ON THE CONSTRUCTION AND ASYMPTOTIC BEHAVIOUR
OF m-DIMENSIONAL SIMPLE EPIDEMIC MODELS

by

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1. Introduction and Summary

In some simple epidemic situations susceptible individuals, "susceptibles" for short, are exposed only to one disease [Bailey 1975]. In this paper we assume that \( n \) "susceptibles" are exposed to \( m \) contagious diseases. We call models that describe the progress of this epidemic among "susceptibles" \( m \)-dimensional simple epidemic models. For \( m = 1 \) this extension reduces to the simple epidemic. We describe an \( m \)-dimensional simple epidemic model by a stochastic process \( X^n(t) = (X^n_1(t), \ldots, X^n_m(t)) \). The components of \( X^n(t) \) represent the number of infected individuals, "infectives" for short, with the respective diseases at time \( t \).

In section 2 we construct a variety of \( m \)-dimensional stochastic processes. Those processes can be used to model the progress of a wide range of epidemics among "susceptibles". However, in the sequel we restrict ourselves to exponential \( m \)-dimensional simple epidemic models. Those models are described by Markovian stochastic processes. In Section 3 we construct the exponential \( m \)-dimensional simple epidemic models. Although the transition rates or the state probabilities associated with exponential models are not used directly, they are presented in Section 3 for the sake of completeness. In Section 4 we introduce exponential \( m \)-dimensional simple epidemic models which describe the progress of an epidemic among infinitely many "susceptibles". Each such model is represented by a stochastic process \( X(t) = (X_1(t), \ldots, X_m(t)) \). The components of \( X(t) \) count the number of "infectives" with the respective diseases at time \( t \) and are independent univariate stochastic processes.

The only previous papers in the area of \( m \)-dimensional simple epidemic models are those by Billard, Lacayo, and Langberg 1978, and by Lacayo and Langberg 1978.
Both of the papers treat some special cases of exponential m-dimensional simple epidemic models. Let $b_r$, $r = 1, \ldots, m$, denote the initial number of "infectives" with disease $r$, let $k = (k_1, \ldots, k_m)$ be a vector with nonnegative integer valued components, let $\alpha$ and $t$, be in $(0, \infty)$, and let $\beta$ be in $[1, \infty)$. Billard, Lacayo, and Langberg 1978 prove that if the transition rates at time $t$ of disease 1 through $m$ are given by

(1.1) $\frac{\alpha_k}{\alpha} X_r^n(t) \left[ n + \sum_{q=1}^{m} b_q - \sum_{q=1}^{m} X_q^n(t) \right]$, for $r = 1, \ldots, m$,

then the following three statements hold,

(1.2) $\lim_{n \to \infty} P(X^n(t) \leq h + k) = P(X(t) \leq h + k)$;

(1.3) $\lim_{n \to \infty} E(X^n(t))^\beta = E(X(t))^\beta$, for $r = 1, \ldots, m$;

(1.4) The vector $X(t)$ has independent negative binomial components with parameters respectively equal to $(e^{-\alpha t}, b_r)$, $r = 1, \ldots, m$.

Let $\alpha_1, \ldots, \alpha_m$ be real numbers in $(0, \infty)$. Lacayo and Langberg 1978, also prove (1.2) and (1.3). They assume that the transition rates at time $t$ of disease 1 through $m$ are given respectively by

(1.5) $\frac{\alpha_k^\prime}{\alpha} X_r^n(t) \cdot \left[ n + \sum_{q=1}^{m} b_q - \sum_{q=1}^{m} X_q^n(t) \right]$, for $r = 1, \ldots, m$.

Assumption (1.5) is weaker than (1.1). Lacayo and Langberg 1978 show that for $t$ in $(0, \infty)$ the vector $X(t)$ has independent negative binomial components with parameters respectively equal to $(e^{-\alpha t}, b_r)$, $r = 1, \ldots, m$. In Section 5 we prove for a variety of exponential m-dimensional simple epidemic models including those cited above. statements (1.2) and (1.3). In all these cases we show that for $t$ in $(0, \infty)$ the random vector $X(t)$ has independent components.

In conclusion we briefly discuss the evaluation of the joint and marginal state probabilities associated with processes $X^n(t)$ and $X(t)$. In Section 6 we
present formulas for these quantities. We apply a method suggested by Billard, Lacayo, and Langberg 1978b and 1978c for the simple epidemic. This method differs from the traditional way of using the differential equations associated with the state probabilities. It utilizes available information about the distributions of sums of independent exponential random variables with arbitrary rates.
2. Preliminaries

We start to observe at time $T_0^n$ "susceptibles" that are exposed to $m$ contagious diseases. We say that the "susceptibles" are exposed to a $m$-dimensional simple epidemic if the following four assumptions are satisfied.

(2 A I) Each "susceptible" can contract only one disease.

(2 A II) Once a "susceptible" enters the $r^{th}$ infective state, $r = 1, \ldots, m$, he remains in that state during the duration of the epidemic.

(2 A III) An "infective" with disease $r$, $r = 1, \ldots, m$, can transmit only that disease.

(2 A IV) At each point in time at most one "susceptible" becomes an "infective".

Next we construct a stochastic process which models the progress of the epidemic among the "susceptibles". Let $b_r$, $r = 1, \ldots, m$, be the number of "infectives" with disease $r$ at time $T_0^n$, and let $b = \sum_{q=1}^m b_q$. We denote by $\tau_{i,j,1}$, $i = 1, \ldots, b$; $j = 1, \ldots, n$, the random time measured with respect to $T_0^n$ until the $i^{th}$ contagious individual causes the $j^{th}$ "susceptible" to become the first "infective". If $T_1^n$ denotes the time measured with respect to $T_0^n$ until the first infection, then

$$T_1^n = \min_{i=1, \ldots, b} \tau_{i,j,1}, \quad j=1, \ldots, n$$

Let $\xi_1^n$ be the random variable, with values in the set $\{1, \ldots, m\}$ designating the cause of the first infection, and let $J_1, r$, $r = 1, \ldots, m$, be the index set of all "infectives" with disease $r$ at time $T_0^n$. Then for $r = 1, \ldots, m$,

$$I(\xi_1^n = r) = I(\min_{i \in J_1, r} \tau_{i,j,1} < \min_{e=1, \ldots, m} \min_{i \in J_1, e} \tau_{i,j,1}), \quad j=1, \ldots, n$$

