CONTRIBUTIONS TO NONPARAMETRIC GENERALIZED
FAILURE RATE FUNCTION ESTIMATION

by.

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ABSTRACT

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For a specified distribution function G with density g, and unknown
distribution function F with density f, the generalized failure rate func-
tion \( \gamma(x) = f(x)/gG^{-1}F(x) \) may be estimated by replacing f and F by \( \hat{f}_n \) and
\( \hat{F}_n \), where \( f_n \) is an empirical density function based on a sample of size
n from distribution function F, and \( F_n(x) = \int_{-\infty}^{x} f_n(t) \, dt \). Under regularity
conditions we show \( \sup_{x \in \mathbb{C}} |\gamma_n(x) - \gamma(x)| \overset{\text{w.p.} 1}{\rightarrow} 0 \), and, under additional re-
strictions \( \sup_{x \in \mathbb{C}} |\gamma_n(x) - \gamma(x)| \overset{\text{w.p.} 1}{\rightarrow} o(n^{-1/3} \beta_n \log n) \), where \( \mathbb{C} \) is a sub-
set of \( \mathbb{R} \) and \( \beta_n \to 0 \). Moreover, asymptotic normality is derived and the
Berry-Esseen rate \( \sup_{t \in \mathbb{R}} |P\left( n^{1/2} (\gamma_n(x) - \gamma(x) - \mu_n(x, f))/\sigma_n(x, f) \leq t \right) - \phi(t) | = O(n^{-1/2} c_n) \) is obtained, where \( \mu_n \) and \( \sigma_n \) are normalizing constants, \( \phi(t) \)
is the standard normal distribution, and \( c_n \) is a sequence of positive
constants related to \( f_n \) and tending to 0 as \( n \to \infty \).

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strong consistency; rate of convergence; asymptotic
normality; Berry-Esseen rate.
1. **Introduction.** Consider the *generalized failure rate function* (GFRF) defined by

\[ \gamma(x) = f(x)/gG^{-1}F(x) \]

for all \( x \) such that \( gG^{-1}F(x) > 0 \), where \( F \) and \( G \) are distribution functions with density functions \( f \) and \( g \) respectively. If \( G \) is the exponential distribution function, then \( \gamma(x) \) reduces to the *failure rate function* \( \gamma_0(x) = f(x)/(1 - F(x)) \). In general, if \( \gamma(x) \) is a nondecreasing function, then \( G^{-1}F(x) \) is convex on the support of \( F \). This motivates the important notion, due to Van Zwet (1964), of convex ordering on the space of distribution functions. The properties of \( \gamma(x) \) and its utility in reliability theory and life data analysis have been extensively discussed in Barlow and Proschan (1975). Usually, the function \( g \) is given but little is assumed known about the underlying density \( f \). Thus it becomes of interest to estimate \( \gamma(x) \) from the data. To this end, both parametric and nonparametric methods have been introduced. See, for instance, Grenander (1956), Watson and Leadbetter (1964 a,b), Barlow and Van Zwet (1969, 1970, 1971), and Ahmad and Lin (1977).

Let \( f_n \) be an empirical density function for estimation on \( f \) based on a sample of size \( n \) from \( f \). We consider a sample analogue estimator \( \gamma_n(x) \) of \( \gamma(x) \) given by

\[
\gamma_n(x) = \begin{cases} 
    f_n(x)/gG^{-1}F_n(x), & \text{if } gG^{-1}F_n(x) \neq 0, \\
    0, & \text{otherwise,}
\end{cases}
\]

where \( F_n(x) \) denotes the associated cdf obtained by integration of \( f_n \).
In the present investigation we study in detail the stochastic behavior of \( \gamma_n(x) \). Specifically, we shall explore the following major properties concerning our estimator: uniform strong convergence, the rate of uniform strong convergence, asymptotic normality in distribution, and the Berry-Esseen rate associated with asymptotic normality.

Let \( f_n \) be a kernel-type estimator of \( f \). A uniform strong convergence result for \( \gamma_n(x) \) was developed by Ahmad (1976). In Section 2 we obtain such a result by a different approach with less conditions. Moreover, we allow greater flexibility in choosing \( f_n \). Assume further that \( f_n \) is the "naive" kernel estimator of \( f \). Then \( f_n \) simplifies to an estimator \( \rho_n \) proposed by Shaked (1979). For this estimator, asymptotic normality was obtained by Shaked. For \( G \) the exponential distribution, uniform strong convergence rate was obtained by Winter (1978), and asymptotic normality result was derived by Watson and Leadbetter (1964a, b), and extended by Ahmad and Lin (1977) for the multivariate case. For the univariate case, our Theorems 2 and 3 essentially generalize and improve the above-mentioned results to general distribution function \( G \) and kernel function \( K \).

