2|, |X_3|(x_1, x_2, x_3)\) given by (3.1), viewed as a function of \(A\), is invariant on the set \(\{A: A = D_e A_0 D_e\}\).
Define \( \text{sgn}(x) = 1 \) if \( x > 0 \); \( = 0 \) if \( x = 0 \); \( = -1 \) if \( x < 0 \).

**Lemma 3.2.** A necessary and sufficient condition that there exists \( D_0 \), a diagonal matrix with elements \( \pm 1 \), so that the off-diagonal elements of \( D_0 A_0 D_0 \) are all negative (positive) is that \( \prod_{i<j} \text{sgn}(\lambda_{ij}) = -1 \) (1), where \( \lambda_{ij} \) is the \( i, j \)th element of \( A_0 \).

**Theorem 3.1.** Let \( (X_1, X_2, X_3)' \sim N_3(0, \Sigma) \). Then a necessary and sufficient condition that the joint density function
\[
f_{|X_1|, |X_2|, |X_3|}(x_1, x_2, x_3) \text{ of } |X_1|, |X_2|, |X_3| \text{ be TP}_2 \text{ in pairs is that } \prod_{i<j} \text{sgn}(\lambda_{ij}) \leq 0, \text{ where } \Lambda = (\lambda_{ij}) = \Sigma^{-1}.
\]

**Proof.**

**Sufficiency.** If \( \prod_{i<j} \text{sgn}(\lambda_{ij}) = -1 \), then by Lemmas 3.1 and 3.2, we may suppose that \( -\lambda_{12} > 0, -\lambda_{13} > 0, -\lambda_{23} > 0 \).

Let
\[
\begin{align*}
u &= -\lambda_{13} x_1 x_3, \\
v &= -\lambda_{23} x_2 x_3,
\end{align*}
\]
and
\[
\alpha = -\lambda_{12}^{-1}/(\lambda_{13} \lambda_{23} x_3^2),
\]
so that \( -\lambda_{12} x_1 x_2 = \alpha uv \). Without loss of generality, we only show \( f_{|X_1|, |X_2|, |X_3|}(x_1, x_2, x_3) \) is \( \text{TP}_2 \) in \( x_1 > 0, x_2 > 0 \)

for \( x_3 > 0 \) fixed. This is equivalent to showing that for \( \alpha > 0 \), \( h_\alpha(u,v) \) is \( \text{TP}_2 \) for \( u > 0, v > 0 \), where
(3.3) \[ h_a(u,v) = P_a(u,v) + \frac{P_a(u,-v) + P_a(-u,v) + P_a(-u,-v)}{2e^\alpha u v \cosh(u + v) + 2e^{-\alpha u v} \cosh(u - v)} \]

and \[ P_a(u,v) = \exp[u + v + \alpha u v]. \]

Let

(3.4) \[ \Delta_a(u,v) = h_a(u,v) \frac{\partial^2 h_a(u,v)}{\partial u \partial v} - \frac{\partial h_a(u,v)}{\partial u} \frac{\partial h_a(u,v)}{\partial v} \]

To verify that \( h_a(u,v) \) is TFз, we verify for \( u > 0, v > 0, \alpha > 0 \) that

(3.5) \( \Delta_a(u,v) \geq 0. \)

(See Karlin [1968], p. 49). Direct calculation yields that

\[
\frac{\partial h_a(u,v)}{\partial u} = (1 + \alpha v)[P_a(u,v) - P_a(-u,v)] + (1 - \alpha v)[P_a(u,-v) - P_a(-u,-v)],
\]

\[
\frac{\partial h_a(u,v)}{\partial v} = (1 + \alpha u)[P_a(u,v) - P_a(u,-v)] + (1 - \alpha u)[P_a(-u,v) - P_a(-u,-v)],
\]