Similarly, for $k = 2, \ldots, n$, let $\tau_{i,j,k}$, $i = 1, \ldots, b + k - 1$; $j=1, \ldots, n - k + 1$, denote the random time, measured with respect to $\sum_{j=1}^{k-1} T_j^n$ until the $i^{th}$ contagious
individual causes the jth "susceptible" to become the kth "infective". If \( T^n_k \)
denotes the time measured with respect to \( \{ T^n_j \} \) until the kth infection, then
for \( k = 2, \ldots, n \),

\[
(2.3) \quad T^n_k = \min_{i=1, \ldots, b+k-1} \min_{j=1, \ldots, n-k+1} \tau_{i,j,k}.
\]

Let \( \xi^n_k, k = 2, \ldots, n \), be the random variable with values in the set \( \{1, \ldots, m\} \)
designating the cause of the kth infection, and let \( J^n_{k,r}, k = 2, \ldots, n; r = 1, \ldots, m \),
be the index set of all "infectives" with disease r at time \( \sum_{j=1}^k T^n_j \). Then for
\( k = 2, \ldots, n \), and \( r = 1, \ldots, m \),

\[
(2.4) \quad I(\xi^n_k = r) = I(\min_{i \in J^n_{k,r}} \tau_{i,k} < \min_{e=1, \ldots, m} \min_{i \in J^n_{e,r}} \tau_{i,k} | j=1, \ldots, n-k+1).
\]

To satisfy assumption (2a A. IV) we require for \( k = 1, \ldots, n \), and for every two
positive distinct integers \( r \) and \( s \) in the set \( \{1, \ldots, m\} \), that

\[
(2a V) \quad P(\min_{i \in J^n_{k,r}} \tau_{i,k} = \min_{i \in J^n_{k,s}} \tau_{i,k}, j=1, \ldots, n-k+1) = 1.
\]

We model the progress of the epidemic among the "susceptibles" by a stochastic
process \( X^n(t) \equiv (X^n_1(t), \ldots, X^n_m(t)) \). The components of \( X^n(t) \) describe the number of
"infectives" with the respective diseases at time \( t \) measured with respect to \( T^n_0 \).
Throughout we use the following notation: \( s^n_o = 0, s^n_k = \sum_{q=1}^k T^n_q, k = 1, \ldots, n; \)
\( C^n_{k,r} = b_r, r = 1, \ldots, m; \) and \( C^n_{k,r} = b_r + \sum_{q=1}^{k-1} I(\xi^n_q = r), k = 2, \ldots, n; r = 1, \ldots, m. \)

Let \( k = o, \ldots, n; r = 1, \ldots, m, \) and let \( t \) be in \( (o, \infty) \). Then

\[
(2.5) \quad (X^n(t) \ge b_r + k) = \bigcup_{q=k}^n (s^n_q \le t < s^n_{q+1}, C^n_{q+1,r} \in k + b_r).
\]

Hence all finite dimensional joint distributions associated with \( X^n(t) \), which deter-
mine the process \( X^n(t) \), are determined by \( (T^n_1, \xi^n_1), \ldots, (T^n_m, \xi^n_m) \) through equation
(2.5). To complete the construction of the process $X^n(t)$ note that the joint distribution of $(\tau_{1,1}^n, \xi_{1,1}^n), \ldots, (\tau_{n,n}^n, \xi_{n,n}^n)$ is determined by the joint distribution of the random variables $\tau_{i,j,k}, \ i = 1, \ldots, b + k - 1; \ j = 1, \ldots, n - k + 1; \ k = 1, \ldots, n$, by equations (2.1) through (2.4). Consequently, the process $X^n(t)$ is well-defined whenever a joint distribution that satisfies (2 AV) is imposed on the random variables $\tau_{i,j,k}, \ i = 1, \ldots, b = k -1; \ j = 1, \ldots, n - k + 1; \ and \ k = 1, \ldots, n$. 
3. Exponential m-Dimensional Simple Epidemic Models

Let \( \mu_n^q(\theta; k; r), \theta, k = 1, \ldots, n, k = 1, \ldots, n, r = 1, \ldots, m, \) be positive real numbers. We say that an \( m \)-dimensional simple epidemic model is exponential if assumptions (3 A I) and (3 A II) given below hold.

(3 A I) The conditional random variables \( \tau_{i,j,k|\xi_q^n}, q = 1, \ldots, n; \)
\[ i = 1, \ldots, b + k - 1; j = 1, \ldots, n - k + 1; k = 1, \ldots, n, \] are independent and exponentially distributed.

(3 A II) For \( r \) in the set \( J_{k,r}, j = 1, \ldots, n - k + 1; \) and \( r = 1, \ldots, m, \) the rate of the conditional exponential random variable \( \tau_{i,j,k|\xi_q^n}, q = 1, \ldots, n, \) is equal to \( \mu_n^{q}(n - k + 1; C_{k,r}^n; r) \).

Note that under (3 A I) and (3 A II) the conditional random variables
\[ \tau_{i,j,k|\xi_q^n}, q = 1, \ldots, n; i = 1, \ldots, b + k - 1; j = 1, \ldots, n - k + 1, \] are independent of \( \xi_{k}^{n}, \ldots, \xi_{n}^{n}, \) for \( k = 1, \ldots, n. \)

First we determine the joint distribution of the random variables
\[ \tau_{i,j,k}, i = 1, \ldots, b + k - 1, j = 1, \ldots, n - k + 1; k = 1, \ldots, n. \] We need the following lemma.

