CHARACTERIZATION OF PARTIALLY ORDERED CLASSES OF LIFE DISTRIBUTIONS.

by

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ABSTRACT

In this paper we obtain characterizations of life distributions $F$ such that (a) $G^{-1}(F)$ is convex (concave) and alternatively (b) $G^{-1}(F)$ is starshaped (antistarshaped), where $G$ is an absolutely continuous life distribution with positive, bounded, right continuous density. These characterizations generalize earlier results for the IFR(DFR) and IFRA(DFRA) classes, and should prove useful in unifying the study of the class of distributions with decreasing density, comparing Weibull (Gamma) distributions with different shape parameters, etc.
1. Introduction.

In a previous paper (Langberg, León, and Proschan, 1973), we obtain characterizations of large classes of standard nonparametric life distributions, such as the IFR(DFR), IFRA(DFRA), etc. (See Section 2 for definitions and notation.) These characterizations are obtained under the weakest possible assumptions that we can make concerning the life distributions being characterized.

In the present paper, we continue our characterization work and also obtain additional results in one direction of implication, but now we focus on more general classes of distributions, many of them of interest and applicable in reliability. We consider classes of distributions $F$ such that $G^{-1}(F)$ is convex (concave) or starshaped (antistarshaped), where $G$ is a known distribution. The assumption that $G$ is known is reasonable in many practical situations, as seen from the following pairs $(F, G)$ such that $G^{-1}(F)$ is convex: (1) $G$ is exponential, $F$ is IFR; (2) $G$ is uniform, $F$ has decreasing density; (3) $G$ is Weibull (Gamma) with shape parameter $\alpha$, $F$ is Weibull (Gamma) with shape parameter $\beta(>\alpha)$; etc. Similar pairs can be displayed for $G^{-1}(F)$ is starshaped. Since we assume $G$ known and $F$ unknown, we make convenient smoothness assumptions for $G$, but as few assumptions about $F$ as possible.

As pointed out in Barlow and Proschan (1975), Barlow and Doksum (1972), Barlow and Van Zwet (1970), the advantage of considering these more general classes is that many results, tests, methods of inference, methods of proof, etc., of use in the IFR(DFR), IFRA(DFRA) classes carry over with minor modifications to the corresponding more general classes.

In Section 2, we present definitions, notation, and elementary properties. In Section 3, we obtain characterization results for the Barlow-Doksum
2. Preliminaries.

Let $F$ be a life distribution, that is, $F(0-) = 0$. We use the following notation and conventions: $F^{-1}(t) \equiv \inf\{x: F(x) > t\}, t \in [0, 1]$; $F^{-1}(1) \equiv \sup\{x: F(x) < 1\}; F \equiv 1 - F; R \equiv -\ln F$. We use "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing". Throughout the paper we assume that $G$ is a fixed absolutely continuous life distribution with positive, bounded, and right continuous density $g$ on the interval $(G^{-1}(0), G^{-1}(1))$. Let $X_1, X_2, \ldots, X_n (Y_1, Y_2, \ldots, Y_n)$ be a random sample of size $n$ from $F(G)$ and let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ $(Y_{1:n} < Y_{2:n} < \ldots < Y_{n:n})$ be the corresponding order statistics.

**Definition 2.1.** The life distribution $F$ is convex with respect to $G$, written $F \prec G$, if either (i) $F$ is degenerate or (ii) $G^{-1}F$ is convex on $(-\infty, F^{-1}(1))$.

**Definition 2.2.** The life distribution $F$ is concave with respect to $G$, written $F \preceq G$ if $G^{-1}F$ is concave on $(F^{-1}(0), \infty)$.

Let $F$ be nondegenerate, strictly increasing on $(F^{-1}(0), F^{-1}(1))$ and $G^{-1}(1) = \infty$. Then $F \preceq G$ if and only if $G \preceq F$. This relationship is the reason only convex ordering is usually defined in the literature (see for example Barlow and Proschan, 1975, p. 106). However without assumptions on $F$, the two orderings are not so easily related.

We define the increasing failure rate (IFR) and shifted decreasing failure rate (SDFR) classes of life distributions.

**Definition 2.3.** The life distribution $F$ is IFR if either (i) $F$ is degenerate or (ii) $R(x)$ is convex on $(-\infty, F^{-1}(1))$.

