A Distribution-Free Test for Concurrence

by

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In this note we introduce a distribution-free signed rank test for the concurrence of two regression lines which are assumed parallel. The test is analogous to a signed rank test of linearity versus convexity developed by Olshen (1967) and a signed rank test for parallelism proposed by Hollander (1970).

KEY WORDS: Concurrence; Regression; Distribution-free test.

1. INTRODUCTION

Consider the model

$$Y_{ij} = \alpha_i + \beta X_{ij} + e_{ij}, \quad i = 1, 2; \quad j = 1, \ldots, N, \quad (1.1)$$

where $\alpha_1$, $\alpha_2$, and $\beta$ are unknown parameters and the $X$'s are known constants. The hypothesis of interest here is

$$H_0: \quad \alpha_1 = \alpha_2, \quad (1.2)$$

that is, the two (assumed) parallel regression lines are actually concurrent. The $e$'s (errors) are taken to be mutually independent unobservable random variables with cumulative distribution functions $P(e_{ij} \leq t) = F_i(t)$. Let $G(\cdot)$ denote the cumulative distribution function of $S_1 - S_2$ when $S_1$, $S_2$ are independent and $S_i$ is distributed according to $F_i$, and let

$$G = \{(F_1, F_2): \quad G \text{ is continuous; } G(t) + G(-t) = 1 \text{ for all } t\} \quad (1.3)$$
We assume that \((F_1, F_2)\) is a member of the class \(G\). This includes the (nonexclusive) cases when \(F_1 = F_2 = F\) and \(F\) is continuous, and when \(F_1, F_2\) are symmetric about the same point with at least one being continuous. In Section 2 we propose a signed rank test of \(H_0\) that is distribution-free when \(H_0\) is true and \((F_1, F_2) \in G\).

Section 3 compares, via Pitman efficiency, the signed rank test and a normal theory \(t\)-test considered by Potthoff (1965).

The test proposed here is analogous to a test of linearity versus convexity developed by Olshen (1967) and a test for parallelism considered by Hollander (1970). Normal theory tests of \(H_0\) for more general designs that include (1.1) have been developed by Potthoff (1965), and normal theory tests for related concurrence problems may be found in Tocher (1952), Williams (1953), (1959), Saw (1966), and Seber (1977).

2. A SIGNED RANK TEST FOR CONCURRENCE

Assume \(N = 2n\), discarding an observation from one or both samples if necessary, and form \(n\) quadruples \((Y_{11}, Y_{12}, Y_{21}, Y_{22})\) with corresponding \(X's\) satisfying \(X_{11} - X_{1u} + X_{2t} - X_{2s} \neq 0\). (We assume the design is such that this can always be done, but note this does not imply the \(X's\) need be distinct.) Each \(Y\) is to be used for one and only one quadruple. The grouping can depend only on the \(X's\) and not on the observed \(Y's\). For each quadruple compute

\[
w = \{(Y_{11} - Y_{21})(X_{2t} - X_{1u}) + (Y_{11} - Y_{2t})(X_{1v} - X_{2s})\}/(X_{1v} - X_{1u} + X_{2t} - X_{2s})
\]

\[(2.1)\]

\[
= [(X_{2t} - X_{1u})(e_{1v} - e_{2s}) + (X_{1v} - X_{2s})(e_{1u} - e_{2t})]/(X_{1v} - X_{1u} + X_{2t} - X_{2s})]
\]

\[+ (a_1 - a_2).\]
Define \( r_i = \text{rank } |w_i| \) in the joint ranking from least to greatest of \( |w_1|, \ldots, |w_n| \), \( \psi(a) = 1 \) for \( a < 0 \) and 0 otherwise, and set

\[
T_n = \sum_{i=1}^{n} r_i \psi(w_i) = \sum_{i<j} \psi(w_i + w_j). \tag{2.2}
\]