**Lemma 3.1** Let \( (3 A I) \) and \( (3 A II) \) are satisfied. Then statements (3.1) and (3.2) given below hold.

(3.1) \[ P(\xi^n_r = r) = b_r \cdot \mu_n^{r}(n; b_r; r) \cdot \left[ \sum_{q=1}^{n} b_q \cdot \mu_n^{q}(n; b_q; q) \right]^{-1}, r = 1, \ldots, m. \]

(3.2) \[ P(\xi^n_k = r|\xi^n_q, q = 1, \ldots, k - 1) = C_{k,r}^{n} \cdot \mu_n^{(n-k+1; C_{k,r}^{n}; r)}, \left[ \sum_{q=1}^{m} C_{k,r}^{n} \cdot \mu_n^{q}(n-q+1; C_{k,r}^{n}; q) \right]^{-1}, \]
\[ q = 1, \ldots, k - 1, r = 1, \ldots, m. \]
Proof. The random variables \( \min_{i \in J_{1,r}} \tau_{i,j,l,r} = 1, \ldots, m \), are independent exponentially distributed and have rates that equal to \( n b_r \mu^u(n; b_r ; r) \). It follows from (2.2) for \( r = 1, \ldots, m \), that \( P(\xi^n_1 = r) \)

\[
= n b_r \mu^u(n; b_r ; r) \int_0^\infty \exp\left\{-\sum_{q=1}^m b_q \mu^u(n; b_q ; q)\right\} du.
\]

Statement (3.1) follows upon a simple integral evaluation. For \( k = 2, \ldots, n \), the conditional random variables

\[
\min_{i \in J_{k,r}} \tau_{i,j,k,r} | \xi^n_q, q = 1, \ldots, k-1 = 1, \ldots, m, \text{ are independent exponentially distributed and have rates that equal respectively to } (n-k+1)C_{k,r}^n \mu^v(n-k+1; C_{k,r}^n, r).
\]

Statement (3.2) follows now from (2.4) and from an argument similar to the one given in the proof of (3.1).

Let \( \tau_{i,j,k}, i = 1, \ldots, b + k - 1; j = 1, \ldots, n - k = 1; k = 1, \ldots, n; \) be in \((0, \infty)\), and let \( r_1, \ldots, r_n \), be positive integers in the set \( \{1, \ldots, m\} \). Then

\[
P(\xi^n_{q} = r_q, q = 1, \ldots, n) = P(\xi^n_1 = r_1) \cdot \prod_{q=2}^n P(\xi^n_q = r_q | \xi^n_j = r_j, j = 1, \ldots, q-1), \text{ and}
\]

\[
P(\tau_{i,j,k} > \tau_{i,j,k}, i = 1, \ldots, b + k - 1; j = 1, \ldots, n - k + 1; k = 1, \ldots, m)
\]

\[
= \sum_{r_q \in \{1, \ldots, m\}} \left[ P(\tau_{i,j,k} > \tau_{i,j,k}, i = 1, \ldots, b + k - 1; j = 1, \ldots, n - k + 1; k = 1, \ldots, m | \xi^n_q = r_q, q = 1, \ldots, n) \right].
\]

Consequently, the joint distribution of the random variables \( \tau_{i,j,k}, i = 1, \ldots, b + k - 1; j = 1, \ldots, n - k + 1; k = 1, \ldots, n, \) is determined by (3.1) through (3.4). Clearly for exponential \( m \)-dimensional simple epidemic models assumption (2 A V) holds. For reference
purposes we note that for \( t_1, \ldots, t_n \) in \((0, \infty)\) equation (3.5) given below holds.

\[
(3.5) \quad P(t^n_q > t_q, q=1, \ldots, n, | \epsilon^n_q, q=1, \ldots, n) = \exp\left( -\sum_{q=1}^{n} (n-q+1) t_q \sum_{r=1}^{q} c^n_q \epsilon^n_{q+r} \right).
\]

The transition rates, or the differential equations associated with the state probabilities, are not used explicitly in the paper. However, we present them for the sake of completeness. Let \( k \equiv (k_1, \ldots, k_m) \) be a vector with nonnegative integer valued components, \( b \equiv (b_1, \ldots, b_m) \), and let \( \delta_*= (\delta_1, \ldots, \delta_m) \), \( i = 1, \ldots, m \), where \( \delta_{i,j} = \left\{ \begin{array}{ll} 1 & i = j, \\
0 & i \neq j \end{array} \right. \). Then for \( t \) in \((0, \infty)\),

\[
(3.6) \quad \lim_{b \to 0} \frac{1}{h} P\{X^h(t+h) = b + k + \delta_*, X^h(t) = b + k\} = (n-k) \mu^n_{n-k; b_r+k} \cdot \mu^n_{n-k; b_r+k}.
\]

In particular, the transition rates of disease 1 through \( m \) at time \( t \) of an exponential \( m \)-dimensional simple epidemic, are given respectively by

\[
(3.7) \quad \frac{d}{dt} P^m_{k}(t) = \sum_{q=1}^{m} \left( b_q k_q \mu^n_{n-k; b_q+k} \right) + \sum_{q:k \geq 1} \sum_{q:k \geq 1} \left( n-k+1 \mu^n_{n-k+1; b_q+k-1} \right).
\]

Let \( k \equiv (k_1, \ldots, k_m) \), \( k = \sum_{q}^m k_q < n \), and let \( P^m_k(t) = P\{X^m(t) = b + k\} \) be the joint probability. Then for \( t \) in \((0, \infty)\)

\[
(3.8) \quad \frac{d}{dt} P^m_k(t) = -P^m_k(t) + (n-k) \sum_{q=1}^{m} \left( b_q k_q \mu^n_{n-k; b_q+k} \right) + \sum_{q:k \geq 1} \sum_{q:k \geq 1} \left( n-k+1 \mu^n_{n-k+1; b_q+k-1} \right) P^m_{k-\delta_*(q)}(t).
\]

Let \( k = 0, \ldots, n \); \( r = 1, \ldots, m \); \( A_{k,r} \equiv \{ \epsilon = (\epsilon_1, \ldots, \epsilon_m) \mid \epsilon_r = k, \sum_{q=1}^{m} \epsilon_q \leq n \} \); and \( \epsilon \)

let \( P^m_{k,r}(t) = P\{X^m(t) = b_r+k\} \) be the marginal state probability. Then for \( t \) in \((0, \infty)\)
\[
(3.9) \quad p_{k,r}^n(t) = \sum_{e \in A_{k,r}} p_e^n(t).
\]

In particular we have for \( t \in (0, \infty) \), \( k = 1, \ldots, n \); and \( r = 1, \ldots, m \), that
\[
(3.10) \quad \frac{dp_{k,r}^n}{dt} = \sum_{e \in A_{k,r}} \frac{dp_e^n(t)}{dt}.
\]

