Optimum Replacement of a System
Subject to Shocks

by

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Abstract

A system is subject to shocks which arrive according to a nonhomogeneous (or homogeneous) Poisson process. Each shock weakens the system and makes it more expensive to run, and hence it is desirable to determine a replacement policy for the system. We consider the possibility of periodic replacement of the system, and exhibit a necessary and sufficient condition for the existence of an optimal finite period. The problem is investigated for both finite and infinite time horizons. Finally, a particular model is studied for a system which fails on the second shock.
Introduction

We consider a system which is subject to shocks, where each shock reduces the effectiveness of the system and makes it more expensive to run. Often it is economical to periodically replace the complete system even though the replacement cost may be quite large.

We interpret the word shock in a very broad sense. In some situations the intensity rate of shocks may increase with the age of the system. For example, consider a large system composed of many components where a shock is interpreted as the failure of one of the components. Although the system continues to operate after a 'shock', it does so under more stress due to increased loading of the other components and is therefore more susceptible to 'shocks'. In another situation where it is appropriate to consider an increasing shock rate, suppose that shocks of a certain type occur at a constant rate but only a proportion of these shocks is harmful to the system. As the system ages, however, the proportion of harmful shocks may increase as a result of a weakening of the system.

There are of course situations where it is reasonable to believe that the intensity rate of shocks decreases with the age of the system. For example the 'shock' to a system may be mishandling or an error of operation of the system by an operator. As time passes, operators become increasingly familiar with the
system and become less likely to mistreat it.

In this paper we perform a cost analysis on a system subject to shocks. We assume that the normal running cost of a new system is $a$ per unit of time, and that each shock to the system increases the running cost by $c$ per unit of time. The cost of replacing the complete system will be $c_0$, where usually $c_0$ is large relative to $c$. We consider periodic replacement of the system with period $T$ (i.e., the system is replaced at times $T, 2T, 3T, ...$), and will attempt to find the value $T_0$ which minimizes the long run expected cost per unit of time. Related cost analysis work for systems where minimal repair is performed between periodic replacements can be found in Barlow and Proschan 1965, and Boland and Proschan 1981. In the final section of this paper we consider a model for a system which necessarily fails on receiving a second shock.

$\lambda(t)$ will denote the intensity rate at which a system of age $t$ is subject to shocks. We assume that $\lambda(t)$ is a continuous positive function for $t > 0$. $A(t) = \int_0^t \lambda(s)ds$ will be the mean value function. We assume that the number of shocks arriving in the age interval $[0, T)$ of the system is a nonhomogeneous Poisson process (or a homogeneous Poisson process) with intensity function $\lambda(t)$. 
§1. Optimal Periodic Replacement for a System Subject to Repeated Shocks

Suppose that the system is subject to \( k \) shocks during the interval \([0, T]\), where the shocks occur at random times \( t_1 < t_2 < \ldots < t_k \). Then the running cost in the interval \([0, T]\) is

\[
\begin{align*}
&= aT + c(t_2 - t_1) + 2c(t_3 - t_2) + \ldots + (k - 1)c(t_k - t_{k-1}) \\
&\quad + kc(T - t_k) \\
&= aT + c[kT - (t_1 + \ldots + t_k)].
\end{align*}
\]

This leads us to the lemma:

**Lemma 1.1.** The conditional expected cost of running the system in the interval \([0, T]\), given that it is subject to \( k \) shocks in this interval, is

\[
aT + \frac{kc}{A(T)} \int_0^T A(s)ds
\]

for \( k = 0, 1, 2, \ldots \).

