ASYMPTOTIC JOINT DISTRIBUTION OF RATIOS
OF SAMPLE QUANTILES TO SAMPLE MEAN

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FSU Statistics Report M492

February, 1979
The Florida State University
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Tallahassee, Florida 32306
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SUMMARY

Based on a sample from an absolutely continuous distribution \( F \) with density \( f \), and with the aid of the Bahadur (Ann. Math. Statist. 37 (1966), 577-580) representation of sample quantiles, the asymptotic joint distribution of three statistics, the sample \( p^{th} \) and \( q^{th} \) quantiles \( (0 < p < q < 1) \) and the sample mean, is obtained. Using the Cramér-Wold device, asymptotic distributions of functions of the three statistics can be derived. In particular, the asymptotic joint distribution of the ratio of sample \( p^{th} \) quantile to sample mean and the ratio of sample \( q^{th} \) quantile to sample mean is presented. Finally, consistent estimators are proposed for the variances and covariances of these limiting distributions.

AMS 1970 Subject Classifications: 62H10; 62E20

Key words and phrases: Absolutely continuous; Bahadur representation of sample quantiles; consistent estimators; Cramér-Wold device; kernel estimate of density; median as a percentage of the mean; nonparametric.
1. INTRODUCTION

The mean, if it exists, and the median are two most commonly recognized location parameters of a univariate population. The two parameters generally differ from each other unless the population is symmetric. To study the extent of the difference between the two parameters, such as in the case of (extremely) skewed populations, an important and meaningful characteristic to use is the ratio of median to mean, if the mean differs from zero. For example, in assessing a property value for tax purposes, one of the methods used by assessors is the median of the assessment—sale ratio. The median as a percentage of the mean is a criterion which provides information on "over assessment" or "under assessment" as compared to the mean. In this situation, the ratio of median to mean plays a key role in settling disputes which might exist between property owners and assessors.

In this paper we obtain the bivariate asymptotic distribution of the ratios of the sample $p^{th}$ quantile to sample mean and sample $q^{th}$ quantile to sample mean. Inference on the difference between the ratios of the $q^{th}$ quantile to mean and $p^{th}$ quantile to mean, or on the ratio of the $p^{th}$ quantile to mean may be performed based on this asymptotic distribution. Note that these ratios are scale-free. The inference is nonparametric in nature since no assumptions are made on the underlying population except that it is absolutely continuous satisfying very mild regularity conditions.

More specifically, let $F$ be an absolutely continuous distribution with density $f$, mean $\mu$, and finite variance $\sigma^2$. Denote by $\xi_p$ the unique $p^{th}$ quantile, $F(\xi_p) = p$, $0 < p < 1$. The following assumptions on $F$ will be made throughout the study:

(A.1) $F'(\xi_p) = f(\xi_p) > 0$, and

(A.2) $F''$ exists and is bounded in a neighborhood $N_p$ of $\xi_p$. 
Based on a random sample \( X_1, \ldots, X_n \) from \( F \), let \( \bar{X} \) be the sample mean and 
\( \hat{\xi}_p = \hat{\xi}_p(X_1, \ldots, X_n) \) the sample \( p \)th quantile, defined by

\[
\hat{\xi}_p = X([np]+1), \quad \text{if } np \neq [np]
\]

\[
= t, \quad X_{(np)} \leq t \leq X_{(np+1)}, \quad \text{if } np = [np],
\]

where \([a]\) stands for the integral part of \(a\) and \(X_{(r)}\) denotes the \(r\)th order statistic. In the next section, the asymptotic joint distribution of 
\((\hat{\xi}_p, \hat{\xi}_q, \bar{X})\) will be obtained, from which the asymptotic distribution of 
\((\hat{\xi}_p/\bar{X}, \hat{\xi}_q/\bar{X})\) is derived. Consistent estimators for the variances and covariances of the asymptotic distributions are proposed in Section 3.