In conclusion we briefly present some examples of exponential \( m \)-dimensional simple epidemic models. Let \( \alpha \) be a positive real number. Then if \( \mu^n(e;k;r) \equiv \alpha \), we obtain exponential homogeneous \( m \)-dimensional simple epidemic models. These models extend Bailey's 1950, homogeneous simple epidemic. For \( \mu^n(e;k;r) \equiv \alpha/n \) we get models called by Billard, Lacyo and Langberg 1978 a the symmetric \( m \)-dimensional models. Let \( \alpha_1, \ldots, \alpha_m \), be positive real numbers, and let \( \mu^n(e;k;r) \equiv \alpha_r/n \). Then we obtain the family of exponential \( m \)-dimensional simple epidemic models introduced by Lacayo and Langberg 1978. Let \( f \) be a real positive function and let \( \mu^n(e;k;r) \equiv \alpha_r f (n) e^{\delta k \lambda} \). Then we get models with transition rates similar to the ones suggested for the simple epidemic by Severo 1969, McNeil 1972, and Langberg 1978 a.
4. Theoretical Exponential m-Dimensional Simple Epidemic Models

Let \( \xi_1, \xi_2, \ldots \) be a sequence of random variables with values in the set \( \{1, \ldots, m\} \), and let \( U_1, U_2, \ldots \) denote a sequence of independent exponential random variables with rates equal to 1. Let

\[ \mu(k, r) = b_r, b_r + 1, \ldots; k = 1, \ldots, m, \]

be positive real numbers. We use the following notation:

\[ C_{k, r} = b_r \sum_{q=1}^{m} \mathbb{I}(\xi_q = r), \quad k = 2, 3, \ldots; r = 1, \ldots, m; \]

\[ S_k = \sum_{q=1}^{k} T_q, \quad k = 1, 2, \ldots \]

We assume throughout that (4 A I) and (4 A II) hold:

(4 A I) \[ P(\xi_1 = r) = \frac{b_r \mu(b_r; r)}{\sum_{q=1}^{m} b_q \mu(b_q; q)} = \frac{\mu_k r}{\mu_k}, \quad r = 1, \ldots, m. \]

(4 A II) \[ P(\xi_k = r | \xi_q, q=1, \ldots, k-1) = \frac{C_{k, r} \mu(C_{k, r}; r)}{\sum_{q=1}^{m} C_{k, q} \mu(C_{k, q}; q)} = \frac{\mu_k r}{\mu_k}, \quad k = 2, 3, \ldots; \]

and \( r = 1, \ldots, m. \)

Note that for every \( k \) positive integers \( r_1, \ldots, r_k \) in the set \( \{1, \ldots, m\} \)

(4.1) \[ P(\xi_q = r_q, q = 1, \ldots, k) = \prod_{q=1}^{k} P(\xi_q = r | \xi_j = r_j, j = 1, \ldots, k-1). \]

Hence, the finite dimensional joint distributions of the sequence \( \xi_1, \xi_2, \ldots \) are uniquely determined by (4 A I) and (4 A II). Consequently, the sequence \( \xi_1, \xi_2, \ldots \) is well defined.

Let \( X(t) = (X_1(t), \ldots, X_m(t)) \) be an m-dimensional stochastic process with components assuming values in the sets \( \{b_r, b_r + 1, \ldots\}, r = 1, \ldots, m, \) respectively.

The sequence \( (T_1, T_2, \xi_1, \xi_2, \ldots) \) determines \( X(t) \) through equation (4.2).
(4.2) \( (X_r(t) \geq b_r + k) = \sum_{q=0}^{k} \left( \sum_{q=1}^{\infty} \Pr \left( S_q \leq t < S_{q+1}, r \geq k + b_r \right) \right), \ k = 0, 1, \ldots, \ r = 1, \ldots, m; \ \text{and} \ t \in (0, \infty). \)

Note that equation (4.2) defines uniquely all finite dimensional joint distributions of \( X(t) \) and thus determines the process \( X(t) \).

The process \( X(t) \) can model the progress of an epidemic among infinity many "susceptibles" who are exposed to \( m \)-contagious diseases.

Next we prove that (i) the processes \( X_r(t), r = 1, \ldots, m, \) are independent, and that (ii) for \( t \in (0, \infty) \), \( k = 0, 1, \ldots, \) and for \( r \) in the set \( \{1, \ldots, m\} \),

\[
P(X_r(t) \geq b_r + k) = \sum_{q=1}^{k} (b_r + q - 1)^{-1} \mu_q^{-1}(b_r + q - 1, r) \ U_q \leq t. \]

Let \( Z_r(t), r = 1, \ldots, m, \) be independent processes, let \( Z(t) = \sum_{q=1}^{m} Z_q(t) \), and let \( V_1, V_2, \ldots, r = 1, \ldots, m, \) be \( m \) independent i.i.d. sequences of exponential random variables with rates equal to 1. We assume that for \( t \in (0, \infty) \), \( k = 1, 2, \ldots, \) and \( r \) in the set \( \{1, \ldots, m\} \), (4 A III) and (4 A IV) hold.

(4 A III) \( \sum_{q=0}^{\infty} P(Z_r(t) = b_r + q) = 1. \)

(4 A IV) \( P(Z_r(t) \geq b_r + k) = \sum_{q=1}^{k} (b_r + q - 1)^{-1} \mu_q^{-1}(b_r + q - 1, r) V_q \leq t. \)

To prove (i) and (ii) it suffices to show that the process \( Z(t) \equiv (Z_1(t), \ldots, Z_m(t)) \) and \( X(t) \) are identically distributed. We need the following notation. \( W_1 = \max(t | Z(t) = b), \ W_{k+1} = \max(t | Z(t + \sum_{q=1}^{k} W_q) = b + k), k = 1, 2, \ldots. \) Let \( \eta_1, \eta_2, \ldots, \) be a sequence of random variables with values in the set \( \{1, \ldots, m\} \), defined by (4.3) and (4.4).

