COHERENT STRUCTURE THEORY: A SURVEY

by

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ABSTRACT.

This is an expository survey of coherent structure theory, written at the invitation of the Editors, N. L. Johnson and S. Kotz, of the forthcoming Encyclopedia of Statistical Sciences. Coherent structure theory provides the foundation for reliability theory and has strong mathematical links with fault-tree analysis currently used by engineers in analyzing system safety and reliability. In addition to surveying (without proofs), basic results in coherent structure theory, the paper presents a very brief discussion of current research extending coherent structure theory (which considers only the two states of functioning and failed) to multistate coherent structure theory (which permits a finite or even infinite number of states, i.e., performance levels).
Coherent Structure Theory

Modern system reliability theory is based on coherent structure theory. In 1961, Birnbaum, Esary, and Marshall, inspired by the brilliant 2-part paper of Moore and Shannon (1956) on relay networks, published a paper laying the foundations of coherent structure theory. The main idea of their paper was to show that practically all engineering systems could be treated in a simple, unified fashion in determining the probability of system functioning in terms of the probabilities of functioning of the components.

Since the publication of this basic paper, some of the definitions have been changed slightly and some of their results have been proven in a different fashion. We summarize the theory using the most recent definitions. A comprehensive discussion of the theory along with a discussion of the key references is presented in Barlow and Proschan (1975); moreover, the intimate connection between coherent structure theory and fault-tree analysis is brought out in the Appendix of the book.

To define a coherent structure having n components, we first indicate the state $x_i$ of component i, setting $x_i = 1$ if component i is functioning and $x_i = 0$ if component i is failed, $i = 1, \ldots, n$. Similarly, the corresponding state $\phi$ of the system is 1 if the system is functioning and 0 if the system is failed. Since the state of the system is determined completely by the states of the components, we write $\phi = \phi(x)$, where $x = (x_1, \ldots, x_n)$; $\phi(x)$ is known as the structure function of the system.

Very few systems are designed with irrelevant components. Component i is irrelevant to the structure $\phi$ if $\phi$ does not really depend on $x_i$; i.e., $\phi(x)$ is constant in $x_i$ for each of the $2^{n-1}$ possible combinations of outcomes.
of the remaining components of the system. Otherwise component $i$ is relevant to the structure.

1. **Definition.** A system of components is coherent if (a) its structure function is nondecreasing in each argument, and (b) each component is relevant.

Requirement (a) states essentially that replacing a failed component by a functioning component will not cause a functioning system to fail—a reasonable requirement. Requirement (b) rules out trivial systems not encountered in engineering practice.

From this deceptively simple definition, a wealth of theoretical results may be derived, many of which yield fruitful applications in reliability practice.

Some basic coherent systems are: (a) A *series* system of $n$ components—the structure function is $\phi(x) = \prod_{1}^{n} x_i \equiv \min(x_1, \ldots, x_n)$, (b) a *parallel* system of $n$ components—the structure function is $\phi(x) = \prod_{i=1}^{n} x_i \equiv \max(x_1, \ldots, x_n)$, (c) A *k-out-of-n-system*; the structure function is $\phi(x) = 1$ if $\sum_{1}^{n} x_i > k$, and 0 if $\sum_{1}^{n} x_i < k$. Note that the series (parallel) system is a special case of the k-out-of-n system with $k = n(1)$. (d) A parallel-series (series-parallel) system; the system consists of a parallel (series) arrangement of series (parallel) subsystems.

Fig. 1. **Diagrammatic Representation of Basic Systems.**

(a) Series system: \[1-2- \ldots -n-\]

(b) Parallel system: \[2-\]

\[\vdots\]

\[n: \]
(c) 2-out-of-3 system (Special case of a k-out-of-n system):

\[
\begin{bmatrix}
1 - 2^n \\
1 - 3 \\
2 - 3^n
\end{bmatrix}
\]

(Note the replication of identical components.)

