THE DISTRIBUTION OF WAITING TIME
IN A MACHINE INTERFERENCE PROBLEM

by

VINCENT HODGSON

ONR Technical Report No. 12

M 45

on

THE DEVELOPMENT OF STATISTICAL METHODS
FOR QUALITY CONTROL AND SURVEILLANCE TESTING

May, 1964
Department of Statistics
Florida State University
Tallahassee, Florida

Research supported by the Army, Navy and Air Force under
Office of Naval Research Contract Number NONR-988(08),
Task Order NR 042-004 with the Florida State University.
Reproduction in whole or in part is permitted for any
purpose of the United States Government.
SUMMARY

This paper considers the probabilistic behavior of a system consisting of one man who repairs a set of m machines one at a time in the order in which they break down. We make the usual assumption of exponentially distributed working time for each machine. Interest centers on the distribution of the length of time a machine is broken down between its working times. Methods and results are a little simpler than those of Takacs [2]. In particular we find a new expression for the probability density function of waiting time in the case of exponentially distributed repair times. It should be noted from the introduction that our definition of waiting time is not the usual one.

V.H.
THE DISTRIBUTION OF WAITING TIME IN A MACHINE INTERFERENCE PROBLEM

by Vincent Hodgson

The Florida State University

1. INTRODUCTION.

We consider the following model of machine interference. A battery of m identical machines is tended by one repairman who can devote all his time to the repair of machines which break down but is only able to repair one machine at a time. If a machine is working at time t the probability that it will break down in the infinitesimal interval \((t, t + \delta t)\) is \(\lambda \delta t + o(\delta t)\); this is equivalent to saying that working times are mutually independent and exponentially distributed with mean \(1/\lambda\). Repair times are independently and identically distributed with (right-continuous) distribution function \(B(t)\) and mean \(1/\mu\). Repairs are performed in order of breakdown.

Takács [2] [3] gives references to published work on this model. However, only two writers take the investigation of machine waiting time in the steady state beyond a derivation of the first two moments of its distribution. Takács [2] [3] gives two arguments leading to a solution for the distribution of waiting time but gives no explicit solutions for special cases of \(B(t)\). In this paper will be found a simpler solution derived through an argument similar to that of [2], which is correct but incomplete. Naor [1] gives the distribution of waiting

---

1Research supported by the Army, Navy, and Air Force under an Office of Naval Research Contract. Reproduction in whole or in part is permitted for any purpose of the United States Government.
time for a model in which repair times are exponentially distributed and there are several repairmen. This paper offers a simpler version of Naor's solution in the special case of one repairman.

A difference in terminology between this and previous papers should be noted. By waiting time we mean the time which a machine is idle from the moment it breaks down to the moment it begins working again. Hence our waiting-time is the sum of what others call waiting time plus a repair time. It could be argued that this new waiting time, which might be called machine downtime, is a more natural variable for the measurement of the effects of machine interference. However we use it for its simplicity; for example, its probability density function will not have any concentrations, arising out of cases in which a machine receives immediate attention because the repairman is idle, unless there are concentrations in the probability density function of repair time.

2. STEADY STATE PROBABILITIES.

Let \( \tau_n \) be the instant at which the \( n^{th} \) repair time begins (\( n = 1, 2 \ldots \)). Let the random variable \( \eta_n \) denote the number of machines working at \( \tau_n + 0 \). The sequence \( \{ \eta_n \} \) forms a Markov chain which is aperiodic, irreducible and finite, and hence ergodic. Let the steady state probabilities associated with this Markov chain be denoted by \( P_i = \lim_{n \to \infty} \text{Prob} ( \eta_n = i ) \) for \( i = 1, 2, \ldots m-1 \). Thus \( P_i \) is the steady state probability that a repair begins with \( i \) machines working.

Let \( \alpha_{ij} \) be the probability that exactly \( j \) machines break down during a repair time which begins with \( i \) machines working. This is the probability of \( j \) breakdowns in \( i \) independent trials, hence
\[ \alpha_{ij} = \int_0^\infty \binom{i}{j} (1 - e^{-\lambda t})^j e^{-\lambda t(i-j)} dB(t), \quad j = 0, 1 \ldots i. \] (1)

The steady state probabilities \( P_i \) satisfy the following equations:

\[ P_i = \sum_{j=1-i}^{m-1} \alpha_{j,j-i+1} P_j \quad i = 1, 2, \ldots m-2, \] (2)

\[ P_{m-1} = \sum_{m-2} \alpha_{j,j-m+2} P_j + \alpha_{m-1,0} P_{m-1}. \] (3)

The extra term in equation (3) corresponds to the case where a period intervenes during which all machines are working.