(4.3) \( \eta_1 = \tau = (Z_q(W_1) = b_q, q = 1, \ldots, m, q \neq r, Z_r(W_1) = b_r + 1), r = 1, \ldots, m, \) and
(4.4) \( (n_{k+1} = r) \equiv \)
\[= \left( Z_q \left( \sum_{j=1}^{r} W_j \right) - Z_q \left( \sum_{j=1}^{k} W_j \right) = 0, \quad q = 1, \ldots, m, \quad q \neq r, \quad Z_r \left( \sum_{j=1}^{k} W_j \right) - Z_r \left( \sum_{j=1}^{k} W_j \right) = 1 \right) \]
\[r = 1, \ldots, m, \quad k = 1, 2, \ldots. \]

Clearly for \( t \) in \((0, \infty)\), \( k = 1, 2, \ldots\), and \( r \) in the set \( \{1, \ldots, m\} \),

(4.5) \( (Z(t) \geq b_r + k) = \frac{q}{q+k} \left( \sum_{j=1}^{q} W_j \geq t \left( \sum_{j=1}^{q} W_j \right) \right) \%
\[I(n_j = r) \geq k \]

Hence, the process \( Z(t) \) is determined by the sequence \((W_1, n_1), (W_2, n_2), \ldots\). To prove that \( Z(t) \) and \( X(t) \) are equivalent processes, it suffices to show that the two sequences \((W_1, n_1), (W_2, n_2), \ldots\), and \((T_1, \xi_1), (T_2, \xi_2), \ldots\), are identically distributed. We need the following:

**Lemma 4.1.** Let \( t \) be in \((0, \infty)\), and let \( r \) be in the set \( \{1, \ldots, m\} \). Then

(4.6) \( P(W_1 > t, n_1 = r) = P(T_1 > t, \xi_1 = r) \).

**Proof.** Let \( t \) be in \((0, \infty)\), and let \( r \) be in the set \( \{1, \ldots, m\} \). Then

\[P(W_1 > t, n_1 = r) = P \left( b_r^{-1} \mu^{-1} (b_r; r) V_{1,r} > t, \quad b_r^{-1} \mu^{-1} (b_r; r) V_{1,r} < \min_{q=1, \ldots, m} b_q^{-1} \mu^{-1} (b_q; q) V_{1,q} \right) = \]

\[P(T_1 > t, \xi_1 = r) \]

Let \( e \) be a positive integer in the set \( \{1, \ldots, m\} \), and let \((T_{e,1}, \xi_{e,1}), (T_{e,2}, \xi_{e,2}), \ldots\) be defined as in the first paragraph of the section, where \( b_e \) is replaced by \( b + 1 \). Further let \( Z_e(t) = Z(t+W_1) \), \( Z_e(0) = b_q \), \( q = 1, \ldots, m \), \( q \neq e \), \( Z_e(0) = b + 1 \), and let \((W_{e,1}, n_{e,1}), (W_{e,2}, n_{e,2}), \ldots\) be defined as before where \( Z_e(t) \) replaces \( Z(t) \). Then by the strong Markov property we obtain (4.7) and (4.8).

(4.7) The sequences \((T_2, \xi_2), (T_3, \xi_3), \ldots, (T_1, \xi_1); \) and \((T_{e,1}, \xi_{e,1}), (T_{e,2}, \xi_{e,2}), \ldots), \ldots\) are identically distributed.
(4.8) The sequences \((W_2, \eta_2), (W_3, \eta_3), \ldots |W_1, \eta_1; \text{ and } (W_{\eta_{1,1}}, \eta_{1,1}), (W_{\eta_{1,2}}, \eta_{1,2}), \ldots \), are identically distributed.

To accomplish our objective we need the following.

Lemma 4.2. Let \(t_1, \ldots, t_k \) be in \((0, \infty)\), and let \(r_1, \ldots, r_k \) be \(k \) positive integers in the set \(\{1, \ldots, m\}\). Then

\[
(4.9) \quad P(W > t_q, \eta_q = r_q, q = 1, \ldots, k) = P(T_q > t_q, \xi_q = r_q, q = 1, \ldots, k).
\]

Proof. By induction on \(k\). In Lemma 4.1 we proved (4.9) for \(k = 1\), and arbitrary positive integers \(b_1, \ldots, b_m\). We assume (4.9) for \(k - 1\) and arbitrary positive integers \(b_1, \ldots, b_m\), and prove it for \(k\). Let \(G = P(W > t_q, \eta_q = r_q, q = 1, \ldots, k)\).

\[
G = \int_{t_1}^{\infty} P(W > t_q, \eta_q = r_q, q = 2, \ldots, k | W_1 = \omega, \eta_1 = r_1) \, dP \{ W_1 = \omega, \eta_1 = r_1 \}. \quad \text{By (4.8) and Lemma (4.1)}
\]

\[
G = \int_{t_1}^{\infty} P(T_1 > t_q, \xi_1 = r_1, q = 2, \ldots, k-1) \, dP \{ T_1 = \omega, \xi_1 = r_1 \}. \quad \text{By the induction assumption (4.7)}
\]

\[
G = \int_{t_1}^{\infty} P(T_q > t_q, \xi_q = r_q, q = 2, \ldots, k | T_1 = \omega, \xi_1 = r_1) \, dP \{ T_1 = \omega, \xi_1 = r_1 \}. \quad \text{Consequently, the conclusion of the lemma follows.}||
\]

For reference purposes we summarize the preceding two Lemmas.

Theorem 4.3. (i) The processes \(X_1(t), \ldots, X_m(t)\), are independent. (ii)

For \(t \) in \((0, \infty)\), \(k = 0, 1, \ldots, \) and \(r \) in the set \(\{1, \ldots, m\}\), \(P(X_r(t) \geq b_r + k) = \)

\[
P\left( \sum_{q=1}^{k} (b_r + q - 1)^{-1} \mu^{-1}(b_r + q - r; \omega) \leq \tau \right).
\]
Proof. From Lemma 4.2 we conclude that the sequences $(T_1, \xi_1), (T_2, \xi_2), \ldots,$ and $(W_1, \eta_1), (W_2, \eta_2), \ldots,$ are identically distributed. Hence the processes $X(t)$ and $Z(t)$ are identically distributed. Since $Z(t)$ has properties (i) and (ii), the process $X(t)$ satisfies (i) and (ii).
5. The Asymptotic Behaviour

Let \( k \equiv (k_1, \ldots, k_m) \) be a vector with nonnegative integer valued components. Throughout this section we assume that (3 A I), (3 A II), and the following assumption hold.

\[ \lim_{n \to \infty} n^{-(n-m)} \sum_{q=1}^{m} q \cdot k_q \cdot r = \mu(k_r; r), \text{ for every vector } k, \text{ and } r = 1, \ldots, m. \]

First we show that for every \( t \) in \((0, \infty)\) the sequence \( X^n(t) \) converges in law as \( n \to \infty \) to \( X(t) \). In particular we obtain that the joint and marginal state probabilities associated with \( X^n(t) \) converge as \( n \to \infty \) to those of \( X(t) \). We need the following two lemmas.