Our strong convergence results are presented in Section 2. Asymptotic normality and associated rates of convergence are developed in Section 3. In Section 4, the stochastic behavior of another estimator \( \hat{\gamma}_n(x) \), obtained by replacing the usual empirical distribution \( F_n \) for \( \hat{F}_n \) in \( \gamma_n(x) \), is explored in terms of the difference \( \hat{\gamma}_n(x) - \gamma_n(x) \). It is shown that \( \hat{\gamma}_n(x) \) is so close to \( \gamma_n(x) \) in stochastic behavior that it preserves all the properties of \( \gamma_n(x) \).
Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with unknown density function $f$. Attention now is confined to estimators of the kernel-type

$$ f_n(x) = n^{-1} \prod_{i=1}^{\lceil \frac{x - X_i}{c_n} \rceil} K \left( \frac{x - X_i}{c_n} \right), $$

where the "kernel" $K$ is density function, and $\{c_n\}$ is a "bandwidth" sequence of positive constants tending to 0. The results of this paper, in some instances, may be extended to other types of nonparametric density estimators.

2. **On uniform strong convergence.** We assume that $f$ and $f_n$ are such that the following conditions hold:

(A1) $\lim_{n \to \infty} ||f_n - f||_\infty^{w,E,1} = 0$, as $n \to \infty$;

(A2) $\lim_{n \to \infty} b_n \cdot ||f_n - f||_\infty^{w,E,1} = 0$, where $b_n \to \infty$;

(A3) $\lim_{n \to \infty} b'_n \cdot ||\hat{f}_n - F||_\infty^{w,E,1} = 0$, where $b'_n \to \infty$.

Conditions for (A1) have been investigated by Nadaraya (1965), Moore and Yackel (1977), and Taylor and Cheng (1979), among others. Conditions for (A2) have been supplied by Reiss (1975), Silverman (1978) and Winter (1978) with $b_n$ sufficiently close to $n^{1/3}$. Conditions for (A3) follow from the work of Winter (1979), who establishes $n^{\frac{1}{2}} ||\hat{f}_n - F||_\infty^{w,E,1} = O(\log \log n)^{\frac{1}{2}}$.

Let $C_\delta$ be the set of $x$ such that $gG^{-1}F(x) > \delta$. Define

$$ ||\gamma_n - \gamma||_C = \sup_{x \in C} |\gamma_n(x) - \gamma(x)|,$$

for any subset $C$ of $\mathbb{R}$, and assume this quantity along with $\hat{f}_n(x)$ to be measurable (this is true at least when $K$ is a continuous function).
THEOREM 1. Let $f$ and $\{f_n\}$ satisfy (A1). Assume that $g G^{-1}$ is a continuous function, and $f$ is a bounded function. Then, for all $\delta > 0$

(1) $||\gamma_n - \gamma||_{C_\delta} \overset{w, \mathbb{P}}{\to} 0$ as $n \to \infty$.

PROOF. By Scheffé's theorem, condition (A1) implies that

$||\hat{\gamma}_n - \gamma||_{\mathbb{P}} \overset{w, \mathbb{P}}{\to} 0$ as $n \to \infty$.

Thus, using uniform continuity of the function $g G^{-1}$ on $[0, 1]$, it follows that

$||g G^{-1} \hat{\gamma}_n - g G^{-1} \gamma||_{\mathbb{P}} \overset{w, \mathbb{P}}{\to} 0$ as $n \to \infty$.