and

\[
\frac{\partial^2 h_a(u,v)}{\partial u \partial v} = a[P_a(u,v) + P_a(-u,-v) - P_a(-u,v) - P_a(u,-v)]
\]
\[
+ (1 + \alpha v)(1 + \alpha u)P_a(u,v)
\]
\[
- (1 - \alpha u)(1 + \alpha v)P_a(-u,v) - (1 + \alpha u)(1 - \alpha v)P(u,-v)
\]
\[
+ (1 - \alpha u)(1 - \alpha v)P_a(-u,-v),
\]

so that after simplification, we have
\[(3.6) \quad \Delta_a(u,v) = 2a[e^{2auv}\cosh(u+v) - e^{-2auv}\cosh(u-v)]
+ 4(2 + a)\sinh(2auv) + 8av[\sinh(2u) + au \cosh(2u)]
+ 8au[\sinh(2v) + av \cosh(2v)].\]

The first term of (3.6) is nonnegative by the monotonicity of \(e^t\) and the monotonicity of \(\cosh(t)\). The remaining 3 terms of (3.4) are nonnegative because \(\cosh t \geq 0\), and \(\sinh t \geq 0\) for \(t \geq 0\). Thus (3.5) holds.

If \(\sum_{i,j} \text{sgn}(\lambda_{ij}) = 0\), then either two or more of the \(\lambda_{ij}\)'s equal to zero, or exactly one of the \(\lambda_{ij}\)'s equal to zero. The case where two or more of the \(\lambda_{ij}\)'s equal to zero follows from the bivariate case discussed in Section 2. If exactly one of the \(\lambda_{ij}\)'s equals to zero, say \(\lambda_{12}\), then in Equation (3.2) divide by the other two \(\lambda_{ij}\)'s, so that \(a = 0\) and then apply a technique similar to the one used when \(a > 0\) to show that the density is \(TP_2\) in pairs for fixed \(x_3\). In this case, to show \(TP_2\) in pairs for fixed \(x_1\) or \(x_2\) the argument would reduce to the bivariate case argument.

**NECESSITY.** Suppose \(\sum_{i,j} \text{sgn}(\lambda_{ij}) = 1\), so that by Lemmas 3.1 and 3.2 we can assume \(\lambda_{12} > 0\), \(\lambda_{13} > 0\), \(\lambda_{23} > 0\). Define \(u, v, a\) as in the proof of the sufficiency, but note \(u, v, a\) and \(a < 0\).

We proceed to show that there exists \(x_3 > 0\) so that
\[f|X_1|, |X_2|, |X_3|(X_1, X_2, X_3)\] has negative second order determinant for certain \(x_1 > 0\), \(x_2 > 0\). To do this, we let
\[ x_3 = \left( \frac{\lambda_{12}}{(\lambda_3 \lambda_{23})} \right)^{\frac{1}{3}}, \] so that \( n = -1 \), and then show that there exists an open set so that \( A_{-1}(u, v) \) defined in (3.4) is negative. To find such an open set, we show that there exists \( t < 0 \) so that \( A_{-1}(t, t) \) is negative and then appeal to the continuity of \( A_{-1}(u, v) \). Note that

\[ A_{-1}(t, t) = 2e^{-2t^2} \left[ 1 - \cosh(2t) \right] - 16t[\sinh(2t) - t \cosh(2t)]. \]

Observe that \( \cosh(2t) \geq 1 \) and that for suitably small negative \( t \), \( \sinh(2t) - t \cosh(2t) < 0 \), so that for suitably small negative \( t \), \( A_{-1}(t, t) < 0 \). Hence, we can conclude that

\[ \{ X_1, \; X_2, \; X_3 \} \text{ is not TP}_2 \text{ in pairs if} \]

\[ i \geq j \text{ sgn}(\lambda_{ij}) = 1. \]

**Remark 3.1.** If \( i \leq j \text{ sgn}(\lambda_{ij}) \neq 0 \), then, using Lemma 2.1 and Theorem 3.1, we have that the random variables \( |X_1|, \; |X_2|, \; |X_3| \) are conditionally increasing in sequence, and associated.

For the general multivariate normal case without absolute values, we note that Barlow and Proschan [1974, Chapter 4] proved that the multivariate normal density function is TP\(_2\) in pairs if and only if \( \lambda_{ij} \leq 0 \) for \( i \neq j \) where \( A = \Sigma^{-1} \).