**Lemma 5.1.** For every \( k \) positive integers \( r_1, \ldots, r_k \), in the set \( \{1, \ldots, m\} \) the following holds.

\[ \lim_{n \to \infty} P(\xi_{q_1} = r_{q_1}, \ldots, \xi_{q_k} = r_{q_k}) = P(\xi_q = r_q, q=1, \ldots, k). \]

**Proof.** Let \( r_1, \ldots, r_k \), be \( k \) positive integers in the set \( \{1, \ldots, m\} \). Then from (3.1) we obtain that \( \lim_{n \to \infty} P(\xi_{q_1} = r_{q_1}) = P(\xi_1 = r_1) \). From (3.2) we obtain for \( q = 2, \ldots, k \), that \( \lim_{n \to \infty} P(\xi_{q_1} = r_{q_1}, \xi_{j_1} = r_{j_1}, \ldots, q-1) = P(\xi_q = r_q, \xi_j = r_j, j=1, \ldots, q-1). \)

Consequently, the conclusion of the lemma follows from (3.3) and (4.1). \( \Box \)

**Lemma 5.2.** For \( t \) in \((0, \infty)\), and for every \( k \) positive integers \( r_1, \ldots, r_k \), in the set \( \{1, \ldots, m\} \) the following holds.

\[ \lim_{n \to \infty} P(S_{k} \leq t | \xi_{q_1} = r_{q_1}, \ldots, k-1) = P(S_{k} \leq t | \xi_q = r_q, q=1, \ldots, k-1). \]

**Proof.** Let \( t \) be in \((0, \infty)\), and let \( r_1, \ldots, r_k \), be \( k \) positive integers in the set \( \{1, \ldots, m\} \). Then from (3.5) it follows that the conditional sequence of random vectors \( (T^n_1, \ldots, T^n_k) | \xi_q = r_q, q = 1, \ldots, k - 1 \) converge in law as \( n \to \infty \) to the conditional random vector \( (T_1, \ldots, T_k) | \xi_q = r_q, q = 1, \ldots, k - 1 \). Consequently, the conclusion of the lemma follows from a well known result [Billingsley 1968, p. 24]. \( \Box \)
Theorem 5.3. For every \( t \in (0, \infty) \), the sequence \( X^n(t) \) converges in law as 
\( n \to \infty \) to \( X(t) \).

Proof. Let \( t \) be in \((0, \infty)\), \( k \equiv (k_1, \ldots, k_m) \), and let \( k \equiv \sum_{q=1}^{m} k_q \). Then by (2.5)

\[
P(X^n(t) = b + k) = P\left(\{S^n_k \leq S^n_{k+1}, C^n_{k+1, r} = \mathbf{b} + k, r = 1, \ldots, m\}\right).
\]

\[
B_k \equiv \left\{s = (s_1, \ldots, s_k) | s_1, \ldots, s_k \in \{1, \ldots, m\}, \text{ and } \sum_{q=1}^{k} I(s_q = r) = k_r, r = 1, \ldots, m\right\}.
\]

Then

\[
P\{S^n_k \leq S^n_{k+1}, C^n_{k+1, r} = \mathbf{b} + k, r = 1, \ldots, m\} = \sum_{s \in B_k} P\{S^n_k \leq s_{k+1} | s_q = b_q, q = 1, \ldots, k\} \cdot P\{s_{k+1} = \mathbf{b} + k\}.
\]

From (5.1), (5.2), and from (4.4) we obtain that (5.3)

\[
P(X^n(t) = b + k) = P(X(t) = b + k).
\]

It is well known that statement (5.3) suffices to insure the conclusion of the theorem. [Billingsley 1968 p. 16].

Corollary 5.4. For every \( t \in (0, \infty) \), and for \( r = 1, \ldots, m \), the sequence \( X^n(t) \) converges in law as \( n \to \infty \) to \( X_r(t) \).

Proof. Let \( t \) be in \((0, \infty)\), \( k \) be a nonnegative integer, and let \( r \) be in the set \( \{1, \ldots, m\} \). Then

\[
P(X^n(t) \geq b_r + k) = P(X^n_r(t) \geq b_r + k, X^n(t) \geq b_q, q = 1, \ldots, m, q \neq r)\).
\]

Clearly the conclusion of the corollary follows from Theorem 5.3. ||

Note that from Theorem 5.3, Corollary 5.4 and Theorem 4.4, we conclude that the marginal state probabilities associated with \( X^n(t) \) converge as \( n \to \infty \) to those associated with \( X(t) \). Further we conclude that the joint state probabilities associated with \( X^n(t) \) converge as \( n \to \infty \) to the product of the limits of the respective marginal state probabilities. Those two results are stated in (5.4) and (5.5) given below. Let \( t \) be in \((0, \infty)\), \( k \equiv (k_1, \ldots, k_m) \), and let \( r \) be in the set \( \{1, \ldots, m\} \). Then

\[
\lim_{n \to \infty} P(X^n(t) = b_r + k_r) = P(X_r(t) = b_r + k_r) \quad \text{and},
\]

\[
\lim_{n \to \infty} P(X^n(t) = b_r + k_r) = P(X(t) = b_r + k_r) \quad \text{and},
\]

\[
\lim_{n \to \infty} P(X^n_r(t) = b_r + k_r) = P(X_r(t) = b_r + k_r) \quad \text{and},
\]

\[
\lim_{n \to \infty} P(X^n(t) \geq b_r + k_r) = P(X(t) \geq b_r + k_r) \quad \text{and},
\]

\[
\lim_{n \to \infty} P(X^n(t) \geq b_r + k_r) = P(X(t) \geq b_r + k_r) \quad \text{and},
\]

\[
\lim_{n \to \infty} P(X^n_r(t) \geq b_r + k_r) = P(X_r(t) \geq b_r + k_r) \quad \text{and}.
\]
\[ (5.5) \quad \lim_{n \to \infty} P(X^n(t) = b + k) = P(X(t) = b + k) = \prod_{q=1}^{m} P(X^r(t) = b_q + k) = \prod_{q=1}^{m} \lim_{n \to \infty} P(X^n_q(t) = b_q + k). \]