(d) A parallel-series system:

\[
\begin{array}{c}
1 \quad 2 \quad \ldots \quad n \\
(n_1 + 1) - (n_1 + 1) - \ldots - (n_1 + n_2) \\
(n_1 + \ldots + n_{k-1} + 1) - \ldots - (n_1 + \ldots + n_k)
\end{array}
\]

The structure function of every coherent system is bounded below by the structure function of the series system formed from its components and bounded above by the structure function of the parallel system formed from its components. Stated formally, we have:

2. **Theorem.** Let \( \phi \) be a coherent system of \( n \) components. Then

\[
\prod_{i=1}^{n} x_i \leq \phi(x) \leq \bigcup_{i=1}^{n} x_i.
\]

Design engineers have long followed the rule: **Redundancy at the component level is superior to redundancy at the system level.** Coherent system theory proves the corresponding

3. **Theorem.** Let \( \phi \) be a coherent system. Then \( \phi(x_1 \sqcup y_1^\dagger, \ldots, x_n \sqcup y_n) \geq \phi(x) \sqcup \phi(y) \). Equality holds for all \( x \) and \( y \) if and only if the structure is parallel.

\[^\dagger x \sqcup y \text{ denotes } 1 - (1 - x)(1 - y).]
The variety of types of coherent systems is very large, especially for large \( n \): Thus it is reassuring to know that every coherent system may be represented as a parallel-series system and as a series-parallel system if replication of components is permitted. (A small scale example of this general result is shown in Fig. 1(c).) These representation results not only conceptually simplify the theory of coherent systems; they also yield simple upper and lower bounds on coherent system reliability, as we shall see shortly.

To describe these representations, we need some terminology and notation.

A minimal (min) path set of a coherent structure is a set of components satisfying (a) if each component in the set functions, the system functions, (b) if all remaining components fail and any one or more of the components of the min path set fails, then the structure fails. The corresponding min path series structure is the series structure formed from the components of the min path set. Given a coherent structure \( \phi \) with \( p \) min paths denote the \( i \)th min path series structure function by \( \rho_i(x) \), \( i = 1, \ldots, p \). The min path representation is given by

\[
\phi(x) = \prod_{i=1}^{p} \rho_i(x),
\]

corresponding to a parallel arrangement of the \( p \) min path series structures.

For example, the 2-out-of-3 system has \( p = 3 \) min path series structures:

\[
\rho_1(x) = x_1x_2, \quad \rho_2(x) = x_1x_3, \quad \rho_3(x) = x_2x_3.
\]
The min path representation is:

\[ \phi_{2|3}(x) = \prod_{i=1}^{3} \rho_i(x), \]

diagrammatically displayed in Fig. 1 (c). Note that each of the components appears twice.

Next we develop the dual min cut representation of a coherent structure. A min cut set is a set of components satisfying (a) if each component in the min cut set fails, the system fails, (b) if all remaining components function and one or more of the components in the min cut set function, then the structure functions. The corresponding min cut parallel structure is the parallel structure formed from the components in the min cut set.

Given a coherent structure \( \phi \) with \( k \) min cuts, denote the \( i^{th} \) min cut parallel structure function by \( \kappa_i(x) \), \( i = 1, \ldots, k \). The min cut representation is given by

\[
\phi(x) = \prod_{i=1}^{k} \kappa_i(x),
\]

(3)

Corresponding to a series arrangement of the \( k \) min cut parallel structures.

In the 2-out-of-3 system,

\[
\kappa_1(x) = x_1 \lor x_2 \quad \kappa_2(x) = x_1 \lor x_3 \quad \kappa_3(x) = x_2 \lor x_3.
\]

The min cut representation is

\[ \phi_{2|3}(x) = \prod_{i=1}^{3} \kappa_i(x). \]

The diagram of the min cut representation is:
Again, each component appears twice. Note further that the diagram just above and the diagram in Fig. 2 (c) are alternative representations of the same system.

Thus far we have confined our discussion to the deterministic aspects of coherent systems. Next we summarize the probabilistic properties of coherent systems. These properties are directly relevant to the prediction (probabilistically) and the estimation (statistically, i.e., from data) of system reliability.

Assume first that component states, \( X_1, \ldots, X_n \), are random, but statistically independent. Thus let \( X_i \) be a Bernouilli random variable indicating the state of component \( i \):

\[
X_i = \begin{cases} 
1 & \text{(component } i \text{ is functioning) with probability } p_i \\
0 & \text{(component } i \text{ is failed) with probability } q_i = 1 - p_i, 
\end{cases}
\]

where \( p_i (0 \leq p_i \leq 1) \) is called the reliability of component \( i \), \( i = 1, \ldots, n \).