Hence, with probability one, and for large $n$, we have

$$\inf_{x \in C_\delta} g G^{-1} \hat{\gamma}_n(x) > 0,$$

and

$$\inf_{x \in C_\delta} g G^{-1} \hat{\gamma}_n(x) \cdot g G^{-1} \gamma(x) > 0.$$

Consequently, the following inequalities are implied with probability one, and for large $n$:

(2) $||\gamma_n - \gamma||_{C_\delta}$

$$\leq ||(f_n / g G^{-1} \hat{\gamma}_n) - (f / g G^{-1} \gamma)||_{C_\delta} + ||(f / g G^{-1} \hat{\gamma}_n) - f / g G^{-1} \gamma||_{C_\delta}$$

$$\leq ||(f_n - f) / g G^{-1} \hat{\gamma}_n||_{C_\delta} + f||_{\infty} \cdot ||(g G^{-1} \hat{\gamma}_n - g G^{-1} \gamma) / g G^{-1} \gamma||_{C_\delta} \cdot g G^{-1} \gamma||_{C_\delta}.$$

Thus (1) follows easily. □

THEOREM 2. Let $f$ and $\{f_n\}$ satisfy (A2) and (A3). Assume that $g G^{-1}$ has a bounded derivative and $f$ is a bounded function. Then for all $\delta > 0$. 
\begin{equation}
\lim_{n \to \infty} \{b_n, b_n'\} \|\gamma_n - \gamma\|_{C_\delta}^w \equiv 0.
\end{equation}

PROOF. Carrying (2) one step further, using Taylor's theorem, we have

\begin{equation}
\|\gamma_n - \gamma\|_{C_\delta}^w \leq \|\frac{(f_n - f)g^{-1}_n}{c_n^+} \|_{C_\delta}^+ \|g^{-1}\|_{\infty} \cdot \|g^{-1}_n \cdot \hat{F}_n \cdot \hat{g}^{-1}_n \|_{C_\delta} \|f\|_{\infty}
\end{equation}

Thus (3) follows easily by using conditions and (4). \hspace{1cm} \square

REMARK. Let \(f_n\) be a kernel estimator or recursive kernel estimator
\(f_n(x) = n^{-1}\sum_{i=1}^{m} c_i^{-1} K((x - X_i)/c_i)\) of \(f\). Then following the work of Winter (1978), we have

\[\|f_n - f\|_{\infty}^w \equiv o(n^{-1/3} \beta_n \log n),\]

and

\[\|\hat{F}_n - F\|_{\infty}^w \equiv o(n^{-1/3} \beta_n \log n),\]

for \(\beta_n \to \infty\), under the assumptions: \(f\) is Lipschitz; \(K\) is uniformly locally Lipschitz and vanishes off finite interval \([-L, L]\); \(c_n = d_n n^{-1/3}\) for \(0 < d_n \leq d_{\overline{d}} < \infty\). Theorem 2 generalizes a conclusion of Winter (1978) who under the above conditions and \(G\) the exponential distribution, showed

\[n^{1/3} \beta_n \log n \|f_n/(1 - \hat{F}_n) - f/(1 - F)\|_{C_\rho}^w \equiv 0 \text{ as } n \to \infty. \hspace{1cm} \square\]

3. Limiting distribution. Let \(x_0\) be a fixed point in the support of \(gG^{-1}F(x)\). In this section, the asymptotic normal distribution of \(\gamma_n(x_0)\) and related rates of convergence are derived. (The joint asymptotic distribution of \((\gamma_n(x_0), \gamma_n(y_0))\), for each pair \((x_0, y_0)\) in the support of \(gG^{-1}F(x)\) then follows by Cramér-Wold device.) The technique used in this section will be to approximate the estimation error \(\gamma_n(x_0) - \gamma(x_0)\) by an appropriate Gateaux differential \(\gamma_{x_0} (f; f_n - f)\) defined by
\[ \gamma_{x_0} (f; f_n - f) = \frac{d}{d\lambda} [f(x_0) + \lambda (f_n(x_0) - f(x_0))] / gG^{-1} [F(x_0) + \lambda (\hat{F}_n(x_0) - F(x_0))] \big|_{\lambda = 0}. \]

Thus if \( gG' \) has derivative, we have

\[ \gamma_{x_0} (f; f_n - f) = ((f_n(x_0) - f(x_0)) \cdot gG^{-1} F(x_0) - (gG^{-1})'(F(x_0)) \cdot (\hat{F}_n(x_0) - F(x_0))). \]

\[ f(x_0)/(gG^{-1} F(x_0)) \]  

for each \( x_0 \). Asymptotic normality follows from an application of the following lemma, and the properties of the sum of double array row independent random variables \( \gamma_{x_0} (f; f_n - f) \). Indeed, \( f_n \) has the following representation:

\[ f_n(x_0) = n^{-1} \sum_{i=1}^{n} f_{ni}(x_0), \]

where \( f_{ni}(x_0) = c_n^{-1} K[(x - X_i)/c_n] \), and thus we have \( \gamma_{x_0} (f; f_n - f) \)

\[ = n^{-1} \sum_{i=1}^{n} \gamma_{x_0} (f; f_{ni} - f). \]

The following lemma is useful.