If we write the \( r \)th binomial moment of the distribution \( P_i \) as

\[ F_r = \sum_{i=r}^{m-1} \binom{i}{r} P_i \quad (r = 0, 1, \ldots m-1). \] (4)

then it follows that

\[ P_i = \sum_{r=i}^{m-1} (-)^{r-i} \binom{r}{i} F_r \quad (i = 1, 2, \ldots m-1). \] (5)

It is possible to transform equations (2) and (3) to

\[ F_r = F_r b_r + F_{r-1} b_{r-1} - \binom{m-1}{r-1} F_{m-1} b_{m-1}, \] (6)

for \( r = 1, 2, \ldots m-1 \), where \( b_r \) is the Laplace-Stieltjes transform of \( B(t) \) with argument \( r\lambda \).
\[ b_r = \int_0^\infty e^{-r\lambda t} \ dB(t). \quad (7) \]

Because of equation (5), a solution of equation (6) for \( F_r \) will provide a solution for \( P_i \). In fact equation (6) has the solution

\[ F_r = \frac{1}{b_r^{K_r}} \cdot \frac{1}{r^{m-1}} \sum_{i=r}^{m-1} \left( \begin{array}{c} m-1 \\ i \end{array} \right) K_i, \quad (8) \]

where \( K_0 = 1 \) and \( K_i = \frac{i}{\prod_{j=1}^{i} (b^{-1} - 1)} \) (i = 1, 2 ...).

The above argument has been only outlined because it is not new.

The use of equations (5) and (8) to find \( P_i \) does not in general lead to simple solutions. However in the special case \( B(t) = 1 - e^{-\mu t} \), where the solution may also be obtained by more direct methods, it is possible to find \( P_i \).

We define a function \( T_k(x) \) which is the generating function of factorial powers of the integer \( k \):

\[ T_k(x) = \sum_{i=0}^{k} (k)_i x^i, \quad (k = 0, 1, 2, \ldots), \quad (10) \]

where \((k)_i = k(k-1) \ldots (k-i+1)\) denotes the \( i^{th} \) factorial power of \( k \).

The function \( T_k(x) \) satisfies the recurrence relationship

\[ T_k(x) = 1 + kx T_{k-1}(x), \quad (k = 1, 2, \ldots). \quad (11) \]
From (15), $P_{m-1} = F_{m-1} = \frac{1 + (m-1) \rho}{T_{m-1}(\rho)}$.  

(18)

This solution agrees with that of Naor [1] in the special case of one repairman. Naor denotes by $p_i$ the steady state probability that $i$ machines are not working at any instant. Now $p_i$ and $P_i$ are related by

$$P_i \propto i P_{m-1} \quad i = 1, 2, \ldots, m-2$$

and $$P_{m-1} \propto m p_o + (m-1) p_1.$$ 

Verbally the argument for the first proportionality is that the event that a repair begins with $i$ machines working ($i = 1, 2, \ldots, m-2$) coincides with a transition in the number of machines working from $i-1$ to $i$ which in the steady state occurs as frequently as a transition in the number working from $i$ to $i-1$; the probability of the latter transition is proportional to the probability of $i$ machines working multiplied by the rate of breakdown when $i$ machines are working. The argument for the second proportionality is similar in principle but a little more intricate in detail.

3. THE DISTRIBUTION OF WAITING TIME.

All machine breakdowns may be classified as being of type $m$ or type $ij$ ($i = 1, 2, \ldots, m-1; j = 1, 2, \ldots, i$). A type $m$ breakdown is one which occurs when all $m$ machines are working; this type of breakdown receives immediate attention from the repairman and hence has a waiting time with distribution function $B(t)$. A type $ij$ breakdown is one which occurs during a repair time which began with $i$ machines working and, further, the breakdown is the $j^{th}$ to occur during this repair time. Thus a type $ij$ breakdown occurs when $i-j+1$ machines are working.
The waiting time of a type $ij$ breakdown is the sum of two mutually independent random variables: first, the time from the occurrence of the type $ij$ breakdown to the end of the repair time then in progress; to this must be added the time taken to repair $m-i+j-1$ machines. We write the distribution function of the first of these random variables as $R_{ij}(t)$; the second random variable's distribution function may be written as $B_{m-i+j-1}(t)$, the $(m-i+j-1)$-fold convolution of the distribution function of repair time. Thus using the symbol $V$ with a suitable subscript to denote the steady state probabilities of the various types of breakdown, the distribution function of the waiting time is

$$W_m(t) = V_m B(t) + \sum_{1 \leq j \leq i \leq m-1} V_{ij} R_{ij}(t) \ast B_{m-i+j-1}(t).$$