**Proof.** For fixed \( k \), let \( \tau_i = \Lambda(t_i) \) for \( i = 1, 2, \ldots, k \). Since the shock process is a nonhomogeneous Poisson process with intensity rate \( \lambda(t) \), we know that given \( k \) shocks in the interval \([0, T]\), the random variables \( \tau_1, \ldots, \tau_k \) are distributed as the order statistics in a random sample of size \( k \) from the uniform distribution on \([0, \Lambda(T)]\) (see for example Parzen, 1962, pp. 139-143 and Barlow and Proschan, 1975, pp. 65-67). Hence the conditional expected
running cost during the interval \([0, T]\) given \(k\) shocks is

\[
E(aT + c[kT - (t_1 + \ldots + t_k)]|k \text{ shocks})
= aT + ckT - cE(\Lambda^{-1}(\tau_1) + \ldots + \Lambda^{-1}(\tau_k))
= aT + ckT - ckE(\Lambda^{-1}(\tau))
\]

(where \(\tau\) is uniformly distributed on \([0, \Lambda(T)]\))

\[
= aT + ckT - c \int_0^T \frac{t \Lambda(t)}{\Lambda(T)} dt
= aT + ckT - c[T - \frac{1}{\Lambda(T)} \int_0^T \Lambda(t) dt]
= aT + \frac{ck}{\Lambda(T)} \int_0^T \Lambda(t) dt.
\]

**Theorem 1.2.** The expected running cost of the system in \([0, T]\) when it is subject to shocks of intensity rate \(\lambda(t)\) and no replacements are made is

\[
aT + c \int_0^T \Lambda(t) dt.
\]

**Proof.** The expected cost in the interval \([0, T]\) can be calculated using Lemma 1.1 and the fact that the probability of \(k\) shocks in the interval \([0, T]\) is \(\frac{\Lambda_T^k}{k!} e^{-\Lambda(T)}\), whereby one obtains

\[
aT + \sum_{k=0}^{\infty} \frac{\Lambda_T^k}{k!} e^{-\Lambda(T)} \left[ \frac{ck}{\Lambda(T)} \int_0^T \Lambda(t) dt \right]
= aT + c \int_0^T \Lambda(t) dt.
\]

Note that as \(\Lambda(t)\) is the expected number of shocks in \([0, t]\),
\(c\Lambda(t)\) is the expected incremental cost at age \(t\) due to shocks.
Hence the expected running cost \( aT + c \int_0^T \Lambda(t) \, dt \) of the system in \([0, T]\) can be interpreted as a standard running cost \( aT \) plus a cost due to an accumulation of expected incremental costs due to shocks.

We consider now the periodic replacement policy with period \( T \) where we replace the system at times \( T, 2T, 3T, \ldots \). We let \( C(T) \) be the long run expected cost per unit of time when using periodic replacement with period \( T \). From Theorem 1.2 it is clear that

\[
C(T) = \left( aT + c \int_0^T \Lambda(t) \, dt + c_0 \right) / T.
\]

We seek the optimal period \( T_0 \), i.e. the value of \( T_0 \) minimizing \( C(T) \). Since \( \Lambda(t) \) is continuous and increasing, we see that \( C(T) \) has a minimum, although possibly at \( T_0 = +\infty \). Now

\[
C'(T) = \left[ \left( c \int_0^T \Lambda(t) \, dt + c_0 \right) / T \right] ' = \left( c(T\Lambda(T) - \int_0^T \Lambda(t) \, dt) - c_0 \right) / T^2.
\]

As \( T\Lambda(T) - \int_0^T \Lambda(t) \, dt \) is positive and increasing, we see that a finite optimal replacement interval exists if and only if

\[
\lim_{T \to \infty} \int_0^T [\Lambda(T) - \Lambda(t)] \, dt > \frac{c_0}{c}.
\]

This leads to

**Corollary 1.3.** Let \( c_0 \) be the replacement cost of the system, and \( c \) be the incremental running cost per unit time due to a
shock to the system. Assume $\Lambda(t)$ is the mean value function of the nonhomogeneous (or homogeneous) shock process to which the system is subject. Then an optimal finite periodic replacement interval $T_0$ exists if and only if

$$\lim_{T \to \infty} \int_0^T [\Lambda(T) - \Lambda(t)] dt > \frac{c_0}{c}. $$

In the case a finite $T_0$ does exist, then $T_0$ is the unique root of

$$T_0^\Lambda(T_0) - \int_0^{T_0} \Lambda(t) dt = \frac{c_0}{c},$$

and otherwise the optimal policy is to never replace the system.