**Definition 2.4.** The life distribution $F$ is SDFR if $R(x)$ is concave on $(F^{-1}(0), \infty)$. 

For a fixed $G$ let $H_{F}^{-1}(t) \equiv \int_{0}^{t} F_{G}^{-1}(u) C_{F}^{-1}(u) \, du$. This transform of $F$ was first introduced in connection with isotonic tests of convex ordering by Barlow and Doksum (1972). Hence we call $H_{F}^{-1}$ the Barlow-Doksum (B-D) transform. When $G$ is the exponential distribution $H_{F}^{-1}$ is the usual total time on test transform studied by Barlow (1977), Barlow and Campo (1975), and Langberg, León, and Proschan (1978), among others. We should remark that Chandra and Singpurwalla (1978) have pointed out the close relationship between the total time on test transform and the Lorenz curve used by econometricians. In this section we develop some properties of $H_{F}^{-1}$ which we use in the proofs of Section 4.

Before stating the first theorem we need two definitions.

Definition 3.1. A point $x$ is a point of increase of $F$ if $F(x - h) < F(x) < F(x + h)$ for every $h > 0$.

Definition 3.2. A sequence $(\xi_{r}, n_{r})_{r=1}^{\infty}$ of ordered pairs of natural numbers is a $t$-sequence $(0 < t < 1)$ if (i) $1 \leq k_{r} \leq n_{r} < n_{r+1}$ for all $r$, and (ii) $k_{r}/n_{r} \rightarrow t$ as $r \rightarrow \infty$.

Let $T_{G}(X_{k:n}) \equiv \sum_{i=1}^{k} g(G^{-1}(i/n))(X_{i:n} - X_{i-1:n})$. If $G$ is the exponential distribution with mean 1, then $T_{G}(X_{k:n}) = n^{-1}T(X_{k:n})$, where $T(X_{k:n}) \equiv \sum_{i=1}^{k} (n - i + 1)(X_{i:n} - X_{i-1:n})$, is the total time on test statistics commonly used in reliability theory (see for example Barlow and Proschan, 1975, p. 61).

If $n$ items are placed on test at time 0 and successive failures are observed at times $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$, then $T(X_{k:n})$ represents the total test time observed between 0 and $X_{k:n}$.

We may now state and prove the following theorem.
Let \( f(x_0) \) denote the right-hand derivative of \( f \) at the point \( x_0 \). We will need the following lemma in the proof of Theorem 3.6 below.

**Lemma 3.5.** Let \( x \) be a point of increase and of continuity of \( F \). Then \( \hat{H}^{-1}_F(F(x)) \) exists and is nonzero if and only if \( \hat{F}(x) \) exists and is nonzero. In this case, \( \hat{F}^{-1}(F(x)) \) exists and is nonzero, and \( \hat{H}^{-1}_F(F(x)) \hat{F}^{-1}(F(x)) = 1 \).

**Proof.** Note that in a neighborhood of \( x \), \( F^{-1} \) behaves like the usual inverse function of \( F \). The result follows using standard differentiation results. ||

The following theorem relates convex (concave) ordering to the B-D transform.

**Theorem 3.6.** Let \( F \) be a life distribution. Then \( F \preceq_G F \) (\( F \preceq_C F \)) if and only if \( H^{-1}_F \) is concave (convex) on \([0, 1]\).

We will need the following simple properties of \( H^{-1}_F \) in the proof of Theorem 3.6.

1. \( H^{-1}_F(0) = F^{-1}(0) \).
2. \( H^{-1}_F(t) \) is increasing on \([0, 1]\).
3. \( H^{-1}_F(s) = y \) if and only if \( P(X_1 = F^{-1}(a)) = b - a \).

In this section we present a series of results concerning convex (concave) ordering and order statistics. Our first theorem gives a sufficient condition for \( F \leq (\xi) G \).

Theorem 4.1. Let \( F \) and \( G \) be life distributions with finite means. Suppose \( F \) is continuous and let \( E(X_{k:n} - X_{k-1:n})/E(Y_{k:n} - Y_{k-1:n}) \) be decreasing (increasing) in \( k \) \( (k = 2, \ldots, n) \) for infinitely many values of \( n \geq 2 \). Then \( F \leq (\xi) G \).

In order to prove Theorem 4.1, we need the following lemma.

Lemma 4.2. Let the conditions of Lemma 4.1 be satisfied. Then the support of \( F \) is the interval \([F^{-1}(0), F^{-1}(1)]\).