The statistic \( T_n \), the Wilcoxon signed rank statistic applied to the \( w \)'s, has the Wilcoxon distribution when \( (F_1, F_2) \in G \) and \( H_0 \) is true. This is a consequence of the independence of the \( w \)'s and the fact that \( (F_1, F_2) \in G \) implies each \( w \) has a distribution which is continuous and symmetric about \( \alpha_1 - \alpha_2 \), the latter being 0 when \( H_0 \) holds. A one-sided test against \( \alpha_2 > \alpha_1 \) rejects for large \( T_n \); the two-sided test against alternatives \( \alpha_1 \neq \alpha_2 \) has a critical region corresponding to large and small values of \( T_n \). Exact critical values can be obtained from tables (e.g., Hollander and Wolfe (1973)) or we may take \( [T_n - (n(n + 1)/4)]/(n(n + 1)(2n + 1)/24)^{1/2} \) as an approximate \( \mathcal{N}(0, 1) \) random variable under \( H_0 \).

Although the null distribution of \( T_n \) does not depend on the specific grouping used to form the quadruples, this is not the case when \( \alpha_1 \neq \alpha_2 \). The question of how to form the quadruples is partially answered by the following proposition. We consider the statistic

\[
U_n = \sum_{i<j} \psi(w_i + w_j), \tag{2.3}
\]

which is asymptotically equivalent to \( T_n \). Let \( E_\theta(U_n) \) denote the expectation of \( U_n \) for model (1.1) with \( \theta = \alpha_2 - \alpha_1 \).

**Proposition 1:** Suppose the \( \{X_{ij}\} \) constants admit a quadruple grouping satisfying
\[ X_{1v} - X_{2s} = X_{2t} - X_{1u} (\neq 0) \] (2.4)

for each quadruple. Then any grouping satisfying (2.4) for each quadruple maximizes the efficacy (cf. Noether (1955)).

\[ \varepsilon (U_n) \overset{\text{def}}{=} \frac{((d/d\theta)\varepsilon_{\theta}(U_n)|_{\theta=\varepsilon})^2}{\text{Var}_0(U_n)} \] (2.5)

of \( U_n \) when \( G \) is a normal distribution with mean 0.

**Proof:** Since

\[ \text{Var}_0(U_n) = n(n - 1)(2n - 1)/24, \] (2.6)

independent of the way in which the quadruples are formed, we need only maximize the numerator in (2.5). From (2.1) and (2.3) we have

\[ \varepsilon_{\theta}(U_n) = \sum_{u < u'} \frac{X_{2t} - X_{1u}}{P(u < u')} \left( \frac{e_{1v} - e_{2s}}{(X_{1v} - X_{2s})(e_{1u} - e_{2t})} + \frac{(X_{2t} - X_{1u})(e_{1v} - e_{2s}) + (X_{1v} - X_{2s})(e_{1u} - e_{2t})}{(X_{1v} - X_{1u} + X_{2t} - X_{2s})} \right) < 2\varepsilon, \] (2.7)

where the summation \( \sum_{u < u'} \) in (2.7) is over all the \( \binom{n}{2} \) pairs of quadruples \((X_{1u}, X_{1v}; X_{2s}, X_{2t}), (X_{1u'}, X_{1v'}; X_{2s'}, X_{2t'})\) taken from the \( n \) quadruples. (We remind the reader that a quadruple is characterized by any one of the four subscripts. Thus in (2.7), s, t, and v are determined once that u has been selected and the quadruple structure has been determined.) Using the fact that the random variable on the left-hand side of the inequality within the probability statement of (2.7) is normal, a direct calculation yields
\[ \frac{d}{ds} \sum_{u<u'} \left( \frac{2}{2\pi} \left( \sigma_{G_{uu'}}^2 \right)^{-\frac{1}{2}} \right) \]

where

\[ A_{uu'} = \frac{(X_{2t} - X_{1u})^2 + (X_{1v} - X_{2s})^2}{(X_{1v} - X_{1u} + X_{2t} - X_{2s})^2} \]

and \( \sigma_G^2 \) is the variance of \( G \).