Note that when $\Lambda(T)$ is unbounded in $T$ (which is always the case if $\lambda(t)$ is nondecreasing in $t$), then there always exists a finite $T_0$. We observe that the solution $T_0$ is a function of the cost ratio $\frac{c_0}{c}$, and not of the individual values.

**Example 1.4.** Suppose the intensity function $\lambda(t)$ takes the form $\lambda(t) = \alpha \beta (\beta t)^{\alpha - 1}$ where $\alpha, \beta > 0$. Then $\Lambda(T) = (\beta T)^\alpha$ and the optimal replacement interval $T_0$ is the solution to

$$T(\beta T)^\alpha - \int_0^T (\beta t)^\alpha dt = \frac{c_0}{c},$$

i.e.,

$$T_0 = \left( \frac{\alpha + 1}{\alpha} \frac{c_0}{\beta^\alpha c} \right)^{1/(\alpha + 1)}.$$

In the particular case when $\alpha = 1$, the shocks are arriving according to a homogeneous Poisson process with constant intensity.
\[ T_0 = \left( \frac{2c_0}{\beta c} \right)^{\frac{1}{2}}. \]

Note that the minimal long run expected cost per unit of time has the form

\[ C(T_0) = a + \frac{c_0}{\alpha + 1} \left\{ \frac{c}{c_0} \frac{1}{\alpha + 1} \right\} \frac{1/\alpha}{\alpha + 1} \frac{1}{\alpha + 1} \left( 1 + \frac{1}{\alpha} \right). \]

\( C(T_0) \) is a concave function of \( \beta \). In the homogeneous case when \( \alpha = 1 \),

\[ C(T_0) = a + \sqrt{2c_0c\beta}, \]

while as \( \alpha \to \infty \) we have

\[ \lim_{\alpha \to \infty} C(T_0) = a + \beta c_0. \]

**Example 1.5.** We assume that the intensity function is a decreasing function of \( t \). Suppose \( \lambda(t) = ae^{-\beta t} \) where \( a, \beta > 0 \). Then

\[ \Lambda(T) = \int_0^T ae^{-\beta t} dt = \frac{a}{\beta}(1 - e^{-\beta T}), \]

which is increasing in \( T \) but is not unbounded. Finding an optimal period \( T_0 \) in this case reduces to solving

\[ T\left( \frac{e^{-\beta T}}{\beta(1 - e^{-\beta T})} - \int_0^T \frac{a}{\beta}(1 - e^{-\beta t}) dt = \frac{c_0}{c} \right), \]
or
\[
\frac{\alpha}{\beta^2}[1 - (e^{-\beta T})(1 + \beta T)] = \frac{c_0}{c},
\]
or
\[
1 - \left(\frac{\beta^2}{\alpha} \frac{c_0}{c}\right) = e^{-\beta T}(1 + \beta T).
\]
This equation has a finite solution \( T_0 > 0 \) if and only if \( \frac{\beta^2 c_0}{\alpha c} < 1 \). When this is not the case (for example the cost of a replacement \( c_0 \) is larger than \( \frac{\alpha}{\beta^2} c \)), then the optimal solution is to never replace. For finite \( T_0 \), the minimal long run expected cost per unit of time has the form
\[
C(T_0) = a + \frac{c_0}{\beta}[1 - e^{-\beta T_0}].
\]

More generally we can suppose our system is subject to shocks of \( n \) types with respective intensity functions \( \lambda_1(t), \ldots, \lambda_n(t) \). We assume the \( n \) shock types act independently according to independent nonhomogeneous (or homogeneous) Poisson processes. Let \( c_i \) represent the increase in operational cost per unit of time of the system due to a shock of type \( i \) for \( i = 1, \ldots, n \). We again try to optimize long run expected cost per unit of time by selecting an optimal \( T_0 \) for periodic replacement.