**LEMMA 1.** Assume that

1. \( x_0 \) is a point such that \( gG^{-1} F(x_0) > 0 \) and \( f(x_0) \) is finite;
2. \( gG^{-1} \) has a bounded derivative in a neighborhood of \( F(x_0) \);
3. \( f_n(x_0) \xrightarrow{P} f(x_0) \) as \( n \to \infty \), and \( n^{1/2} c_n^{1/2} |\hat{F}_n(x_0) - F(x_0)| \xrightarrow{P} 0 \) as \( n \to \infty \), for \( nc_n \to \infty \) as \( n \to \infty \). Then

\[ n^{1/2} c_n^{1/2} |\gamma_n(x_0) - \gamma(x_0) - \gamma_{x_0}(f; f_n - f)| \xrightarrow{P} 0 \text{ as } n \to \infty. \]
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PROOF. Let $I = (F(x_o) - \delta, F(x_o) + \delta)$ be a neighborhood of $F(x_o)$ such that $(gG^{-1})'(y) < \infty$ for all $y \in I$. Condition (iii) yields

\begin{equation}
\lim_{n \to \infty} P(\hat{F}_n \in I) = 1.
\end{equation}

Now,

\begin{align*}
P\left(n^{\frac{1}{\alpha}} c_3^{\frac{1}{2}} \left| \gamma_n(x_o) - \gamma(x_o) - \gamma_{x_o} ((f; f_n - f) > \varepsilon \right) \right. & \\
\leq P\left(n^{\frac{1}{\alpha}} c_3^{\frac{1}{2}} \left| \gamma_n(x_o) - \gamma(x_o) - \gamma_{x_o} ((f; f_n - f) > \varepsilon \right), \text{ and } \hat{F}_n(x_o) \notin I \right) & + P(\hat{F}_n(x_o) \notin I) \\
= S_{n1} + S_{n2}, \text{ say.}
\end{align*}

By (6) it follows that $S_{n2} \to 0$ as $n \to \infty$. Regarding $S_{n1}$, we assume that $\hat{F}_n \in I$ and write

\begin{align*}
|\gamma_n(x_o) - \gamma(x_o) - \gamma_{x_o} ((f; f_n - f)| & \\
= |(gG^{-1}F(x_o)f_n(x_o) \cdot (-gG^{-1})'(t_n(x_o)) |\hat{F}_n(x_o) - F(x_o)| + (gG^{-1})'(F(x_o)) \cdot gG^{-1}\hat{F}_n(x_o) | \\
\cdot (\hat{F}_n(x_o) - F(x_o)) \cdot f(x_o) |/gG^{-1}\hat{F}_n(x_o) \cdot (gG^{-1}F(x_o))^2|,
\end{align*}

where $t_n(x_o)$ lies between $\hat{F}_n(x_o)$ and $F(x_o)$. Since $gG^{-1}$ is a continuous function on $I$. Utilizing Slutsky's theorem and conditions (ii) and (iii) we have

$S_{n1} \to 0$ as $n \to \infty$.

This completes the proof. $\square$
REMARKS. (1) It is easy to see from Parzen (1962) that if \( K \) is a Borel function satisfying \( ||K||_\infty < \infty \), \( \lim_{y \to \infty} yK(y) = 0 \), and \( n c_n \to \infty \) as \( n \to \infty \), then
\[
 f_n(x_0) \overset{p}{\to} f(x_0) \text{ as } n \to \infty,
\]
for each continuity point of \( f(x_0) \).