The corresponding system reliability \( h \) is given by

\[
h = P[\phi(X) = 1] \equiv E\phi(X).
\]

Since component states \( X_1, \ldots, X_n \) are mutually independent, system reliability \( h \) is completely determined by component reliabilities \( p_1, \ldots, p_n \); thus, we define \( h = h(p) \), where \( p = (p_1, \ldots, p_n) \). In the special case of interest, \( p_1 = p_2 = \ldots = p_n = p \), we write \( h(p) \). We call \( h(p) \) (or \( h(p) \)) the reliability function; it expresses system reliability as a function of component reliabilities (common component reliability).
As examples, for the series system, \( h(p) = \prod_{i=1}^{n} (1 - p_i) \); and for the k-out-of-n system with common component reliability \( p \), \( h(p) = \sum_{i=k}^{n} p^i (1 - p)^{n-i} \), the binomial right hand tail.

Basic properties of the reliability functions \( h(p) \) and \( h(p') \) are:

(a) \( h(p) \) is multilinear in \( p_1, \ldots, p_n \).
(b) \( h(p) \) is a polynomial in \( p \), with all coefficients nonnegative.
(c) \( h(p) \) is strictly increasing in each \( p_i \) on the domain \( 0 < p_i < 1 \), \( i = 1, \ldots, n \).
(d) \( h(p_1, \ldots, p_n) = h(p_1', \ldots, p_n') \geq h(p) = h(p') \).

Equality holds for all \( p \) and \( p' \) if and only if the system is parallel.

Property (d) states that redundancy at the component level yields higher system reliability than redundancy at the system level. This is the probabilistic version of Theorem 3, which gives the deterministic version of this familiar design engineer's rule.

A basic problem is to compute system reliability in terms of component reliabilities. Alternative exact methods are:

(a) By means of min cut and min path representations, based on (2) and (3):

\[
(4) \quad h(p) = E \prod_{i=1}^{p} \prod_{i \in P_j} x_i \equiv E \prod_{j=1}^{k} \prod_{i \in k_j} x_i,
\]

where \( P_j \) denotes the \( j \)th min path set and \( k_j \) the \( j \)th min cut set.

(b) By examining all \( 2^n \) possible outcomes of the \( n \) components and using the obvious formula:

\[
(5) \quad h(p) = \sum_{X} \phi(X) \prod_{i} p_i^{x_i} q_i^{1-x_i},
\]
the summation being taken over all $2^n$ vectors $x$ with 0 or 1 coordinates.

(c) By using the special features of certain types of systems: (c-1) Many systems are designed to consist of distinct subsystems which in turn consist of distinct sub-subsystems, etc. By computing the reliability of each of the lowest level groupings, by then computing the reliability of each of the next level groupings from the lowest level groupings, etc., it becomes possible to compute system reliability.

(c-2) If the components have common reliability $p$, we may use the formula:

$$h(p) = \sum_{i=1}^{n} A_i \binom{n}{i} p^i (1-p)^{n-i},$$

where $A_i$ denotes the number of vectors $x$ for which $\phi(x) = 1$.

(c-3) By eye, for simple or small systems, using basic probability rules.

Clearly there is a need for approximations and bounds for system reliability, since the exact computations for large systems can become formidable or even intractible. Next we list bounds and methods for obtaining them, some of which apply even when component states may be mutually statistically dependent.

(a) The inclusion-exclusion method: Let $E_r$ be the event that all components in min path set $P_r$ work. Then $P[E_r] = \prod_{i \in P_r} p_i$. System success corresponds to the event $\bigcup_{r=1}^{P} E_r$. Thus $h = P[\bigcup_{r=1}^{P} E_r]$. Let

$$S_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq P} P[E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}].$$
By the inclusion-exclusion principle (Feller, 1968, pp. 98-101),

\[ h = \sum_{i=1}^{p} (-1)^{i-1} \text{S}_{k}, \]

and

\[ h \leq \text{S}_{1} = \sum_{r=1}^{p} \prod_{i \in \text{P}_{r}} p_{i}, \]

\[ h \geq \text{S}_{1} - \text{S}_{2}, \]

\[ h \leq \text{S}_{1} - \text{S}_{2} + \text{S}_{3}, \]

\[ h \geq \text{S}_{1} - \text{S}_{2} + \text{S}_{3} - \text{S}_{4}, \]

and so on. Although it is not true that the successive upper bounds decrease necessarily and the successive lower bounds increase necessarily, in practice it may be necessary to calculate only a few \( \text{S}_{k} \)'s to obtain a close approximation.