4. **The Association of Bivariate and Trivariate \( \chi^2 \) and \text{t} Distributions.**

In this section we use the results of the previous sections to obtain the association of certain bivariate, and trivariate, \( \chi^2 \), \text{t}, and \text{F} distributions.
To prove the results of this section, we make use of the following two lemmas which by themselves are quite interesting and useful.

**Lemma 4.1.** Let $U_1, \ldots, U_a$ be positive random variables. If $U_1, \ldots, U_a$ are associated, then $U_1^{-1}, \ldots, U_a^{-1}$ are associated.

**Proof:** For any nondecreasing functions $f$ and $g$, we have

$$\text{Cov}(f(U_1^{-1}, \ldots, U_a^{-1}), g(U_1^{-1}, \ldots, U_a^{-1})) = \text{Cov}(-f(U_1^{-1}, \ldots, U_a^{-1}),$$

$$-g(U_1^{-1}, \ldots, U_a^{-1})) = 0,$$

because $-f(U_1^{-1}, \ldots, U_a^{-1}), -g(U_1^{-1}, \ldots, U_a^{-1})$ are nondecreasing functions in $U_1, \ldots, U_a$. []

**Lemma 4.2.** Suppose that the nonnegative random variables $U_1, \ldots, U_a$ are independent of the nonnegative random variables $V_1, \ldots, V_a$. If $U_1, \ldots, U_a$ are associated and $V_1, \ldots, V_a$ are associated, then $U_1V_1, \ldots, U_aV_a$ are associated.

**Proof:** Independence implies $U_1, \ldots, U_a, V_1, \ldots, V_a$ are associated (P2 of [4]) and, hence, $U_1V_1, \ldots, U_aV_a$ which are nondecreasing functions of $U_1, \ldots, U_a, V_1, \ldots, V_a$ are associated (P4 of [4]). []

**Theorem 4.1.** (a) For $p = 2$, $s_1^2 + s_2^2$, $s_2^2 + s_3^2$ are associated random variables.

(b) For $p = 3$, if $\sum_{k,j} \text{sgn}(v_1^{-1})_{kj} = 0$,

$i = 1, \ldots, n$, then $s_1^2 + s_2^2$, $s_2^2 + s_3^2$ and $s_3^2 + s_3^2$ are associated random variables, where $(v_1^{-1})_{kj}$ denotes $k, j$th element of $v_1^{-1}$. 
PROOF of (a): By Lemma 2.1, Theorem 2.1 and the invariance of association under nondecreasing transformations (P₄ of [4]) we have, for i = 1, ..., n, that \( z_{1i}^2, z_{2i}^2 \) are associated. Because \((g_i), s_{1i}^2, s_{2i}^2\) are independent, we have that \( z_{11}^2, z_{21}^2, ..., z_{1n}^2, z_{2n}^2 \) are associated (P₂ of [4]). Since
\[ s_{1i}^2 + s_{1i}^2, s_{2i}^2 + s_{2i}^2 \]
are nondecreasing functions of the
\[ z_{1i}^2, s_{1i}^2, s_{2i}^2, \]
we obtain \( s_{1i}^2 + s_{1i}^2, s_{2i}^2 + s_{2i}^2 \) are associated.

PROOF of (b): Using Theorem 3.1 and Lemma 2.1, we can prove (b) in a similar fashion to (a) with the obvious modifications.

REMARK 4.1. Note that the condition that \( I_i \) is of the form
\[(i_k, a_j)\] implies that \( k_{ij} \text{sgn}(v_i^{-1}) k_j \leq 0. \) (See Khatri [1967]).

COROLLARY 4.1. (a) For \( p = 2, (s_{1i}^2 + s_{1i}^2)^{-\frac{1}{2}}, (s_{2i}^2 + s_{2i}^2)^{-\frac{1}{2}} \) are associated random variables.