(2) If \( f \) has a bounded derivative, \( \int zK(z)dz = 0 \), \( \int z^2K(z)dz < \infty \), and \( n c_n^5 = o(1) \), then following Cheng (1979) we have, for each \( x_0 \),
\[
 n c_n E[F_n(x_0) - F(x_0)]^2 = o(1) \text{ as } n \to \infty,
\]
and thus for each \( x_0 \),
\[
 n c_n^{\frac{4}{5}} [F_n(x_0) - F(x_0)] \overset{p}{\to} 0 \text{ as } n \to \infty,
\]
by Markov inequality. Furthermore, if \( K \) has bounded support, we need only require that \( f \) has a bounded derivative in a neighborhood of \( x_0 \). □

Define:
\[
 \mu_n(x_0, f) = E_{\gamma_{x_0}}(f; f_n - f), \quad \sigma_n^2(x_0, f) = \text{Var}_{\gamma_{x_0}}(f; f_n - f),
\]
and
\[
 \alpha_n^3(x_0, f) = E|\gamma_{x_0}(f; f_n - f) - \mu_n(x_0, f)|^3.
\]

THEOREM 3. Assume that
(i) \( x_0 \) is a point such that \( gG^{-1}F(x_0) > 0 \) and \( f(x_0) < \infty \);
(ii) \( f \) has a bounded derivative, and \( gG^{-1} \) has a bounded derivative
in a neighborhood of \( F(x_0) \);
(iii) \( K \) is a bounded Borel function such that \( \int zK(z)dz = 0 \), and
\( \int z^2K(z)dz < \infty \);
(iv) \( n c_n \to \infty \) as \( n \to \infty \), and \( n c_n^5 \to 0 \) as \( n \to \infty \).

Then
\[
 n^{\frac{8}{3}} (\gamma_n(x_0) - \gamma(x_0) - \mu_n(x_0, f)) / \sigma_n(x_0, f) \overset{d}{\to} N(0,1) \text{ as } n \to \infty,
\]
where \( \gamma_n(x_0) \) and \( \gamma(x_0) \) are defined in (8).
where $\mu_n(x_0, f) \to 0$ as $n \to \infty$, and $c_n \sigma_n^2(x_0, f) + \gamma_n(x_0) f^{-1}(x_0) \int K^2(z) \, dz \neq 0$ as $n \to \infty$.

PROOF. Using Theorem 1A of Parzen 1962), we have

$$\mu_n(x_0, f) \to 0 \text{ as } n \to \infty.$$ 

Also,

$$c_n \sigma_n^2(x_0, f) + \gamma_n(x_0) f^{-1}(x_0) \int K^2(z) \, dz \text{ as } n \to \infty$$

and

$$E|\gamma_n x_0(f; f_{n1} - f)|^3$$

$$\leq c_2 \left\{ E|f_{n1}(x_0) - f(x_0)|/gG^{-1}F(x_0)|^3 + E|f(x_0) \cdot [\int_{-\infty}^{x_0} f_{n1}(t) \, dt - F(x_0)]/ (gG^{-1}F(x_0))^2\right\}^3$$

$$\sim c_2 \cdot c_n^{-2} \cdot f(x_0) \cdot (gG^{-1}F(x_0))^3 \cdot \int K^2(z) \, dz,$$

for some $c_2 > 0$. The proofs are routine and may be found in Cheng (1979). Consequently,

$$\alpha_n^3(x_0, f)/n \sigma_n^3(x_0, f) = e_n x_0^3(x_0, f)/n \cdot c_n^{-1} \sigma_n^{-2} (x_0, f) + 0$$

as $n \to \infty$, and hence by Liapounov's theorem in connection with Lemma 1, we have

$$n^{\frac{1}{2}} (\gamma_n(x_0) - \gamma(x_0) - \mu_n(x_0, f))/\sigma_n(x_0, f) \overset{d}{\to} N(0, 1) \text{ as } n \to \infty.$$

REMARKS. (1) If $K(\cdot)$ has bounded support, then we may require that $f$ has a bounded derivative merely in a neighborhood of $x_0$.

(2) If $f_n$ is a recursive kernel estimator, then (8) still holds under modified conditions on $c_n$. □
Elimination of the quantity \( \mu_n(x_o, f) \) in (8) is worthwhile, but requires further smoothness restrictions on \( f \). The following corollary is useful in typical practical applications.

**COROLLARY 1.** Assume the conditions of Theorems 1 and 3. If \( f'' \) exists and is bounded (If \( K \) has bounded support then we need only assume that \( f'' \) is bounded in a neighborhood of \( x_o \)), then

\[
(9) \quad \left( n_c \right)^{\frac{4}{3}} \left( \gamma_n(x_o) - \gamma(x_o) \right) / \left( \gamma_n^2(x_o) f_{n}^{-1}(x_o) \int K^2(z) dz \right) \xrightarrow{\mathcal{D}} N(0, 1) \text{ as } n \to \infty.
\]

**PROOF.** It suffices to show that

\[
n^{\frac{4}{3}} \left( \mu_n(x_o, f) / \sigma_n(x_o, f) \right) \to 0 \quad \text{as} \quad n \to \infty.
\]

and the proof is routine. \( \square \)

In what follows, a Berry-Esseen type result is obtained which provides a sharper conclusion than (8). First, we observe a useful lemma.