We add the following assumption

\[ (5 \text{ A II}) \quad n \mu_n(n - \sum_{q=1}^{m} k_q) \leq \mu(k_r; r), \text{ for } k \equiv (k_1, \ldots, k_m), \sum_{q=1}^{m} k_q < n; \quad r = 1, \ldots, m; \text{ and } n = 1, 2, \ldots. \]

Next we show that for \( t \in (0, \infty), \beta \in [1, \infty), \) and for \( r \) in the set \( \{1, \ldots, m\} \)

the \( \beta \)-th moments of \( X^n_r(t), \) and of \( X^n(t) = \sum_{q=1}^{m} X^n_q(t), \) converge as \( n \to \infty \) to the \( \beta \)-th moment of \( X_r(t), \) and of \( X(t) = \sum_{q=1}^{m} X_q(t), \) provided \( E(X(t))^\beta < \infty. \) In particular we

obtain that the means and variances of \( X^n_r(t), r = 1, \ldots, m, \) and of \( X^n(t), \) converge as \( n \to \infty \) to the means and variances of \( X_r(t), \) and of \( X(t), \) whenever \( E(X(t))^2 < \infty. \)

We need the following lemma.

**Lemma 5.5.** Let \( t \) be in \( (0, \infty), \) and \( \beta \) in \( (1, \infty). \) Then

\[ (5.6) \quad \lim_{n \to \infty} E(X^n(t))^\beta \geq E(X(t))^\beta. \]

**Proof.** Let \( t \) be in \( (0, \infty), \) and let \( \beta \) be in \( [1, \infty). \) Then \( E(X^n(t))^\beta = \int_0^\infty y^{\beta-1} P(X^n(t) > y)dy. \) The conclusion of the lemma follows Theorem 5.3. and

Fano's lemma [Chung 1974 p. 42].

**Theorem 5.6.** Let \( t \) be in \( (0, \infty), \) and \( \beta \) in \( [1, \infty). \) Then

\[ (5.7) \quad \lim_{n \to \infty} E(X^n(t))^\beta = E(X(t))^\beta. \]

**Proof.** Let \( t \) be in \( (0, \infty), \) and let \( \beta \) be in \( [1, \infty). \) Then \( E(X^n(t))^\beta = \int_0^\infty y^{\beta-1} P(X^n(t) > y)dy, \) and \( E(X(t))^\beta = \int_0^\infty y^{\beta-1} P(X(t) > y)dy. \) Since from

(5 A II) we obtain that \( P(X^n(t) > y) \leq P(X(t) < y), \) the conclusion of the theorem,
for the case in which \( E\{X(t)\}^\beta < \infty \) follows the dominated convergence theorem [Chung 1974, p. 42]. The conclusion of the theorem for the case in which 
\( E\{X(t)\}^\beta = \infty \) follows from (5.6).

**Theorem 5.7.** Let \( t \) be in \((0, \infty)\), \( \beta \) be in \([1, \infty)\), and let \( r \) be in the set \( \{1, \ldots, m\} \). If \( E\{X(t)\}^\beta < \infty \), then

\[
\lim_{n \to \infty} E\{X_r^n(t)\}^\beta = E\{X_r^\infty(t)\}^\beta.
\]

(5.8)

**Proof.** Let \( t \) be in \((0, \infty)\), \( \beta \) be in \([1, \infty)\), and let \( r \) be in the set \( \{1, \ldots, m\} \). Then \( \sup_n E\{X_r^n(t)\}^\beta \leq \sup_n E\{X^n(t)\}^\beta < \infty \). The conclusion of the theorem follows from corollary 5.4 and a well known result [Chung 1974, Th 4.5.2, p. 95].

In particular we obtain from the two preceding theorems, the following

**Corollary 5.8.** Let \( t \) be in \((0, \infty)\), and let \( r \) be in the set \( \{1, \ldots, m\} \). If

\( E\{X(t)\}^2 < \infty \), then the following four statements hold.

\[
\lim_{n \to \infty} E\{X^n(t)\} = E\{X(t)\}.
\]

(5.9)

\[
\lim_{n \to \infty} E\{X_r^n(t)\} = E\{X_r^\infty(t)\}.
\]

(5.10)

\[
\lim_{n \to \infty} \text{Var}\{X^n(t)\} = \text{Var}\{X(t)\}.
\]

(5.11)

\[
\lim_{n \to \infty} \text{Var}\{X_r^n(t)\} = \text{Var}\{X_r^\infty(t)\}.
\]

(5.12)

The results obtained by Billard, Lacayo and Langberg 1978 a,

select \( \mu(k; r) \) to be equal respectively to \( \alpha \) or \( \alpha_r \), \( k = 1, 2, \ldots, r = 1, \ldots, m \).

In these cases, \( X_r(t) \), \( r = 1, \ldots, m \), is distributed as a negative binomial random variable with parameters \((e^{-\alpha t}b, \alpha_r)\) or \((e^{-\alpha r}b, \alpha_r)\) respectively. [See Section 6].

Consequently, \( E\{X^2(t)\} < \infty \), and the results of the two cited references follow.
6. Formulas For The State Probabilities

Let $k \equiv (k_1, \ldots, k_m)$, and let $b \equiv (b_1, \ldots, b_m)$. In this section we present formulas for the joint state probabilities: $P_k^n(t) \equiv P\{X^n(t) = b+k\}$, $P_k(t) = P\{X(t) = b+k\}$, and for the marginal state probabilities: $P_{k,r}^n(t) \equiv P\{X^n_r(t) = b_r+k\}$, $P_{k,r}(t) \equiv P\{X_r(t) = b_r+k\}$. Those formulas are calculated without the traditional use of the differential equations associated with the state probabilities, which are presented in Section 3.

We note that the use of the formulas for $P_k^n(t)$, and $P_{k,r}^n(t)$ is restricted to the cases in which the initial number of "susceptibles" $n$, and the initial number of "infectives" with the different diseases $b_1, \ldots, b_m$, are known. It is quite conceivable that for large groups of "susceptibles" $n$ is unknown. However, in such situations one can approximate $P_k^n(t)$ and $P_{k,r}^n(t)$ by the respective state probabilities associated with the process $X(t)$, if one can obtain the values of $b_1, \ldots, b_m$.