(b) Bounds for series and parallel systems: If \( X_{1}, ..., X_{n} \) are associated \(^{†}\) random indicators of the respective component states, then

\[ (7) \quad \prod_{i=1}^{n} P(X_{i} = 1) \geq \prod_{i=1}^{n} P[X_{i} = 1] \]

\[ (8) \quad \prod_{i=1}^{n} P[X_{i} = 1] \leq \prod_{i=1}^{n} P[X_{i} = 1]. \]

Note that (7) ((8)) states that the reliability of a series (parallel) systems of positively dependent components is bounded below (above) by system reliability computed under the assumption of independent components.

\(^{†}\)See the discussion of "associated random variables" in this Encyclopedia.
(c) Crude bounds for coherent systems: Let \( \phi \) be a coherent structure of associated components with respective reliabilities \( p_1, \ldots, p_n \). Then

\[
\prod_{i=1}^{n} p_i \leq P[\phi(X) = 1] \leq \prod_{i} p_i.
\]

These bounds are generally rather crude since each arises as the result of two successive bounding operations.

Next we present improved bounds on system reliability using additional information, namely, the minimal path and minimal cut representation of the structure given in (2) and (3) respectively.

(d) Bounds for coherent structures based on min path and min cut sets:

(d-1) Let \( \phi \) be a coherent structure of associated components. Let

\( \rho_1(x), \ldots, \rho_k(x) \) be the minimal path series structures, and \( \kappa_1(x), \ldots, \kappa_k(x) \)

be the minimal cut parallel structures of \( \phi \). Then

\[
\prod_{j=1}^{k} P[\kappa_j(x) = 1] \leq P[\phi(x) = 1] \leq \prod_{j=1}^{k} P[\rho_j(x) = 1].
\]

(d-2). If components are independent, then the bounds become more explicit:

\[
\prod_{j=1}^{k} \prod_{i \in \kappa_j} p_i \leq P[\phi(x) = 1] \leq \prod_{j=1}^{k} \prod_{i \in \rho_j} p_i.
\]

(e) Min-max bounds for coherent structures: (e-1) Regardless of

the joint distribution of component states, the following bounds hold:

\[
\max_{1 \leq r \leq p} P[\min_{i \in \rho_r} X_i = 1] \leq P[\phi(X) = 1] \leq \min_{1 \leq s \leq k} P[\max_{i \in \kappa_s} X_i = 1].
\]

(e-2) If components are associated, then the more explicit bounds hold:

\[
\max_{1 \leq r \leq p} \prod_{i \in \rho_r} p_i \leq P[\phi(X) = 1] \leq \min_{1 \leq s \leq k} \prod_{i \in \kappa_s} p_i.
\]
Next we examine system reliability $h(p)$ as a function of common component reliability $p$. We find that for the case of independent components, the reliability $h(p)$ of a coherent system without path sets or cut sets of size 1 is an S-shaped function of $p$; i.e., there exists a value $p_0$, $0 < p_0 < 1$, such that $h(p_0) = p_0$, $h(p) \leq p$ for $0 < p \leq p_0$, and $h(p) \geq p$ for $p_0 \leq p \leq 1$. This represents a generalization of a similar result obtained in the fundamental 2-part paper of Moore and Shannon (1956) on relay networks.

The practical implications of the S-shapedness of $h(p)$ are as follows:

(a) For a coherent system with redundancy, when component reliability $p$ is high, system reliability $h(p)$ is even higher.

(b) When component reliability is sufficiently low, system reliability is even lower.

(c) Relay networks and safety systems are subject to two types of failure to respond properly. A well designed relay network or safety system has considerably lower risk of improper response of either type than does a single relay in the case of a relay network or a single component in the case of a safety system.

Coherent structure theory is currently being generalized so that instead of only the functioning and failed states being possible for both components and system, a finite or even infinite number of states are now possible. These states correspond to levels of performance of component and system. Different axiomatic treatments are presented by El-Neweihi, Proschan, and Sethuraman (1978), Barlow (1978), and Ross (1979). This current research on multistate coherent systems represents just the initial phase of a flood of research to come.
References.


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