Let \( c \) denote a generic constant, not necessarily the same at each appearance.

**LEMMA 2.** Assume that \( \| K \|_\infty < \infty, \int zK(z)dz = 0, \int z^2 K(z)dz < \infty, \|
f'\|_\infty < \infty, \|
f''\|_\infty < \infty, n_c^4 = O(1), \text{ and } n^{-1} c^{-2} = O(1). \) If \( gG^{-1}F(x_o) \to 0, f(x_o) < \infty, \) and \( (gG^{-1})'' \) is bounded in a neighborhood \( I \) of \( F(x_o) \), then

\[
(10) \quad P(\sqrt{n} \sigma_n(x_o, f) \gamma_n(x_o) - \gamma(x_o) - \gamma(x_o)(f; f_n - f) > a_n) = o(a_n),
\]

for \( a_n = n^{-\frac{4}{3}} c^{-\frac{1}{2}} \).

**PROOF.** First we need some background. By a parallel argument to that in Lemma 1 of Cheng and Serfling (1979), we have
\[ E|\hat{F}_n(x_o) - F(x_o)|^{2k} = O(n^{-k} + c_n^{4k}), \]
and
\[ E|f_n(x_o) - f(x_o)|^{2k} = O(n^{-k} \cdot c_n^{-2k+1} + c_n^{4k}), \]
and thus
\[ E(f_n(x_o))^{2k} \leq c, \text{ for large } n. \]

Now, applying Taylor's expansion for \( gG^{-1}\hat{F}_n(x_o) \), with respect to \( F(x_o) \), to the second order, we obtain the following representation:

\[
\gamma_n(x_o) - \gamma(x_o) - \gamma_{x_o} (f; f_n - f)
\]

\[ = (gG^{-1})'F(x_o)[\hat{F}_n(x_o) - F(x_o)]^2 \cdot f_n(x_o) \cdot (gG^{-1})'(t_n(x_o)) - gG^{-1}F(x_o) \cdot (qG^{-1})'(F(x_o)). \]

\[ [\hat{F}_n(x_o) - F(x_o)] [f_n(x_o) - f(x_o)] - \frac{1}{2} (gG^{-1}F(x_o)) \cdot f_n(x_o) \cdot (gG^{-1})''(t_n(x_o)) \cdot [\hat{F}_n(x_o) - F(x_o)]^2 \]

\[ /gG^{-1}\hat{F}_n(x_o) \cdot (gGF(x_o))^2, \]

where \( t_n(x_o) \) and \( \tilde{t}_n(x_o) \) lie between \( \hat{F}_n(x_o) \) and \( F(x_o) \). (If \( \hat{F}_n(x_o) \not\in I \), we define \( (gG^{-1})'(E_n(x_o)) \), and \( (gG^{-1})''(t_n(x_o)) \) arbitrarily.) Since for

\[ E_n = \{ \hat{F}_n(x_o) \in I = [F(x_o) - \delta, F(x_o) + \delta] \}, P(E_n^c) = O(a_n) \text{ as } n \to \infty, \text{ hence for every event } A_n, P(A_n) = O(a_n) \text{, if } P(A_n E_n) = O(a_n). \text{ Therefore to prove our assertion, it suffices to show}

\[ P(\sqrt{n}|(\gamma_n(x_o) - \gamma(x_o) - (f; f_n - f)) / \sigma_n(x_o, f) > a_n, \hat{F}_n(x_o) \in I) \]

\[ = O(a_n). \]