2. ASYMPOTIC DISTRIBUTIONS.

Various methods are available for deriving the well-known asymptotic
distribution of \(\hat{\xi}_p\). See, e.g., Rao (1973), p. 423. In the following
theorem we will obtain the asymptotic joint distribution of 
\((\hat{\xi}_p, \hat{\xi}_q, \bar{X})\), 
\(0 < p < q < 1\), utilizing the Bahadur (1966) representation of \(\hat{\xi}_p\), viz.,

\[
\hat{\xi}_p = \xi_p + [1 - F_n(\xi_p) - (1 - p)]/f(\xi_p) + R_{n,p},
\]

where \(F_n\) is the empirical distribution function and \(R_{n,p}\) is a remainder term, \(0 < p < 1\). For ease of notation, define

\[
\xi_p = \int_0^\infty xf(x)dx, \quad 0 < p < 1.
\]
THEOREM 2.1. Let $X_1, \ldots, X_n$ be a random sample from an absolutely continuous distribution $F$ with density $f$ satisfying assumptions (A.1) and (A.2). Then

$$n^4(\hat{\xi}_p - \xi_p, \hat{\xi}_q - \xi_q, \bar{X} - \mu, \bar{Y} - \mu) \overset{L}{\longrightarrow} N_4(\mathbf{0}, \Sigma), \quad n \to \infty,$$

where $\mathbf{Q}_4 = (0, 0, 0, 0)'$ and $\Sigma = (\sigma_{ij})$ with

$$\sigma_{11} = p(1-p)/[f(\xi_p)]^2, \quad \sigma_{12} = p(1-q)/[f(\xi_p)f(\xi_q)],$$

$$\sigma_{22} = q(1-q)/[f(\xi_q)]^2, \quad \sigma_{13} = [\xi_p - \mu(1-p)]/f(\xi_p),$$

$$\sigma_{33} = \sigma^2, \quad \sigma_{23} = [\xi_q - \mu(1-q)]/f(\xi_q).$$

PROOF. For $i = 1, \ldots, n$, and $0 < p < q < 1$, define

$$\mathbf{W}_i = (I(X_i \geq \xi_p), I(X_i \geq \xi_q), X_i)',$$

where $I(.)$ denotes the usual indicator function. Then $\mathbf{W}_1, \ldots, \mathbf{W}_n$ are independent and identically distributed random vectors with mean $\mathbf{\gamma} = (1-p, 1-q, \mu)'$ and covariance matrix $\Gamma = (\gamma_{ij})$, where

$$\gamma_{11} = p(1-p), \quad \gamma_{12} = p(1-q),$$

$$\gamma_{22} = q(1-q), \quad \gamma_{13} = \xi_p - \mu(1-p),$$

$$\gamma_{33} = \sigma^2, \quad \gamma_{23} = \xi_q - \mu(1-q).$$

It follows from a multivariate central limit theorem (see, e.g., Rao (1973), p. 128) that
\[
\begin{bmatrix}
1 - F_n(\xi_p) - (1 - p) \\
1 - F_n(\xi_q) - (1 - q) \\
\bar{X} - \mu
\end{bmatrix} = n^{-\frac{1}{2}} \sum_{i=1}^{n} (W_i - \chi)
\]

\[
\frac{L}{N_3(\phi_3, \Gamma)}, \quad \text{as } n \to \infty.
\]

Now, applying the Bahadur (1966) representation for \(\hat{\xi}_p\) and \(\hat{\xi}_q\), we have

\[
\begin{bmatrix}
\hat{\xi}_p - \xi_p - R_{n,p} \\
\hat{\xi}_q - \xi_q - R_{n,q} \\
\bar{X} - \mu
\end{bmatrix} \xrightarrow{L} N_3(\phi_3, \Sigma), \quad \text{as } n \to \infty.
\]

The theorem is established by noting a result of Ghosh (1971), i.e.,

\[
n^{\frac{1}{2}} R_{n,p} \to 0, \quad \text{in probability, as } n \to \infty, \quad \text{for any } p, 0 < p < 1.
\]

Asymptotic distributions of functions of \(\hat{\xi}_p\), \(\hat{\xi}_q\), and \(\bar{X}\) may be obtained using the Cramér-Wold device. We will present, in the following theorem, the limiting distribution of one such function which has important applications.