Proof. The support of a continuous distribution is a closed set without isolated points (see Chung, 1974, p. 10). It follows that if \( S \), the support of \( F \), is not an interval, then we can find \( a, b, \) and \( \varepsilon \) such that \((a - \varepsilon, a] \subset S\), \((a, b) \subset (x: x \leq s)\), and \([b, b + \varepsilon] \subset S\). Let \( t = F(a) = F(b)\), \( t_1 = t + F(a - \varepsilon)/2\).

Also let \( h > 0 \) be small enough so that \([t_1 - h, t_2 + h] \subset (t, F(b + \varepsilon))\) and \([t_2 - h, t_2 + h] \subset (t, F(b + \varepsilon))\).

By hypothesis, \( E G^{-1}(\frac{i-1}{n})(X_{i:n} - X_{i-1:n})/E G^{-1}(\frac{i-1}{n})(Y_{i:n} - Y_{i-1:n}) \) is decreasing (increasing) in \( i \) \( (i = 2, 3, \ldots, n) \) for infinitely many values of \( n \). Now observe that if \( \{a_i\}_{i=2}^n \) and \( \{b_i\}_{i=2}^n \) are sequences of positive real numbers such that \( a_i/b_i \) is decreasing (increasing) in \( i \) \( (i = 2, \ldots, n) \), then \( \sum_{i=k}^{k+j} a_i/\sum_{i=k}^{k+j} b_i \) is decreasing (increasing) in \( k \) \( (k = 2, \ldots, n - j) \) for each \( j \) \( (j = 1, \ldots, n - 1) \). Thus we obtain for each one of the infinitely many \( n \) that
Since (4.3) is true for all $t_1, t_2$, and $h$ satisfying the above constraints, $H_F^{-1}$ must be concave (convex) on $[0, 1)$. By Theorem 3.6, this implies that $F < (\leq) G$. 

Theorem 4.3 is a partial converse to Theorem 4.1.

Theorem 4.3. Let $F$ and $G$ be life distributions with finite means and suppose $F < (\leq) G$. Then

$$
\lim_{n \to \infty} \frac{E(X[n(t+h)]:n) - X[n(t)]:n)}{E(Y[n(t+h)]:n) - Y[n(t)]:n)}
$$

is decreasing (increasing) in $t$ ($0 < t < t + h < 1$) for all $h$ ($0 < h < 1$).

Proof. Let $F < G$. Note that every element of $(F^{-1}(0), F^{-1}(1))$ is a point of increase of $F$ since $F < G$ and every element of $(G^{-1}(0), G^{-1}(1))$ is a point of increase of $G$. Thus $X[n]:n \to F^{-1}(t)$ a.s. and $Y[n]:n \to G^{-1}(t)$ a.s. as $n \to \infty$ (see Rao, 1973, p. 423). We show $X[n]:n$ is uniformly integrable.

We have $P(X[n]:n > x) = P(B(n, \bar{F}(x)) > n - [nt] + 1)$, where $B(n, \bar{F}(x))$ denotes a binomial random variable with parameters $n$ and $\bar{F}(x)$. Thus

$$
P(X[n]:n > x) \leq \frac{n}{n - [nt] + 1} \bar{F}(x)
$$

since $F(Z > A) \leq EZ/A$ for any nonnegative random variable $Z$ and any $A > 0$.

Hence

$$
E_X[n]I[X[n]:n > A] = \int_A^\infty P(X[n]:n > x)dx + AP[X[n]:n > A]
$$

[by integration by parts]

$$
\leq \frac{n}{n - [nt] + 1} \left( \int_A^\infty \bar{F}(x)dx + A\bar{F}(A) \right)
$$

[by (4.5)]

$$
\leq \frac{1}{1 - [nt] + 1} \left( EX[I[X \geq A] \right).
$$
(ii) \( h(n - i, n) \) changes sign at most \( k \) times as a function of \( n = 1, 2, \ldots \); if \( h(n - i, n) \) actually does change sign in \( n \) exactly \( k \) times, then the changes occur in the same order as do those of \( h(x) \).

Before stating our result, we observe that \( F \lesssim G \) and \( \text{EY} \lesssim \infty \) imply that \( \text{EX} \lesssim \infty \) and consequently that \( \text{EX}_{i:n} \lesssim \infty \) for all \( i \) and \( n \) (\( i = 1, 2, \ldots, \ n; \ n \geq 1 \)).