Now,
\[ P(n^{\frac{1}{2}}|\gamma(x_o) - \gamma(x_n) - \gamma(f; f - f)|/\sigma_n(x_n, f) > a_n, \hat{F}_n(x_o) \in I) \]

\[ \mathbb{P} \left[ \sum_{i} \frac{1}{n} \left| \hat{F}_n(x_o) - F(x_o) \right|^2 / \sigma_n(x_o, f) \cdot g^{-1} \hat{F}_n(x_o) \cdot (g^{-1}F(x_o))^2 \right] > a_n / 3, \]

\[ \hat{F}_n(x_o) \in I \right] + \mathbb{P} \left[ \sum_{i} \frac{1}{n} \left| f_n(x_o) - f(x_o) \right| / \sigma_n(x_o, t) \cdot g^{-1} \hat{F}_n(x_o) \cdot (g^{-1}F(x_o))^2 \right] > a_n / 3, \hat{F}_n(x_o) \in I \right] + \mathbb{P} \left[ \sum_{i} \frac{1}{n} \left| f_n(x_o) \right| \cdot [\hat{F}_n(x_o) - F(x_o)]^2 \right] / \sigma_n(x_o, f) \cdot g^{-1} \hat{F}_n(x_o) \cdot (g^{-1}F(x_o))^2 > a_n / 3, \hat{F}_n(x_o) \in I \right]\]

= \sum_{i=1}^{3} S_{n1} + S_{n2} + S_{n3}, say.

Thus making use of our preliminary work and the Markov inequality,

\[ S_{n1} = O(n^{-2k + c_n^8k}) / a_n^{2k} \sigma_n(x_o, f) \cdot n^{-k} + O(n^{-k} + c_n^{4k}), \]

\[ S_{n2} = O(n^{-k} (-2k+1)/2) / a_n^{2k} + O(n^{-k} + c_n^{4k}), \]

and

\[ S_{n3} = O(n^{-2k} c_n^{2k} + n^{-2k} c_n^{18k} / a_n^{2k} + O(n^{-k} + c_n^{4k}). \]

The proofs can be found in Cheng (1979). Consequently,

\[ S_{n1} + S_{n2} + S_{n3} = O(n^{-k} (-2k+1)/2) / a_n^{2k} + O(n^{-k} + c_n^{4k}), \]

\[ = O(a_n) \]

for large \( n \), and hence (10) follows. \( \Box \)

Let \( \Phi(x) \) be the standard normal distribution.
THEOREM 4. Assume the conditions of Theorem 3. If \( nc_n^4 \to 0 \) as \( n \to \infty \),
\[ \|f^*\|_\infty < \infty \quad \text{and} \quad (g^{-1})^* \] is bounded in a neighborhood \( I \) of \( F(x_0) \), then
\[ \text{(11)} \sup_{t \in \mathbb{R}} |P\left(n^{\frac{1}{c_n}}(Y_n(x_0) - Y(x_0) - \mu_n(x_0, f)/\sigma_n(x_0, f) \leq t\right) - \Phi(t)\| = O(n^{-4c_n^3}) \]
as \( n \to \infty \).

PROOF. We apply the following device,
\[ \text{(12)} \sup_{t \in \mathbb{R}} |P\left(n^{\frac{1}{c_n}}(Y_n(x_0) - Y(x_0) - \mu_n(x_0, f)/\sigma_n(x_0, f) \leq t\right) - \Phi(t)\|
= \sup_{t \in \mathbb{R}} P\left(n^{\frac{1}{c_n}}(Y_n(f; n - f) - \mu_n(x_0, f))/\sigma_n(x_0, f) \leq t\right) - \Phi(t)\|
+ P\left(n^{\frac{1}{c_n}}|\sigma_n(x_0) - \sigma(x_0) - Y_{n^1}(f; n - f)|/\sigma_n(x_0, f) \geq a_n\right) + O(a_n), \]
for a sequence of nonnegative constants \( a_n \). The proof of (11) is elementary.

Now, according to Liapounov's condition for the classical Berry-Esseen theorem, we have
\[ \sup_{t \in \mathbb{R}} P\left(n^{\frac{1}{c_n}}(Y_n(f; n - f) - \mu_n(x_0, f))/\sigma_n(x_0, f) \leq t\right) - \Phi(t)\|
= O(n^{-4c_n^3}). \]
By this, (10) and (12), our assertion follows. \( \Box \)

REMARKS. (1) In addition to the conditions of Theorem 4, if \( n^{1-c_n^3} \to 0 \)
as \( n \to \infty \), then \( \mu_n(x_0, f) \) can be deleted in (11).