We see that if \( k_i \) shocks of type \( i \) occur during the age interval \( [0, T] \) at times \( t_1^i, t_2^i, \ldots, t_{k_i}^i \), then the running cost during the interval \( [0, T] \) is
\[
aT + \sum_{i=1}^{n} c_i \sum_{j=1}^{k_i} j(t_{j+1}^i - t_j^i),
\]
where we interpret $t_{k_i+1}^i$ as $T$ for $i = 1, \ldots, n$. Proceeding as in the proofs of Lemma 1.1 and Theorem 1.2 we obtain

**Theorem 1.6.** Let a system be subject to shocks of $n$ types with respective intensity rates $\lambda_1(t), \ldots, \lambda_n(t)$. If the shock processes are independent and nonhomogeneous (or homogeneous) Poisson processes, then the optimal periodic replacement interval $T_0$ is a solution to

$$
\sum_{i=1}^{n} c_i(T\Lambda_i(T) - \int_0^T \Lambda_i(t) dt) = c_0.
$$

If at least one of the shock processes has unbounded mean function $\Lambda_i(T)$, then there must be a unique finite solution $T_0$.

**Example 1.7.** Suppose a system is subject to two types of shocks which occur independently according to homogeneous Poisson processes with rates $\lambda_1$ and $\lambda_2$. If $c_1$ and $c_2$ are the respective incremental costs to the system, then the optimum replacement interval is

$$
T_0 = [2c_0/(c_1\lambda_1 + c_2\lambda_2)]^{1/2}
$$

and the minimal long run expected cost per unit time is

$$
C(T_0) = a + \sqrt{2c_0(c_1\lambda_1 + c_2\lambda_2)}.
$$

**Remark 1.8.** Our considerations up to this point have dealt with optimization for an infinite time horizon. However we may also consider the problem of minimizing total expected cost over a finite time horizon $[0, t_0)$. Again we would like to find a
replacement interval $T_0$ which minimizes total expected cost over the interval $[0, t_0)$. We use the same cost notation as before, and assume that the system is subject to shocks of $n$ types which act independently. For period $T$, where $mT < t_0 < (m + 1)T$ for some integer $m$, we have by using Theorem 1.6 that the total expected cost in $[0, t_0)$ is

$$C_{t_0}(T) = a t_0 + m(c_0 + \sum_{i=1}^{n} c_i \int_0^{t_0} \Lambda_i(t) dt)$$

$$+ \sum_{i=1}^{n} c_i \int_{0}^{t_0 - mT} \Lambda_i(t) dt.$$ 

Hence

$$C_{t_0}^*(T) = m \sum_{i=1}^{n} c_i [\Lambda_i(T) - \Lambda_i(t_0 - mT)] \geq 0$$

since each $\Lambda_i$ is an increasing function of $t$. Therefore we may conclude that the minimum of $C_{t_0}(T)$ must occur at one of the points $\{t_0, t_0/2, t_0/3, \ldots\}$. This is of practical significance since $C_{t_0}(t_0/j) \to \infty$ as $j \to \infty$, and hence it is generally computationally easy to find the optimum period $T_0$.

For example suppose a system is subject to one type of shock where the shocks occur with intensity $\alpha \beta(t)^{\alpha-1}$. Then the optimal replacement interval $T_0$ during the time interval $[0, t_0)$ is of the form $t_0/m$, where $m$ is the integer minimizing

$$\left\{ at_0 + (m - 1)c_0 + \frac{\beta^\alpha t_0^{\alpha+1}}{(\alpha + 1)m^\alpha} \right\}_{m=1}^{\infty}.$$

In this section we model a system which fails at the second shock. It then must be replaced by a new system at a cost of \(c_R\) (which is larger than the cost \(c_0\) of a planned replacement). We let \(a\) and \(c\) have the same meaning as before, but in this section we consider only the shock model where shocks of one type arrive according to a homogeneous Poisson process with constant intensity \(\lambda\). Our goal is to find the periodic replacement interval \(T_0\) which minimizes long run expected cost per unit of time.