**Theorem 4.5.** Let \( F \lesssim (\lesssim) G \), \( F \) be continuous at \( F^{-1}(1) \), and \( \text{EY} \lesssim \infty \) (continuous at \( F^{-1}(0) \), \( \text{EY} \lesssim \infty \), and \( \text{EX} \lesssim \infty \)). Then (i) for all \( a \geq 0 \) and \( b \geq 0 \),

\[
\text{a EX}_{i:n} - \text{EY}_{i:n} - b \text{ changes signs at most twice in } i = 1, 2, \ldots n(n = 1, 2, \ldots),
\]

and if twice, from negative to positive to negative (positive to negative to positive); (ii) for all \( a \geq 0, \ b \geq 0 \),

\[
\text{a EX}_{n-i:n} - \text{EY}_{n-i:n} - b \text{ changes signs at most twice in } n = 1, 2, \ldots, \text{and if twice, from negative to positive to negative (positive to negative to positive).}
\]

**Proof.** Let \( F \lesssim G \) and let \( \phi(x) = G^{-1}F(x) \). Then \( \phi \) is convex. Thus for \( a \geq 0, \ b \geq 0, (ax - b) - \phi(x) \) changes signs at most twice, and if twice, from negative to positive to negative. Hence by Lemma 4.4(i),

\[
h(i, n) \equiv \int_0^\infty (ax - b - \phi(x))dF_{i:n}
\]

\[
= a\text{EX}_{i:n} - b - \text{EY}_{i:n}
\]

changes sign at most twice in \( i = 1, 2, \ldots, n(n = 1, 2, \ldots) \), and if twice, from negative to positive to negative. Thus (i) follows.

A similar argument using part (ii) of Lemma 4.4 yields (ii). For the case \( F \lesssim G \), the proof is similar. ||

We now present a converse to Theorem 4.5 (ii).
\[
\frac{Y_{i+1:n} - Y_{i:n}}{Y_{i:n} - Y_{i-1:n}} \geq \frac{X_{i+1:n} - X_{i:n}}{X_{i:n} - X_{i-1:n}}
\]

for \(i = 2, 3, \ldots, n - 1; \ n \geq 2\). Since

\[
\frac{Y_{i+1:n} - Y_{i:n}}{Y_{i:n} - Y_{i-1:n}} \ngeq \frac{Y_{i+1:n} - Y_{i:n}}{Y_{i:n} - Y_{i-1:n}},
\]

the conclusion follows in the case \(F \lessdot G\).

A similar argument yields the conclusion when \(F \lessdot G\). \(\|\)

If \(F \lessdot (\lessdot)G\) it is reasonable to expect that information about the order statistics \(Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}\) yields information about the order statistics \(X_{1:n}, X_{2:n}, \ldots, X_{n:n}\). Theorem 4.9 shows one way this expectation is fulfilled.

Other examples will follow.

**Theorem 4.9.** Let \(F \lessdot (\lessdot)G\) and the support of \(F\) be an interval. Let \(1 \leq i < j < \ell \leq n\), \(i < k \leq \ell\), and \(a > 0\). Then

\[
P[Y_{\ell:n} - Y_{j:n} \geq a(Y_{k:n} - Y_{i:n})] \geq (\leq) P[X_{\ell:n} - X_{j:n} \geq a(X_{k:n} - X_{i:n})].
\]

**Proof.** Let \(F \lessdot G\) and \(Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}\) be as in the proof of Theorem 4.5. Let \(\phi(y)\) be the concave function \(F^{-1}G\). Then for \(1 \leq i \leq j < \ell \leq n\) and \(i < k \leq \ell\),

\[
\frac{\phi(Y_{k:n}^i) - \phi(Y_{i:n}^i)}{Y_{k:n}^i - Y_{i:n}^i} \geq \frac{\phi(Y_{\ell:n}^j) - \phi(Y_{j:n}^j)}{Y_{\ell:n}^j - Y_{j:n}^j}
\]

(see Royden, 1968, p. 108). Hence
5. Starshaped (Antistarshaped) Ordering and Order Statistics.

In this section we consider another ordering, namely starshaped (antistarshaped) ordering. The first result gives a necessary and sufficient condition in terms of the order statistics for two life distributions to be related under the starshaped (antistarshaped) ordering.

Theorem 5.1. Let F and G be continuous life distributions with finite means. Assume that the supports of both F and G are intervals and that

\[ G(0) = F(0) = 0. \]

Then \( F \preceq_a G \) if and only if \( \operatorname{EX}_{i:n}/\operatorname{EY}_{i:n} \) is decreasing (increasing) in \( i(i = 1, 2, \ldots, n) \) for infinitely many \( n \).