**Theorem 2.2.** Let the assumptions of Theorem 2.1 be satisfied and assume \(\mu \neq 0\). Then

\[
\begin{bmatrix}
\hat{\xi}_p/\bar{X} - \xi_p/\mu \\
\hat{\xi}_q/\bar{X} - \xi_q/\mu
\end{bmatrix} \xrightarrow{L} N_2(\phi_2, \Lambda), \quad \text{as } n \to \infty,
\]
where \( \mathbf{q}_2 = (0, 0)' \) and \( \Lambda = (\lambda_{ij}) \) with

\[
\lambda_{11} = \frac{[(\xi_p/\mu)^2 \sigma^2 - 2(\xi_p/\mu)\sigma_{13} + \sigma_{11}]/\mu}{\mu^2}
\]

\[
\lambda_{12} = \frac{[(\xi_p/\mu)^2 \sigma^2 - (\xi_p/\mu)\sigma_{23} - (\xi_q/\mu)\sigma_{13} + \sigma_{12}]/\mu}{\mu^2}
\]

\[
\lambda_{22} = \frac{[(\xi_q/\mu)^2 \sigma^2 - 2(\xi_q/\mu)\sigma_{23} + \sigma_{22}]/\mu}{\mu^2}
\]

and the \( \sigma_{ij} \)'s are given in Theorem 2.1.

**PROOF.** Let \( \mathbf{h}(u_1, u_2, u_3) = (u_1/u_3, u_2/u_3) = (h_1, h_2) \), say. Then \( \mathbf{h} \) is totally differentiable and

\[
H = \left( \begin{array}{c}
\frac{\partial h_1}{\partial u_1} \\
\frac{\partial h_1}{\partial u_2}
\end{array} \right) = \left( \begin{array}{ccc}
1/u_3 & 0 & -u_1/u_3^2 \\
0 & 1/u_3 & -u_2/u_3^2
\end{array} \right).
\]

It is easily seen that \( \Lambda = HH' \). The theorem follows immediately by an application of the Cramér-Wold device (see, e.g., Rao (1973), p. 388).

An interesting special case of Theorem 2.2 is the limiting distribution of \( \hat{\xi}_q/\overline{X} \) which may be used for inference on \( \xi_q/\mu \). An important application of this distribution is clear as discussed in Section 1.

Limiting distributions of many other functions of \( \hat{\xi}_p, \hat{\xi}_q, \) and \( \overline{X} \) may be obtained using the approach of Theorem 2.2, but they will not be derived here.
3. CONSISTENT ESTIMATORS OF VARIANCES AND COVARIANCES.

Variances and covariances of the limiting distributions obtained in Section 2 depend on the parameters $\xi_p$, $\mu$, $f(\xi_p)$, and $\xi_p$. We need consistent estimators of these parameters for the purpose of making inferences in which the limiting distributions are utilized. It is well known that $\hat{\xi}_p$ and $\hat{\mu}$ are strongly consistent for $\xi_p$ and $\mu$, respectively; we now propose consistent estimators for $f(\xi_p)$ and $\xi_p$.

Let $k(u)$ be a kernel function defined on the real line $\mathbb{R}$ such that

$$\sup_{u \in \mathbb{R}} |k(u)| < \infty,$$

$$\lim_{u \to \infty} |uk(u)| = 0, \quad \text{and}$$

$$\int_{-\infty}^{\infty} k(u) du = 1.$$

For example, the standard normal and double exponential densities are two such functions. Let $\{a_n\}$ be a sequence of nonnegative constants converging to 0 as $n \to \infty$. Then, based on a random sample $X_1, \ldots, X_n$, a kernel estimate of $f(x)$, for fixed $x$, is given by

$$\hat{f}(x) = (na_n)^{-1} \sum_{i=1}^{n} k[(x - X_i)/a_n].$$

This estimate was proposed by Rosenblatt (1956) and later studied by Parzen (1962) and Nadaraya (1965), among others. It is natural to propose $\hat{f}(\xi_p)$ as a consistent estimator for $f(\xi_p)$, $0 < p < 1$; the following theorem establishes its strong consistency.
THEOREM 3.1. Let \( X_1, \ldots, X_n \) be a sample from \( F \) satisfying assumptions (A.1) and (A.2). Let \( k(u) \) be a kernel function of bounded variation and assume, for any \( \varepsilon > 0 \),

\[
\sum_{n=1}^{\infty} \exp(-\varepsilon n^2) < \infty.
\]