We will now show that when (2.4) holds for each quadruple, each term in (2.8) is maximized. From (2.8) and (2.9) we see that it is equivalent to show that the choice (2.4) minimizes

\[ B_u \overset{\text{def}}{=} \frac{(X_{2t} - X_{1u})^2 + (X_{1v} - X_{2s})^2}{(X_{1v} - X_{1u} + X_{2t} - X_{2s})^2}. \]

Consider the means inequality (cf. Hardy, Littlewood, and Pólya (1952), Theorem 16)

\[ \left( \sum_{i=1}^{m} \frac{(a_i)^t}{m} \right)^{1/t} \geq \left( \sum_{i=1}^{m} \frac{(a_i)^s}{m} \right)^{1/s}, \ t > s, \]

where the \( a_i \)'s are assumed positive and the inequality is strict unless all the \( a_i \)'s are equal. We have

\[ B_u \geq \frac{(X_{2t} - X_{1u})^2 + (X_{1v} - X_{2s})^2}{(|X_{2t} - X_{1u}| + |X_{1v} - X_{2s}|)^2} \geq \frac{1}{2} \]

where the right-hand inequality in (2.12) is obtained by taking \( m = 2, t = 1, s = \frac{1}{2}, a_1 = (X_{2t} - X_{1u})^2 \), and \( a_2 = (X_{1v} - X_{2s})^2 \) in (2.11). The condition for equality on the right-hand side of (2.12) is \( (X_{1v} - X_{2s})^2 = (X_{2t} - X_{1u})^2 \). Since (2.4) ensures this, and also yields \( B_u = \frac{1}{2} \), the proof is complete.
We remark that condition (2.4) allows the common value of $X_{1v} - X_{2s}$ and $X_{2t} - X_{1u}$ to vary from quadruple to quadruple. Also, even if the design under consideration does not admit a grouping satisfying (2.4), an "optimal" grouping may be defined as one which maximizes $\sum_{u < u'}^{n} (A_{uu'})^{-1/2}$ where $A_{uu'}$ is given by (2.9). However, for large values of $n$ the determination of such a grouping will be tedious.

Although the proof of the preceding proposition made use, via (2.8), of the assumption that $G$ is a normal distribution, I believe that the proposition is true for a larger class of $G$'s, such as those having a square integrable density.

3. A COMPARISON WITH A $t$-TEST PROPOSED BY POTTHOFF

Potthoff (1965) has developed $t$-tests of $H_0$ for more general designs that include (1.1). He assumes normal error distributions $F_1 = N(0, \sigma_1^2), F_2 = N(0, \sigma_2^2)$ but does not require that $\sigma_1^2$ equal $\sigma_2^2$.

Potthoff's "arbitrary" $t$-statistic for testing $H_0$, when specialized to model (1.1), is given by (3.2) below.

Set $Y_{iN} = \left( Y_{i1}, \ldots, Y_{iN} \right)^T$, $X_{iN} = \left( X_{i1}, \ldots, X_{iN} \right)^T$,

$Y_i = (N \sum_{j=1}^{N} Y_{ij}/N), X_1 = \left( X_{i1} - X_{11}, \ldots, X_{iN} - X_{1N} \right)^T$, $X_2 = \left( X_{i1} - X_{11}, \ldots, X_{iN} - X_{1N} \right)^T$ where $i = 1, 2$ and the superscript "T" denotes transpose. Let $\tau_i^2 = (X_{iN}^T X_i^T/N), s = 1 + [(X_{2i} - X_{1i})^2]/(\tau_1 + \tau_2)^2$, and let $D_1, D_2$ be $N \times (N - 2)$ matrices such that

$$D_1^T D_1 = I, D_1^T X_i = 0, i = 1, 2; \quad (3.1)$$
In (3.1) \( \xi_N = (1, \ldots, 1)^T \) is an \( N \times 1 \) vector of 1's, \( I \) is an identity matrix, and \( 0 \) is a zero vector. Finally, define

\[
\text{t}_N = \hat{\delta} \cdot (sV_\perp^TV/(N - 2))^{-\frac{1}{2}},
\]