(19)

Now as we noted in the last paragraph of the previous section the number of transitions from $i-1$ to $i$ in the number of machines working is, in the steady state, equal to the number of transitions from $i$ to $i-1$, for $i = 1, 2, \ldots m$; hence, summing over $i$, the total number of repairs is equal to the total number of breakdowns. (More precisely they differ by at most $m$.) Consider then a large number $N$ of breakdowns and repairs. We have

$$NV_{ij} = NP \sum_{k=j}^{i} \alpha_{ik}.$$ 

(20)

In words, the expected number of type $ij$ breakdowns is equal to the expected number of repair times in which such a breakdown is possible, multiplied by the probability that it actually occurs. Further, repair times can begin with $m-1$ machines working only as a consequence of one of two types of breakdown; hence we
have
\[ NV_m = NP_{m-1} - NV_{m-1,1} = NP_{m-1} b_{m-1}, \] (21)

Finally to find \( R_{ij}(t) \) we consider two independent random variables: \( A_{ij} \) is the time to the occurrence of the \( j^{th} \) breakdown among \( i \) machines when broken down machines are not repaired; \( B \) is a repair time and has distribution function \( B(t) \). The probability density function of \( A_{ij} \) is given by
\[ a_{ij}(t)dt = \alpha_{ij-1}(t) \lambda (i-j+1) dt, \] (22)

where
\[ \alpha_{ij}(t) = \binom{i}{j} (1 - e^{-\lambda t})^j e^{-\lambda t (i-j)} \] (23)

is the probability that exactly \( j \) breakdowns occur in an interval of length \( t \) which begins with \( i \) machines working and during which no repairs are completed. 

Now \( R_{ij}(t) \) is the distribution function of \( B - A_{ij} \) conditional upon \( B - A_{ij} > 0 \); therefore
\[ R_{ij}(t) = \frac{\int_0^\infty [B(t+x) - B(x)] a_{ij}(x) dx}{\int_0^\infty [1 - B(x)] a_{ij}(x) dx} \] (24)

We note that the denominator is the probability that \( j \) or more breakdowns occur during a repair time which begins with \( i \) machines working, which is \( \sum_{k=j}^i \alpha_{ik} \).

It follows that equation (19) may, with a slight change in the summation variables, be written
\[ W_m(t) = p_{m-1} b_{m-1} B(t) + \sum_{1 \leq j \leq i \leq m-1} \lambda_j p_i \int_0^\infty [B(t+x) - B(x)] \alpha_{i,j-i}(x) dx * B_{m-j}(t). \] (25)
\[ w_m(t) = F_{m-1} b_{m-1} b(t) + \sum_{1 \leq j \leq r \leq m-1} \lambda_j(-)^{r-j} \binom{r}{j} F_r \int_0^\infty b(t+x) e^{-\lambda x r} dx \ast b_{m-j}(t) \]  

(28)

4. SPECIAL CASES.

When \( b(t) = \mu e^{-\mu t} \) equation (28) may be written

\[ w_m(t) = F_{m-1} b_{m-1} b(t) + \sum_{1 \leq j \leq r \leq m-1} j(-)^{r-j} \binom{r}{j} F_r b_r \rho b_{m-j+1}(t) \]

\[ = \frac{1}{T_{m-1}(\rho)} \left[ b(t) + \sum_{1 \leq j \leq r \leq m-1} j \rho (-)^{r-j} \binom{r}{j} \binom{m-1}{r} T_{m-1-r}(\rho) b_{m-j+1}(t) \right] \]

from equation (15). Rearranging the product of binomial coefficients and using equation (13) to sum over \( r \) gives

\[ v_m(t) = \frac{\mu e^{-\mu t}}{T_{m-1}(\rho)} \left[ 1 + \sum_{j=1}^{m-1} \binom{m-1}{j-1} (\lambda t)^{m-j} \right] . \]

\[ v_m(t) = \frac{\mu e^{-\mu t} (1 + \lambda t)^{m-1}}{T_{m-1}(\rho)} . \]

(29)

This is a simpler version of the result obtained by Naor [1] in the special case of one repairman.

In other cases we have not been able to express \( w_m(t) \) in closed form.

For example, consider what is probably the next simplest case:

\[ b(t) = \frac{(k\mu)^k}{(k-1)!} \frac{e^{-k\mu t}}{e^{-kt}} . \]