Then, for any \( p, 0 < p < 1 \),

\[
\hat{f}(\xi_p) \rightarrow f(\xi_p) \quad \text{w.p.1 as } n \rightarrow \infty.
\]

PROOF. It is clear that

\[
|\hat{f}(\xi_p) - f(\xi_p)| \leq |\hat{f}(\hat{\xi}_p) - f(\hat{\xi}_p)| + |f(\hat{\xi}_p) - f(\xi_p)|
\]

\[
\leq \sup_{x \in \mathbb{R}} |f(x) - f(x)| + |\hat{\xi}_p - \xi_p| \cdot \sup_{x \in \mathbb{R}} |f'(x)|
\]

\[
= A + B, \quad \text{say.}
\]

As \( n \rightarrow \infty \), \( A \rightarrow 0 \) w.p.1 by Nadaraya (1965), and \( B \rightarrow 0 \) w.p.1, since \( \hat{\xi}_p \rightarrow \xi_p \) w.p.1 and (A.2) is assumed.

From the above proof it is obvious that any statistic \( f_n(\hat{\xi}_p) \), say, for which \( \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \rightarrow 0 \) w.p.1 as \( n \rightarrow \infty \), is a strongly consistent estimator of \( f(\xi_p) \).

As for \( \xi_p \) we propose the following estimator: Let

\[
\hat{\xi}_p = \int_{\xi_p}^{\infty} x dF_n(x) = (1/n) \sum_{i=1}^{n} X_i I(X_i > \hat{\xi}_p).
\]

To show the consistency of \( \hat{\xi}_p \) we will need the following lemma, which is
also of independent interest.

**Lemma 3.2.** Let \( X_1, \ldots, X_n \) be independent and identically distributed random variables with common distribution \( F \) and density \( f \). Assume that \( E|X_1|^\delta < \infty \) for some \( \delta > 2 \). Then

\[
\sup_{t \in \mathbb{R}} \left| \frac{1}{n} \sum_{i=1}^{n} X_i 1(X_i \geq t) - \int_t^\infty x f(x) dx \right| \longrightarrow 0 \quad \text{w.p.1,}
\]

as \( n \to \infty \).

**Proof.** Note that, for any \( t \in \mathbb{R} \) and any \( c > 0 \),

\[
\left| \int_t^\infty x dF_n(x) - \int_t^\infty x dF(x) \right| \\
\leq \left| \int_t^\infty x dF_n(x) - \int_t^\infty x 1(|x| \leq c) dF_n(x) \right| \\
+ \left| \int_t^\infty x 1(|x| \leq c) dF_n(x) - \int_t^\infty x 1(|x| \leq c) dF(x) \right| \\
+ \left| \int_t^\infty x dF(x) - \int_t^\infty x 1(|x| \leq c) dF(x) \right| \\
= A_1(t) + A_2(t) + A_3(t), \quad \text{say.}
\]

The lemma will be established by showing, for \( i = 1, 2, \) and 3,

\[
\sup_{t \in \mathbb{R}} A_i(t) \longrightarrow 0 \quad \text{w.p.1, as } n \to \infty.
\]

Now, define \( c_i = i^{1/\delta} \), \( i = 1, \ldots, n \), and set \( c = c_n \). Since \( |E_{X_1}|^\delta < \infty \) for some \( \delta > 2 \), it follows that \( \sum_{i=1}^{n} P(|X_1| \geq i^{1/\delta}) < \infty \),

which in turn implies that

\[
\sup_{t \in \mathbb{R}} A_1(t) \leq \sup_{t \in \mathbb{R}} \int_{t}^{\infty} |x| [1 - I(|x| \leq c)] dF_n(x)
\]

\[= \int_{-\infty}^{\infty} |x| [1 - I(|x| \leq c)] dF_n(x)\]

\[= \left(\frac{1}{n}\right) \sum_{i=1}^{n} \left[ |X_i| - |X_i| I(|X_i| \leq c) \right]\]

\[\leq \left(\frac{1}{n}\right) \sum_{i=1}^{n} \left[ |X_i| - |X_i| I(|X_i| \leq c) \right]\]

\[\rightarrow 0 \quad \text{w.p.l.},\]

as \(n \to \infty\), by Theorems 3.2.1 and 5.2.1 of Chung (1974).