Let $k \equiv (k_1, \ldots, k_m)$, $k = \sum_{q=1}^m k_q$, $B_k \equiv \{s=(s_1, \ldots, s_k) | s_1, \ldots, s_k \in \{1, \ldots, m\}$, and $\sum_{q=1}^m I(s_q = k_q, r=1, \ldots, m}$, $A_{k,r} \equiv \{e=(e_1, \ldots, e_m) | e_1, \ldots, e_m \in \{0, 1, \ldots\}$, and $\min_{e} = k_r\}$. $r = 1, \ldots, m$, and let $U_1, U_2, \ldots, U_t$ be an i.i.d sequence of exponential random variables with rates equal to 1. Then for $t$ in $(0, \infty)$, and for $r$ in the set $\{1, \ldots, m\}$, the following four equations given below hold.

\begin{align*}
(6.1) \quad P_k^n(t) &= \sum_{s \in \mathcal{B}_k} P\{S_k^n \leq t < S_{k+1}^n | \epsilon_q^n = s_q, q=1, \ldots, k\} \cdot P\{\epsilon_q^n = s_q, q=1, \ldots, k\}.
(6.2) \quad P_{k,r}^n(t) &= \sum_{e \in \mathcal{A}_{k,r}} P\{S_k^n \leq t < S_{k+1}^n | \epsilon_q^n = e_q, q=1, \ldots, m\} \cdot P\{\epsilon_q^n = e_q, q=1, \ldots, m\}.
(6.3) \quad P_{k,r}(t) &= \mathbb{P}\left\{\sum_{q=1}^{k_r} (b_r + q - 1) - 1 \mu^{-1} (b_r + q - 1; r) U_q < t < \sum_{q=k_r+1}^{k_r+1} (b_r + q - 1; r) U_q \right\}.
\end{align*}
(6.4) \( P_k(t) = \prod_{q=1}^{m} P_{k,q}(t) \).

Hence, to compute the state probabilities it suffices to evaluate

\[ P\left\{ \sum_{q=1}^{k} \frac{a^{-1}}{q} U_q \leq t < \sum_{q=1}^{k+1} \frac{a^{-1}}{q} U_q \right\}, \text{ for } k = 1, 2, \ldots, \]

where \( a_1, a_2, \ldots \) is a sequence of positive real numbers.

Let \( f_k(t) \), \( k = 1, 2, \ldots \), denote the density function of \( \sum_{q=1}^{k} \frac{a^{-1}}{q} W_q \). Then for \( t \in (0, \infty) \), and \( k = 1, 2, \ldots \),

\[
(6.5) \quad P\left\{ \sum_{q=1}^{k} \frac{a^{-1}}{q} U_q \leq t < \sum_{q=1}^{k+1} \frac{a^{-1}}{q} U_q \right\} = \int_{0}^{t} e^{-a_{k+1}(t-u)} f_k(u) \, du = \int_{0}^{\frac{t}{a_{k+1}}} f_{k+1}(u) \, du.
\]

Consequently, to compute the state probabilities it suffices to evaluate \( f_k(t) \), for \( k = 1, 2, \ldots \), and \( t \in (0, \infty) \). A formula for \( f_k(t) \) in the case where \( a_1, \ldots, a_k \), are arbitrary can be found in Box 1954, and in Billard, Lacyo, and Langberg 1978c. For the case in which \( a_1, \ldots, a_k \), are distinct an expression for \( f_k(t) \) can be found in Rényi 1970. If \( a_1, \ldots, a_k \), repeat in pairs, a situation typical to some epidemic models, a formula for \( f_k(t) \) can be found in Billard, Lacyo, and Langberg 1978b.

To obtain a formula for \( P_{k,r}(t) \), \( k = 0, 1, \ldots; r = 1, \ldots, m \), thus also for \( P_k(t) \), in the cases considered by Billard, Lacyo and Langberg 1978a, and by Lacyo and Langberg 1978, we evaluate \( f_k(t) \), \( k = 1, 2, \ldots \), for \( a_k = (k + a - 1) \), \( k = 1, 2, \ldots \), where \( a \) is a positive integer. Langberg 1978b proved that

\[
\sum_{q=1}^{k} \frac{a^{-1}}{q} U_q \quad \text{and} \quad Y_{a+k-1,k} \text{ are identically distributed, where } Y_{a+k-1,k} \text{ is the}
\]

\[
\frac{1}{Y_{a+k-1,k}} \quad \text{and} \quad Y_{a+k-1,k} \text{ are identically distributed, where } Y_{a+k-1,k} \text{ is the}
\]
$k$th order statistic in a sample of size $a + k - 1$ taken from an exponential population with rate to $1$. This result is reproven in Billard, Lacayo, and Langberg 1978. Consequently, it follows from (6.5) that for $t$ in $(0, \infty)$, and $r = 1, \ldots, m$, $X_r(t)$ is a negative binomial random variables with parameters $(e^{-at}, b_r)$ in the case considered by Billard, Lacayo and Langberg 1978, and is a negative binomial random variable with parameters $(e^{-\alpha t}, b_r)$ in the case considered by Lacayo and Langberg 1978.
On the Construction and Asymptotic Behaviour of m- Dimensional Simple Epidemic Models

by

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ABSTRACT

We consider a group of n susceptible individuals who are exposed to m contagious diseases. The progress of the epidemic among the individuals is modeled by a stochastic process $\mathbf{X}^n(t) = (X^n_1(t), \ldots, X^n_m(t))$. The components of $\mathbf{X}^n(t)$ describe the number of infected individuals with the respective diseases at time t.

For epidemics modeled by Markovian processes, we present formulas for the joint and marginal state probabilities associated with $\mathbf{X}^n(t)$. The method applied does not follow the traditional approach of solving differential equations. Rather, it utilizes available information about the distributions of sums of independent exponential random variables.

For a class of epidemics modeled by Markovian processes, and for every $t$ in $(0, \infty)$ we prove the convergence in law as $n \to \infty$ of the random vector $\mathbf{X}^n(t)$ to an identifiable random vector $\mathbf{X}(t)$. This result includes and extends the ones obtained by Billard, Lacayo and Langberg 1978, and by Lacayo and Langberg 1978.

Even though only epidemics described by Markovian processes are analyzed we construct stochastic process that can be used to model the progress of a variety of epidemics.

Key Words: m-dimensional simple epidemic models, exponential m-dimensional simple epidemic models, stochastic processes, exponential distributions, and convergence in law.
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