Proof. We prove \( F \preceq G \) if and only if \( \operatorname{EX}_{i:n}/\operatorname{EY}_{i:n} \) is decreasing in \( i(i = 1, 2, \ldots, n) \) for infinitely many \( n \). The counterpart result for \( F \preceq_a G \) has a similar proof. The "only if" part is Theorem 3.6 of Barlow and Proschan (1966).

To show the "if" part recall that in the proof of Theorem 4.4 we showed that \( \operatorname{EX}_{[nt]:n} \rightarrow F^{-1}(t) \) and \( \operatorname{EY}_{[nt]:n} \rightarrow G^{-1}(t) \) as \( n \rightarrow \infty \). Thus if

\[ \operatorname{EX}_{[nt]:n}/\operatorname{EY}_{[nt]:n} \]

is decreasing in \( t (0 < t < 1) \), then \( F^{-1}(t)/G^{-1}(t) \) is decreasing in \( t (0 < t < 1) \). Equivalently, \( F^{-1}(F(x))/G^{-1}(F(x)) = x/G^{-1}F(x) \)

is decreasing (increasing) in \( x(0 < x < F^{-1}(1)) \). The "if" part follows. ||

Corollary 5.2. (Theorem 5.6 of Langberg, León, and Proschan). Let F be a continuous life distribution with finite mean. Assume that the support of F is an interval and that \( F(0) = 0 \). Then F is IFRA(DFRA) if and only if

\[ \operatorname{EX}_{i:n}/\sum_{k=1}^{n} (n - k + 1)^{-1} \]

is decreasing (increasing) in \( i (i = 1, 2, \ldots, n) \) for infinitely many \( n \).
Hence \( X_{i:n} \overset{\text{st}}{\geq} (\overset{\text{st}}{\leq}) a X_{j:n} \).

**Theorem 5.5.** Let \( F \leq (\leq)_a G \) and the support of \( F \) be an interval. Let \( 1 \leq i < j \leq n \) and \( a > 0 \). Then

\[
P(Y_{j:n} \geq aY_{i:n}) \geq (\leq) P(X_{j:n} \geq a X_{i:n}).
\]

**Proof.** Let \( F \leq G \) and \( Y_{1:n}', Y_{2:n}', \ldots, Y_{n:n}' \) be as in the proof of Theorem 4.8. Let \( \phi(y) \) be the antistarshaped function \( F^{-1}G \). Then for \( 1 \leq i \leq j \leq n \),

\[
\frac{\phi(Y_{i:n}')}{Y_{i:n}'} \geq \frac{\phi(Y_{j:n}')}{Y_{j:n}'}.
\]

Hence

\[
\frac{Y_{j:n}}{Y_{i:n}'} \geq \frac{X_{j:n}}{X_{i:n}}.
\]

The conclusion follows as in the proof of Theorem 4.9.

If \( F \leq (\leq)_a G \), the proof is similar.

It is clear that a corollary to Theorem 5.5 can be fashioned along the lines of Corollary 4.10. This corollary can be used for nonparametric tests for \( F \leq (\leq)_a G \); in particular, for tests for IFRA and DFRA.

We prove a converse of Theorem 5.4.

**Theorem 5.6.** Let the support of \( F \) be an interval. Suppose

\[
EY_{i:n} \geq (\leq) a EY_{j:n} \implies EX_{i:n} \geq (\leq) a EX_{j:n}
\]

for all \( a(0 < a < 1) \) and all \( i, j, n \) \((1 \leq i < j \leq n)\). Then \( F \leq (\leq)_a G \).

**Proof.** Suppose \( F \leq G \) is not true. Then there exist an \( a \) \((0 < a < 1)\) and an \( x \geq 0 \) such that \( G^{-1}F(ax) \geq a G^{-1}F(x) \). Therefore there exists a \( y > x \) such that \( G^{-1}F(ax) > a G^{-1}F(y) \). Hence for \( n \) sufficiently large, \( EY_{[nF(ax)]:n} > a EY_{[nF(y)]:n} \). By hypothesis, this implies that for \( n \) sufficiently large, \( E[X_{[nF(ax)]:n}] = a E[X_{[nF(y)]:n}] \). Consequently \( F^{-1}(F(ax)) \geq a F^{-1}(F(y)) \); that is, \( ax \geq ay \) - a contradiction.
REFERENCES


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