To show that \(\sup_{t \in \mathbb{R}} A_2(t) \to 0 \text{ w.p.l as } n \to \infty\), we need to consider three cases: \(t > c\), \(|t| \leq c\), and \(t < -c\).

(i) When \(t > c\), \([t, \infty) \cap [-c, c] = \emptyset\), the empty set, and \(A_2(t) = 0 \text{ w.p.l}, \) for all \(t > c\) and \(n \geq 1\).

(ii) When \(|t| \leq c\), \([t, \infty) \cap [-c, c] = [t, c]\) and, upon integration by parts, we have

\[A_2(t) \leq c |F_n(c) - F(c)| + |t| |F_n(t) - F(t)|\]

\[+ (c - t) \sup_{x \in \mathbb{R}} |F_n(x) - F(x)|\]

\[\leq 4c \cdot \sup_{x \in \mathbb{R}} |F_n(x) - F(x)|.\]

Since \(c = n^{1/\delta} \), \(\delta > 2\), it follows that \(\sup_{t \in \mathbb{R}} A_2(t) \to 0 \text{ w.p.l as } n \to \infty\).
(iii) When \( t < -c \), \( [t, \infty) \cap [-c, c] = [-c, c] \). The desired result follows upon integration by parts as in Case (ii).

It remains to show that \( \sup_{t \in \mathbb{R}} A_3(t) \rightarrow 0 \) w.p.1 as \( n \rightarrow \infty \). To this end we note that

\[
\sup_{t \in \mathbb{R}} A_3(t) = \sup_{t \in \mathbb{R}} \left| \int_{t}^{\infty} x dF(x) - \int_{t}^{\infty} x I(|x| \leq c) dF(x) \right|
\]

\[
\leq \int_{-\infty}^{\infty} x |1 - I(|x| \leq c)| dF(x)
\]

\[
= \int_{-\infty}^{-c} x dF(x) + \int_{c}^{\infty} x dF(x)
\]

\[
\leq \left( \frac{1}{c} \right) \int_{-\infty}^{\infty} x^2 dF(x)
\]

\[\longrightarrow 0 \quad \text{as} \quad n \rightarrow \infty,\]

since \( E|X_i|^2 < \infty \) and \( c = n^{1/\delta}, \delta > 2. \)

The next theorem establishes the consistency of \( \hat{\zeta}_p \).

**Theorem 3.3.** Assume the conditions of Lemma 3.2 and \( \sup_{x \in \mathbb{N}_p} |xf(x)| < \infty. \)

Then, as \( n \rightarrow \infty \),

\[\hat{\zeta}_p \longrightarrow \zeta_p \quad \text{w.p.1.}\]

**Proof.** As in the proof of Theorem 3.1, we have

\[
|\hat{\zeta}_p - \zeta_p| \leq \left| (1/n) \sum_{i=1}^{n} X_i I(X_i \geq \hat{\xi}_p) - \int_{\hat{\xi}_p}^{\infty} xf(x) dx \right|
\]

\[\hspace{2cm} + \left| \int_{\hat{\xi}_p}^{\infty} xf(x) dx - \int_{\xi_p}^{\infty} xf(x) dx \right|\]
\[ \leq \sup_{t \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} X_i I(X_i \geq t) - \int_{t}^{\infty} xf(x)dx \]

\[ + |\xi_p - \xi_p| \sup_{x \in \mathbb{N}_p} |xf(x)| \]

\[ \rightarrow 0 \text{ w.p.l.} \]

as \( n \to \infty \), by Lemma 3.2 and the additional assumption.

ACKNOWLEDGEMENT.

The authors would like to acknowledge helpful conversation with Kuang-Fu Cheng.
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