(b) For \( p = 3, \text{if } k_{ij} \text{sgn}(v_i^{-1}) k_j \leq 0, \)
\[ i = 1, ..., n, \]
then \( (s_{1i}^2 + s_{1i}^2)^{-\frac{1}{2}}, (s_{2i}^2 + s_{2i}^2)^{-\frac{1}{2}}, \) and \( (s_{3i}^2 + s_{3i}^2)^{-\frac{1}{2}} \)
are associated random variables.

PROOF: The proof follows from Theorem 4.1 and the square root analogue of Lemma 4.1.

THEOREM 4.2. (a) For \( p = 3, \) the random variables \( |X_1|/(s_{1i}^2 + s_{1i}^2)^{\frac{1}{2}} \)
and \( |X_2|/(s_{2i}^2 + s_{2i}^2)^{\frac{1}{2}} \) are associated.
(b) For $p = 3$, if $i < j$ sgn($\lambda_{ij}$) $\leq 0,$

\[ \prod_{k < j} \text{sgn}(y_{ik}^{-1})_{kj} \leq 0, \quad i = 1, 2, \ldots, n, \]  

then $|X_1|/(S_1^2 + S_1^{*2})^{1/4},$ 

$|X_2|/(S_2^2 + S_2^{*2})^{1/4},$ 

$|X_3|/(S_3^2 + S_3^{*2})^{1/4}$ are associated random variables.

**PROOF:** The proof of the theorem follows from Theorem 2.1, Theorem 3.1, Lemma 4.2, and Corollary 4.1. \[ \square \]

Up to constants, a bivariate and a trivariate $F$ random vector can be defined by:

\[ F(2) = (S_1^2/S_1^{*2}, S_2^2/S_2^{*2})', \]

and

\[ F(3) = (S_1^2/S_1^{*2}, S_2^2/S_2^{*2}, S_3^2/S_3^{*2})'. \]

**THEOREM 4.2.** (a) $S_1^2/S_1^{*2}, S_2^2/S_2^{*2}$ are associated random variables.

(b) If $\prod_{k < j} \text{sgn}(y_{ik}^{-1})_{kj} \leq 0, \quad i = 1, \ldots, n,$ then the random variables $S_1^2/S_1^{*2}, S_2^2/S_2^{*2}, S_3^2/S_3^{*2}$ are associated.

**PROOF:** The proof of the theorem follows immediately from Theorem 4.1, Lemma 4.1 and Lemma 4.2. \[ \square \]

**REMARK 4.2.** Theorems 4.1, 4.2, 4.3 and Corollary 4.1 remain true as long as $S_1^{*2}$ and $S_2^{*2}$ are any pair of positive independent random variables such that $(X_1, \ldots, X_p)', (\xi_1), S_1^{*2}, S_2^{*2}$ are all mutually independent sets of random variables.
We conclude this paper with the following conjecture for the TP$_2$ in pairs of the multivariate absolute normal, of dimension larger than 3.

**Conjecture.** Let $f|x_1|,\ldots,|x_p|$ ($x_1,\ldots,x_p$) be the p.d.f. of the multivariate absolute normal, $p > 3$. A necessary and sufficient condition for it to be TP$_2$ in pairs is that there exists $D_0$, a diagonal matrix with elements $\pm 1$, such that the off-diagonal elements of $D_0^{-1}D_0$ are all negative.

Note that if this conjecture were true, then the corresponding result concerning the multivariate t-distribution could be proved directly in the same fashion as Theorem 4.2.

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REFERENCES


Positive Dependence of the Bivariate and Trivariate Absolute Normal, t, $\chi^2$ and F Distributions

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Total positivity, positive quadrant dependence, conditionally increasing in sequence, association, multivariate t distribution, multivariate f distribution, multivariate normal distribution.

It is shown that the bivariate density of the absolute normal distribution is totally positive of order 2. Necessary and sufficient conditions are given for the trivariate density of the absolute normal distribution to be totally positive of order 2 in pairs of arguments. These results are then used to show that certain generalized bivariate and trivariate t, $\chi^2$ and F random variables are associated.