(3.2)

where

\[
\hat{\delta} = (Y_2, -Y_1) - (X_2, -X_1)[\frac{x_2^TY_2}{N\tau_2} + \frac{x_1^TY_1}{N\tau_1}]/(\tau_2 + \tau_1),
\]

(3.3)

and

\[
V = N^{-\frac{1}{2}}(D_1^TY_1 + D_2^TY_2).
\]

(3.4)

Potthoff shows that when \( F_1 = N(0, \sigma_1^2) \), \( F_2 = N(0, \sigma_2^2) \) and \( H_0 \) is true, \( t_N \) has Student's t-distribution based on \( N - 2 \) degrees of freedom. He establishes a minimax property of the estimator \( \hat{\delta} \) but also points out that many experimenters will consider the non-uniqueness of the matrices \( D_1, D_2 \) a serious disadvantage of the test based on \( t_N \). A similar undesirable element of arbitrariness is present in the \( T_N \) test, namely the choice of a particular grouping to form the \( n \) quadruples.

To get an indication of the relative performance of \( T_N \) versus \( t_N \), consider an equally spaced design where on line \( i \) there are \( N = 2kn' \) observations with \( n' \) observations at each of the \( 2k \) points

\[
-d(2k - 1), \ldots, -3d, -d, d, 3d, \ldots, (2k - 1)d.
\]

(3.5)

For this design we can form the quadruples so that (2.4) is satisfied.

Assume \( F_1 \) has a square integrable density \( f_1 \) and let \( h_1 \) denote the density of \( \sum_{i=1}^{4} e_{ij} \), \( i = 1, 2 \). Calculations similar to those in Olshen (1967) and Hollander (1970) show that the Pitman efficiency
of $T_N$ (based on any grouping satisfying (2.4)) with respect to $t_N$ for design (3.5) is

$$e(T_N, t_N) = 24(\sigma_1^2 + \sigma_2^2)\int h_1(t)h_2(t)dt^2.$$  \hspace{1cm} (3.6)

Note that (3.6) is independent of $d$ and $k$. In the special case where $F_1 = F_2 = F$, and thus $h_1 = h_2 = h$, (3.6) becomes

$$e(T_N, t_N) = 12\sigma_H^2\int h^2(t)dt^2,$$  \hspace{1cm} (3.7)

where $\sigma_H^2$ is the variance of the distribution of $H$. The results of Hodges and Lehmann (1956) show the right-hand-side of (3.7) cannot be less than .864 for all $F$. Some values are

<table>
<thead>
<tr>
<th>$F$</th>
<th>Normal</th>
<th>Uniform</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e(T_N, t_N)$</td>
<td>.955</td>
<td>.919</td>
<td>1.172</td>
</tr>
</tbody>
</table>

We briefly mention another $t$-test of $H_0$ derived by Potthoff (1965). The statistic $t$, given by equation (37) of Potthoff's paper, uses a non-random pairing of the samples. Potthoff shows this test statistic also follows Student's $t$-distribution with $N - 2$ degrees of freedom when (1) holds, $F_1 = N(0, \sigma_1^2)$, $F_2 = N(0, \sigma_2^2)$ and $H_0$ is true. The statistic $t$ removes the arbitrary feature of the $D_1$, $D_2$ matrices present in the formation of $t_N$ (3.2). There remains however some arbitrariness in Potthoff's test based on $t$ as his pairing rule only covers designs where no two $X_1$'s and no two $X_2$'s are equal.

The exactness of Potthoff's $t$-tests requires more restrictive distributional assumptions than those governing the $T_N$ test presented here but, unlike $T_N$, the $t$-tests do not require equal
numbers of observations for each line. We have not been able to obtain a reasonably efficient unconditional distribution-free test for unequal sample sizes. The idea of randomly discarding some observations and then applying $T_N$ may be tolerable when the sample sizes are nearly equal, but in general it sacrifices too much power.

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20. ABSTRACT
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