(2) If \( K \) is assumed to have bounded support, then the conditions on \( f \) can be weakened. It suffices to assume that functions \( f'(x) \) and \( f'(x) \) are bounded in a neighborhood of \( x_0 \). \( \Box \)
3. An alternative GFRF estimator. Let $F_n$ denote the usual empirical distribution function. In this section, stochastic properties of an alternative GFRF estimator,

$$\hat{\gamma}_n(x) = f_n(x)/g^{G^{-1}}_{n}(x),$$

are investigated. We show that $\hat{\gamma}_n(x)$ and $\gamma_n(x)$ are so closely related in behavior that $\hat{\gamma}_n(x)$ shares the same asymptotic properties with $\gamma_n(x)$. Moreover, $\hat{\gamma}_n(x)$ is simpler in practical computation than $\gamma_n(x)$.

The following lemmas show that the difference $\hat{\gamma}_n(x) - \gamma_n(x)$ is negligible in many senses. The results of this section in conjunction with Theorems 1, 2, and 3 and corollary 1 lead immediately to the extension of (1), (3), (8) and (9) for $\hat{\gamma}_n(x)$ in place of $\gamma_n(x)$.

**Lemma 3.** Assume that $f$ is a bounded function, $\|f_n - f\|_\infty^{W.P.1} \rightarrow 0$ as $n \rightarrow \infty$, and $g^{G^{-1}}$ is a continuous function. Then

$$\|\hat{\gamma}_n - \gamma_n\|_{C_\delta}^{W.P.1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

for each $\delta > 0$.

**Lemma 4.** Assume that $g^{G^{-1}}$ has a bounded derivative, $\|f_n - f\|_\infty^{W.P.1} \rightarrow 0$ as $n \rightarrow \infty$, and $\lim_{n \rightarrow \infty} b_n^{'}\|\hat{F}_n - F\|_\infty^{W.P.1} \rightarrow 0$, for $b_n^{'} \rightarrow \infty$ as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty}(b_n^{'} n^\delta/(\log \log n)^{\gamma'})\|\hat{\gamma}_n - \gamma_n\|_{C_\delta}^{W.P.1} \rightarrow c$$

for some $c > 0$ and each $\delta > 0$.

**Remark.** By directly applying the argument of Theorem 2, we obtain
\[ \lim_{n \to \infty} \frac{b_n \cdot n^{\frac{1}{2}}}{(\log \log n)^{\frac{1}{2}}} \| \hat{\gamma}_n - \gamma \|_{C^0} \overset{w}{\longrightarrow} 1 \]

without requiring any conditions on \( F_n \). \( \square \)

**Lemma 5.** Let \( x_o \) be a point such that \( g(G^{-1}F(x_o) > 0, f_n(x_o) \overset{P}{\to} f(x_o) \),
and \( (nc_n)^{\frac{1}{2}}(\hat{\gamma}_n(x_o) - F(x_o)) \overset{P}{\to} 0 \) as \( n \to \infty \). Then

\[ (nc_n)^{\frac{1}{2}}|\hat{\gamma}_n(x_o) - \gamma_n(z_o)| \overset{P}{\to} 0 \text{ as } n \to \infty. \]

The proofs are routine and may be found, with related results in Cheng (1979).

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**References**


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Abstract
For a specified distribution function $G$ with density $g$, and unknown distribution function $F$ with density $f$, the generalized failure rate function $\gamma(x) = f(x)/gG^{-1}F(x)$ may be estimated by replacing $f$ and $F$ by $\hat{f}_n$ and $\hat{F}_n$, where $\hat{f}_n$ is an empirical density function based on a sample of size $n$ from distribution function $F$, and $\hat{F}_n(x) = \int_{-\infty}^{x} \hat{f}_n(t)dt$. Under regularity conditions we show $\sup_{x \in C} |\gamma_n(x) - \gamma(x)| \overset{w}{\rightarrow} 0$, and, under additional
restrictions \( \sup_{x \in C} |\gamma_n(x) - \gamma(x)| \leq \text{w.p.1} o(n^{-1/3} \beta_n \log n) \), where \( C \) is a subset of \( \mathbb{R} \) and \( \beta_n \rightarrow 0 \). Moreover, asymptotic normality is derived and the Berry-Esséen rate

\[
\sup_{t \in \mathbb{R}} |P(n^{1/2}(\gamma_n(x) - \gamma(x)) - u_n(x, f))/\sigma_n(x, f) \leq t) - \Phi(t)| = O(n^{-1/2} + c_n^{1/2})
\]

is obtained, where \( u_n \) and \( \sigma_n \) are normalizing constants, \( \Phi(t) \) is the standard normal distribution, and \( c_n \) is a sequence of positive constants related to \( f_n \) and tending to 0 as \( n \rightarrow \infty \).