Suppose we are considering using the periodic replacement interval \(T\), and that \(k\) is the number of shocks occurring in the interval \([0, T]\). Let \(t_1 < t_2 < \ldots < t_k\) be the times of the shocks. The cost incurred depends on these times and is

\[
aT + mc_R + c \sum_{i=1}^{m} (t_{2i} - t_{2i-1}) \quad \text{if } k = 2m
\]

and

\[
aT + mc_R + c \left( \sum_{i=1}^{m} (t_{2i} - t_{2i-1}) + T - t_{2m+1} \right) \quad \text{if } k = 2m + 1.
\]

Hence given \(k\) shocks the expected cost is

\[
\left\{ \begin{array}{ll}
aT + mc_R + \frac{cmT}{2m+1}\quad & \text{if } k = 2m \\
aT + mc_R + \frac{c(m+1)T}{2(m+1)}\quad & \text{if } k = 2m + 1.
\end{array}\right.
\]
Hence summing over all possible values of \( k \), we find that the expected running cost in \([0, T]\) is

\[
\sum_{m=0}^{\infty} \frac{\lambda T^{2m}}{(2m)!} e^{-\lambda T} \left[ aT + mc_R + \frac{cmT}{2m + 1} \right]
\]

\[
+ \sum_{m=0}^{\infty} \frac{\lambda T^{2m+1}}{(2m + 1)!} e^{-\lambda T} \left[ aT + mc_R + \frac{c}{2} T \right]
\]

\[
= aT + cR^M(T) + c \left[ \sum_{m=0}^{\infty} \frac{\lambda T^{2m}}{(2m)!} e^{-\lambda T} \frac{mT}{2m + 1} + \sum_{m=0}^{\infty} \frac{\lambda T^{2m+1}}{(2m + 1)!} e^{-\lambda T} \frac{T}{2} \right]
\]

(\text{where } M(T) = \frac{\lambda T}{2} - \frac{1}{4} + \frac{1}{4} e^{-2\lambda T} \text{ is the renewal function, i.e., the expected number of system failures})

\[
= -\frac{1}{4}(c_R + \frac{c}{\lambda}) + T(a + \frac{cR^\lambda}{2} + \frac{c}{2}) + \frac{1}{4} e^{-2\lambda T}(c_R + \frac{c}{\lambda})
\]

after a bit of calculation. Therefore the long run expected cost per unit of time using the replacement interval \( T \) is

\[
C(T) = [c_0 - \frac{1}{4}(c_R + \frac{c}{\lambda}) + T(a + \frac{cR^\lambda}{2} + \frac{c}{2}) + \frac{1}{4} e^{-2\lambda T}(c_R + \frac{c}{\lambda})]/T
\]

\[
= [c_0 - b + 2\lambda T(a + b) + e^{-2\lambda T}b]/T,
\]

where we have made the substitution \( b = \frac{1}{4}(c_R + \frac{c}{\lambda}) \). We note that

\[
\lim_{T \to 0} C(T) = +\infty \text{ and } \lim_{T \to \infty} C(T) = a + 2\lambda b. \text{ As } C(T) \text{ is continuous, there is a value of } T_0 \text{ (possibly } \infty) \text{ for which } C(T) \text{ is minimized.}
\]

Note that if \( c_0 \geq b \) (i.e., \( c_0 \geq \frac{1}{4}(c_R + \frac{c}{\lambda}) \)), which is the case if for instance the cost of planned replacement is more than \( \frac{1}{4} \) the cost of unplanned replacement), then \( C(T) \) is a decreasing function of \( T \) and the optimal replacement interval is \( T_0 = +\infty \).
Let us therefore consider the case when \( c_0 < b = \frac{1}{4}(c_R + \frac{c}{\lambda}) \) (This would be the case if for example the cost of unplanned replacement were > 4 (cost of a planned replacement)). Now

\[
C(T) = a + 2\lambda b + \frac{c_0 - b(1 - e^{-2\lambda T})}{T}.
\]

Differentiating \( C(T) \), we see that \( T_0 \) is a root of \( C'(T) = 0 \) (and a minimum point of \( C(T) \)) if and only if \( T_0 \) is a root of

\[
e^{-2\lambda T}(1 + 2\lambda T) = \frac{b - c_0}{b}.
\]

Hence when \( c_0 < b \), the optimal replacement interval \( T_0 \) is the unique solution to

\[
e^{-2\lambda T}(1 + 2\lambda T) = \frac{b - c_0}{b},
\]

and the resulting long run expected cost per unit of time is

\[
C(T_0) = a + 2\lambda b[1 - e^{-2\lambda T_0}].
\]


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